Math2343: Problem Set 1

1. Let $\mathbb{Z}$ be a universal set. Let A be the set of integers whose elements are multiples of 3, and B the set of integers whose elements are the multiples of 5. Find the intersection $A \cap B$, union $A \cup B$, complement $\overline{A}$, relative complement $A \smallsetminus B$, and the symmetric difference $A \oplus B = (A \smallsetminus B) \cup (B \smallsetminus A)$.

2. For arbitrary sets A, B, C, show that $A \smallsetminus (B \smallsetminus C) = (A \smallsetminus B) \cup C$ if and only if $C \subseteq A$.

3. Let $X = \{0, 1, 2, 3, 4\}$ and $Y = \{0, 2, 4\}$.
   (a) How many ordered pairs are in $X \times Y$, $Y \times X$, and $Y \times X \times Y$ respectively?
   (b) List the elements in the set $\{(a, b, c) \in X \times Y \times X : a < b < c\}$.

4. Let $\Sigma = \{a, b\}$, $A = \{w \in \Sigma^* : 1 \leq \text{length}(w) \leq 2\}$, $B = \{w \in \Sigma^* : \text{length}(w) \geq 2\}$.
   (a) Determine the sets $A \cap B$, $A \smallsetminus B$, $B \smallsetminus A$, and $A \oplus B$.
   (b) Determine the set $C = \{w \in A^* : \text{length}(w) \leq 2\}$.
   (c) List the elements of the power set $\mathcal{P}(\Sigma)$ of Σ, and the power set $\mathcal{P}(\mathcal{P}(\Sigma))$ of the set $\mathcal{P}(\Sigma)$.

5. Let A be a set and let $A_i, i \in I$ be a family of sets. Show that
   (a) $A \cap \left(\bigcup_{i \in I}^{n} A_i\right) = \bigcup_{i \in I}(A \cap A_i)$.
   (b) $A \cup \left(\bigcap_{i \in I} A_i\right) = \bigcap_{i \in I}(A \cup A_i)$.
   (c) If $A \subseteq A_i$ for all $i \in I$, then $A \subseteq \bigcap_{i \in I} A_i$.
   (d) If $A_i \subseteq A$ for all $i \in I$, then $\bigcup_{i \in I} A_i \subseteq A$.

6. Let $f : X \to Y$ be a function. Given subsets A_1, A_2, and a family subsets A_i of X with $i \in I$. And given subsets B_1, B_2, and a family subsets B_j of Y with $j \in J$. Show the following statements.
   (a) If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$.
   (b) If $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.
   (c) $f\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} f(A_i)$.
   (d) $f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i)$.
   (e) $f^{-1}\left(\bigcap_{j \in J} B_j\right) = \bigcap_{j \in J} f^{-1}(B_j)$.
   (f) $f^{-1}\left(\bigcup_{j \in J} B_j\right) = \bigcup_{j \in J} f^{-1}(B_j)$.

7. Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be functions defined respectively by
$$f(n) = \frac{(-1)^n + 1}{2} \quad \text{and} \quad g(n) = \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor.$$
   (a) Find the domain and codomain of f and g;
   (b) Find $g \circ f$ and $f \circ g$;
   (c) Show that f and g are characteristic functions of some subsets of $\mathbb{Z}$.

8. Let A be a nonempty set, and let $A^{[n]}$ be the set of all functions from $[n] = \{1, 2, \ldots, n\}$ to A. Find a bijection from $A^{[n]}$ to the Cartesian product A^n.

9. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show the following statements.
   (a) If f and g are one-to-one, then $g \circ f$ is one-to-one;
   (b) If f and g are onto, then $g \circ f$ is onto;
   (c) If $g \circ f$ is one-to-one, then f is one-to-one;
   (d) If $g \circ f$ is onto, then g is onto.

10. Let $f : \mathbb{R} \smallsetminus \{0, 1\} \to \mathbb{R} \smallsetminus \{0, 1\}$ be defined by $f(x) = 1 - x$. Verify that f is a bijection and determine the set $\{f^k : k \in \mathbb{Z}\}$, where f^0 is the identity function.