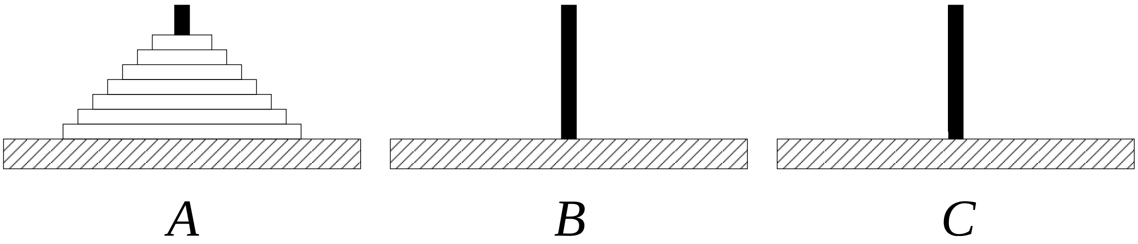
# Recurrence Relations

## 1 Infinite Sequences

An infinite sequence is a function from the set of positive integers to the set of real numbers or to the set of complex numbers.

**Example 1.1.** The game of **Hanoi Tower** is to play with a set of disks of graduated size with holes in their centers and a playing board having three spokes for holding the disks.




The object of the game is to transfer all the disks from spoke $\boldsymbol{A}$ to spoke $\boldsymbol{C}$ by moving one disk at a time without placing a larger disk on top of a smaller one. What is the minimal number of moves required when there are $n$ disks?

*Solution.* Let $a_n$ be the minimum number of moves to transfer $n$ disks from one spoke to another. In order to move $n$ disks from spoke $\boldsymbol{A}$ to spoke $\boldsymbol{C}$, one must move the first $n-1$ disks from spoke $\boldsymbol{A}$ to spoke $\boldsymbol{B}$ by $a_{n-1}$ moves, then move the last (also the largest) disk from spoke $\boldsymbol{A}$ to spoke $\boldsymbol{C}$ by one move, and then remove the $n-1$ disks again from spoke $\boldsymbol{B}$ to spoke $\boldsymbol{C}$ by $a_{n-1}$ moves. Thus the total number of moves should be

$$a_n = a_{n-1} + 1 + a_{n-1} = 2a_{n-1} + 1.$$

This means that the sequence $\{a_n \mid n \geq 1\}$ satisfies the recurrence relation

$$\left\{ \begin{array}{rcl} a_n & = & 2a_{n-1} + 1, \;\; n \geq 1 \\ a_1 & = & 1. \end{array} \right. \tag{1}$$

Applying the recurrence relation again and again, we have

$$
\begin{aligned}
a_1 &= 2a_0 + 1 \\
a_2 &= 2a_1 + 1 = 2(2a_0 + 1) + 1 \\
&= 2^2 a_0 + 2 + 1 \\
a_3 &= 2a_2 + 1 = 2(2^2 a_0 + 2 + 1) + 1 \\
&= 2^3 a_0 + 2^2 + 2 + 1 \\
a_4 &= 2a_3 + 1 = 2(2^3 a_0 + 2^2 + 2 + 1) + 1 \\
&= 2^4 a_0 + 2^3 + 2^2 + 2 + 1 \\
&\vdots \\
a_n &= 2^n a_0 + 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \\
&= 2^n a_0 + 2^n - 1.
\end{aligned}
$$

Let $a_0 = 0$. The general term is given by

$$a_n = 2^n - 1, \quad n \geq 1.$$

Given a recurrence relation for a sequence with initial conditions. Solving the recurrence relation means to find a formula to express the general term $a_n$ of the sequence.

## 2 Homogeneous Recurrence Relations

Any recurrence relation of the form

$$x_n = a x_{n-1} + b x_{n-2} \tag{2}$$

is called a **second order homogeneous linear recurrence relation**.

Let $x_n = s_n$ and $x_n = t_n$ be two solutions, i.e.,

$$s_n = a s_{n-1} + b s_{n-2} \quad \text{and} \quad t_n = a t_{n-1} + b t_{n-2}.$$

Then for constants $c_1$ and $c_2$

$$
\begin{aligned}
c_1 s_n + c_2 t_n &= c_1(a s_{n-1} + b s_{n-2}) + c_2(a t_{n-1} + b t_{n-2}) \\
&= a(c_1 s_{n-1} + c_2 t_{n-1}) + b(c_1 s_{n-2} + c_2 t_{n-2}).
\end{aligned}
$$

This means that $x_n = c_1 s_n + c_2 t_n$ is a solution of (2).

**Theorem 2.1.** *Any linear combination of solutions of a homogeneous recurrence linear relation is also a solution.*

In solving the first order homogeneous recurrence linear relation

$$x_n = ax_{n-1},$$

it is clear that the general solution is

$$x_n = a^n x_0.$$

This means that $x_n = a^n$ is a solution. This suggests that, for the second order homogeneous recurrence linear relation (2), we may have the solutions of the form

$$x_n = r^n.$$

Indeed, put $x_n = r^n$ into (2). We have

$$r^n = ar^{n-1} + br^{n-2} \quad \text{or} \quad r^{n-2}(r^2 - ar - b) = 0.$$

Thus either $r = 0$ or

$$r^2 - ar - b = 0. \tag{3}$$

The equation (3) is called the **characteristic equation** of (2).

**Theorem 2.2.** *If the characteristic equation* (3) *has two distinct roots $r_1$ and $r_2$, then the general solution for* (2) *is given by*

$$x_n = c_1 r_1^n + c_2 r_2^n.$$

*If the characteristic equation* (3) *has only one root $r$, then the general solution for* (2) *is given by*

$$x_n = c_1 r^n + c_2 n r^n.$$

*Proof.* When the characteristic equation (3) has two distinct roots $r_1$ and $r_2$ it is clear that both

$$x_n = r_1^n \quad \text{and} \quad x_n = r_2^n$$

are solutions of (2), so are their linear combinations.

Recall that $r = \frac{a \pm \sqrt{a^2+4b}}{2}$. Now assume that (2) has only one root $r$. Then

$$a^2 + 4b = 0 \quad \text{and} \quad r = a/2.$$

Thus
$$b = -\frac{a^2}{4} \quad \text{and} \quad r = \frac{a}{2}.$$
We verify that $x_n = nr^n$ is a solution of (2). In fact,
$$\begin{aligned} ax_{n-1} + bx_{n-2} &= a(n-1)\left(\frac{a}{2}\right)^{n-1} + \left(-\frac{a^2}{4}\right)(n-2)\left(\frac{a}{2}\right)^{n-2} \\ &= [2(n-1)-(n-2)]\left(\frac{a}{2}\right)^{n} = n\left(\frac{a}{2}\right)^{n} = x_n. \end{aligned}$$

□

**Remark.** There is heuristic method to explain why $x_n = nr^n$ is a solution when the two roots are the same. If two roots $r_1$ and $r_2$ are distinct but very close to each other, then $r_1^n - r_2^n$ is a solution. So is $(r_1^n - r_2^n)/(r_1 - r_2)$. It follows that the limit
$$\lim_{r_2 \to r_1} \frac{r_1^n - r_2^n}{r_1 - r_2} = nr_1^{n-1}$$
would be a solution. Thus its multiple $x_n = r_1(nr_1^{n-1}) = nr_1^n$ by the constant $r_1$ is also a solution. Please note that this is *not* a mathematical proof, but a mathematical idea.

**Example 2.1.** Find a general formula for the **Fibonacci sequence**
$$\begin{cases} f_n &= f_{n-1} + f_{n-2} \\ f_0 &= 0 \\ f_1 &= 1 \end{cases}$$

*Solution.* The characteristic equation $r^2 = r + 1$ has two distinct roots
$$r_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad r_2 = \frac{1-\sqrt{5}}{2}.$$
The general solution is given by
$$f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Set

$$\begin{cases} 0 = c_1 + c_2 \\ 1 = c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right). \end{cases}$$

We have $c_1 = -c_2 = \frac{1}{\sqrt{5}}$. Thus

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n, \quad n \geq 0.$$

**Remark.** The Fibonacci sequence $f_n$ is an integer sequence, but it "looks like" a sequence of irrational numbers from its general formula above.

**Example 2.2.** Find the solution for the recurrence relation

$$\begin{cases} x_n &= 6x_{n-1} - 9x_{n-2} \\ x_0 &= 2 \\ x_1 &= 3 \end{cases}$$

*Solution.* The characteristic equation

$$r^2 - 6r + 9 = 0 \Longleftrightarrow (r-3)^2 = 0$$

has only one root $r = 3$. Then the general solution is

$$x_n = c_1 3^n + c_2 n 3^n.$$

The initial conditions $x_0 = 2$ and $x_1 = 3$ imply that $c_1 = 2$ and $c_2 = -1$. Thus the solution is

$$x_n = 2 \cdot 3^n - n \cdot 3^n = (2-n)3^n, \ \ n \geq 0.$$

**Example 2.3.** Find the solution for the recurrence relation

$$\begin{cases} x_n &= 2x_{n-1} - 5x_{n-2}, \ \ n \geq 2 \\ x_0 &= 1 \\ x_1 &= 5 \end{cases}$$

*Solution.* The characteristic equation

$$r^2 - 2r + 5 = 0 \Longleftrightarrow (x-1-2i)(x-1+2i) = 0$$

has two distinct complex roots $r_1 = 1+2i$ and $r_2 = 1-2i$. The initial conditions imply that

$$c_1 + c_2 = 1 \quad c_1(1+2i) + c_2(1-2i) = 5.$$

So $c_1 = \frac{1-2i}{2}$ and $c_2 = \frac{1+2i}{2}$. Thus the solutions is

$$\begin{aligned} x_n &= \frac{1-2i}{2} \cdot (1+2i)^n + \frac{1+2i}{2} \cdot (1-2i)^n \\ &= \frac{5}{2}(1+2i)^{n+1} + \frac{5}{2}(1-2i)^{n+1}, \;\; n \geq 0. \end{aligned}$$

**Remark.** The sequence is obviously a real sequence. However, its general formula involves complex numbers.

**Example 2.4.** Two persons A and B gamble dollars on the toss of a fair coin. A has \$70 and B has \$30. In each play either A wins \$1 from B or loss \$1 to B. The game is played without stop until one wins all the money of the other or goes forever. Find the probabilities of the following three possibilities:

(a) A wins all the money of B.

(b) A loss all his money to B.

(c) The game continues forever.

*Solution.* Either A or B can keep track of the game simply by counting their own money. Their position $n$ (number of dollars) can be one of the numbers $0, 1, 2, \ldots, 100$. Let

$$p_n = \text{probability that A reaches 100 at position } n.$$

After one toss, A enters into either position $n+1$ or position $n-1$. The new probability that A reaches 100 is either $p_{n+1}$ or $p_{n-1}$. Since the probability of A moving to position $n+1$ or $n-1$ from $n$ is $\frac{1}{2}$. We obtain the recurrence relation

$$\begin{cases} p_n &= \frac{1}{2}p_{n+1} + \frac{1}{2}p_{n-1} \\ p_0 &= 0 \\ p_{100} &= 1 \end{cases}$$

**First Method:** The characteristic equation

$$r^2 - 2r + 1 = 0.$$

has only one root $r = 1$. The general solutions is

$$p_n = c_1 + c_2 n.$$

Applying the boundary conditions $p_0 = 0$ and $p_{100} = 1$, we have

$$c_1 = 0 \quad \text{and} \quad c_2 = \frac{1}{100}.$$

Thus

$$p_n = \frac{n}{100}, \quad 0 \le n \le 100.$$

Of course, $p_n = \frac{n}{100}$ for $n > 100$ is nonsense to the original problem. The probabilities for (a), (b), and (c) are 70%, 30%, and 0, respectively.

**Second Method:** The recurrence relation $p_n = \frac{1}{2}p_{n+1} + \frac{1}{2}p_{n-1}$ can be written as

$$p_{n+1} - p_n = p_n - p_{n-1}.$$

Then

$$p_{n+1} - p_n = p_n - p_{n-1} = \cdots = p_1 - p_0.$$

Since $p_0 = 0$, we have $p_n = p_{n-1} + p_1$. Applying the recurrence relation again and again, we obtain

$$p_n = p_0 + np_1.$$

Applying the conditions $p_0 = 0$ and $p_{100} = 1$, we have $p_n = \frac{n}{100}$.

## 3 Higher Order Homogeneous Recurrence Relations

For a **higher order homogeneous recurrence relation**

$$x_{n+k} = a_1 x_{n+k-1} + a_2 x_{n+k-2} + \cdots + a_{n-k} x_n, \quad n \ge 0 \tag{4}$$

we also have the **characteristic equation**

$$t^k = a_1 t^{k-1} + a_2 t^{k-1} + \cdots + a_{n-k+1} t + a_{n-k} \tag{5}$$

or

$$t^k - a_1 t^{k-1} - a_2 t^{k-1} - \cdots - a_{n-k+1} t - a_{n-k} = 0.$$

**Theorem 3.1.** *For the recurrence relation* (4), *if its characteristic equation* (5) *has distinct roots* $r_1, r_2, \ldots, r_k$, *then the general solution for* (4) *is*

$$x_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n$$

*where* $c_1, c_2, \ldots, c_k$ *are arbitrary constants. If the characteristic equation has repeated roots* $r_1, r_2, \ldots, r_s$ *with multiplicities* $m_1, m_2, \ldots, m_s$ *respectively, then the general solution of* (4) *is a linear combination of the solutions*

$$\begin{array}{llll} r_1^n, & nr_1^n, & \ldots, & n^{m_1-1} r_1^n; \\ r_2^n, & nr_2^n, & \ldots, & n^{m_2-1} r_2^n; \\ & & \ldots; & \\ r_s^n, & nr_s^n, & \ldots, & n^{m_s-1} r_s^n. \end{array}$$

**Example 3.1.** Find an explicit formula for the sequence given by the recurrence relation

$$\begin{cases} x_n = 15x_{n-2} - 10x_{n-3} - 60x_{n-4} + 72x_{n-5} \\ x_0 = 1,\ x_1 = 6,\ x_2 = 9,\ x_3 = -110,\ x_4 = -45 \end{cases}$$

*Solution.* The characteristic equation

$$r^5 = 15r^3 - 10r^2 - 60r + 72$$

can be simplified as

$$(r-2)^3(r+3)^2 = 0.$$

There are roots $r_1 = 2$ with multiplicity 3 and $r_2 = -3$ with multiplicity 2. The general solution is given by

$$x_n = c_1 2^n + c_2 n 2^n + c_3 n^2 2^n + c_4(-3)^n + c_5 n(-3)^n.$$

The initial condition means that

$$\left\{ \begin{array}{rrrrrcl} c_1 & & & +c_4 & & = & 1 \\ 2c_1 & +2c_2 & +2c_3 & -3c_4 & -3c_5 & = & 1 \\ 4c_1 & +8c_2 & +16c_3 & +9c_4 & +18c_5 & = & 1 \\ 8c_1 & +24c_2 & +72c_3 & -27c_4 & -81c_5 & = & 1 \\ 16c_1 & +64c_2 & +256c_3 & +81c_4 & +324c_5 & = & 1 \end{array} \right.$$

Solving the linear system we have

$$c_1 = 2,\ c_2 = 3,\ c_3 = -2,\ c_4 = -1,\ c_5 = 1.$$

## 4 Non-homogeneous Equations

A recurrence relation of the form

$$x_n = ax_{n-1} + bx_{n-2} + f(n) \tag{6}$$

is called a **non-homogeneous recurrence relation**.

Let $x_n^{(s)}$ be a solution of (6), called a **special solution**. Then the general solution for (6) is

$$x_n = x_n^{(s)} + x_n^{(h)}, \tag{7}$$

where $x_n^{(h)}$ is the general solution for the corresponding homogeneous recurrence relation

$$x_n = ax_{n-1} + bx_{n-2}. \tag{8}$$

**Theorem 4.1.** *Let $f(n) = cr^n$ in (6). Let $r_1$ and $r_2$ be the roots of the characteristic equation*

$$t^2 = at + b. \tag{9}$$

(a) *If $r \neq r_1$, $r \neq r_2$, then $x_n^{(s)} = Ar^n$;*

(b) *If $r = r_1$, $r_1 \neq r_2$, then $x_n^{(s)} = Anr^n$;*

(c) *If $r = r_1 = r_2$, then $x_n^{(s)} = An^2r^n$;*

*where $A$ is a constant to be determined in all cases.*

*Proof.* We assume $r \neq 0$. Otherwise the recurrence relation is homogeneous.

(a) Put $x_n = Ar^n$ into (6). We have

$$Ar^n = aAr^{n-1} + bAr^{n-2} + cr^n.$$

Thus

$$A(r^2 - ar - b) = cr^2.$$

Since $r$ is not a root of the characteristic equation (9), then $r^2 - ar - b \neq 0$. Hence

$$A = \frac{cr^2}{r^2 - ar - b}.$$

(b) Since $r = r_1 \neq r_2$, it is clear that $x_n = nr^n$ is not a solution for its corresponding homogeneous equation (8), i.e.,

$$\begin{aligned} nr^2 - a(n-1)r - b(n-2) &= n(r^2 - ar - b) + ar + 2b \\ &= ar + 2b \neq 0. \end{aligned}$$

Put $x_n = Anr^n$ into (6). We have

$$Anr^n = aA(n-1)r^{n-1} + bA(n-2)r^{n-2} + cr^n,$$

Thus $A(nr^2 - a(n-1)r - b(n-2)) = cr^2$. Therefore

$$A = \frac{cr^2}{ar + 2b}.$$

(c) Since $r = r_1 = r_2$, then $a^2 + 4b = 0$ (discriminant of $r^2 - ar - b = 0$ must be zero), $r = a/2$, and $x_n = n^2r^n$ is not a solution of the corresponding homogeneous equation (8), i.e.,

$$\begin{aligned} & n^2r^2 - a(n-1)^2r - b(n-2)^2 \\ &= n^2(r^2 - ar - b) + 2n(ar + 2b) - ar - 4b \\ &= -ar - 4b \neq 0. \end{aligned}$$

Put $x_n = An^2r^n$ into (6). We have

$$Ar^{n-2}\left(n^2r^2 - a(n-1)^2r - b(n-2)^2\right) = cr^n.$$

Thus

$$A = -\frac{cr^2}{ar + 4b}.$$

□

**Example 4.1.** Consider the non-homogeneous equation

$$\begin{cases} x_n = 3x_{n-1} + 10x_{n-2} + 7 \cdot 5^n \\ x_0 = 4 \\ x_1 = 3 \end{cases}$$

*Solution.* The characteristic equation is

$$t^2 - 3t - 10 = 0 \iff (t-5)(t+2) = 0.$$

We have roots $r_1 = 5$, $r_2 = -2$. Since $r = 5$, then $r = r_1$ and $r \neq r_2$. A special solution can be of the type $x_n = An5^n$. Put the solution into the non-homogeneous relation. We have

$$An5^n = 3A(n-1)5^{n-1} + 10A(n-2)5^{n-2} + 7 \cdot 5^n$$

Dividing both sides by $5^{n-2}$,

$$An5^2 = 3A(n-1)5 + 10A(n-2) + 7 \cdot 5^2.$$

Thus

$$-35A + 7 \cdot 25 = 0 \Longrightarrow A = 5.$$

So

$$x_n = n5^{n+1}.$$

The general solution is

$$x_n = n5^{n+1} + c_1 5^n + c_2(-2)^n.$$

The initial condition implies $c_1 = -2$ and $c_2 = 6$. Therefore

$$x_n = n5^{n+1} - 2 \cdot 5^n + 6(-2)^n.$$

**Example 4.2.** Consider the non-homogeneous equation

$$\begin{cases} x_n &= 10x_{n-1} - 25x_{n-2} + 8 \cdot 5^n \\ x_0 &= 6 \\ x_1 &= 10 \end{cases}$$

*Solution.* The characteristic equation is

$$t^2 - 10t + 25 = 0 \Longleftrightarrow (t-5)^2 = 0.$$

We have roots $r_1 = r_2 = 5$, then $r = r_1 = r_2 = 5$. A special solution can be of the type $x_n = An^2 5^n$. Put the solution into the non-homogeneous relation. We have

$$An^2 5^n = 10A(n-1)^2 5^{n-1} - 25A(n-2)^2 5^{n-2} + 8 \cdot 5^n$$

Dividing both sides by $5^{n-2}$,

$$An^2 5^2 = 10A(n-1)^2 5 - 25A(n-2)^2 + 8 \cdot 5^2.$$

Since $An^2 5^2 = 10An^2 5 - 25n^2$, we have

$$10A(-2n+1)5 - 25A(-4n+4) + 8 \cdot 5^2 = 0 \Longrightarrow A = 4.$$

So a nonhomogeneous solution is

$$x_n = 4n^2 5^n.$$

The general solution is

$$x_n = 4n5^n + c_1 5^n + c_2 n 5^n.$$

The initial condition implies $c_1 = 6$ and $c_2 = -8$. Therefore

$$x_n = (4n^2 - 8n + 6)5^n.$$

## 5 Divide-and-Conquer Method

Assume we have a job of size $n$ to be done. If the size $n$ is large and the job is complicated, we may divide the job into smaller jobs of the same type and of the same size, then conquer the smaller problems and use the results to construct a solution for the original problem of size $n$. This is the essential idea of the so-called **Divide-and-Conquer** method.

**Example 5.1.** Assume there are $n$ $(= 2^k)$ student files, indexed by the student ID numbers as

$$A = \{a_1, a_2, \ldots, a_n\}.$$

Given a particular file $a \in A$. What is the number of comparisons needed in worst case to find the position of the file $a$?

*Solution.* Let $x_n$ denote the number of comparisons needed to find the position of the file $a$ in worst case. Then the answer depends on whether or not the files are sorted.

Case I: The files in $A$ are not sorted. Then the answer is at most $n$ comparisons.

Case II: The files in $A$ are sorted in the order $a_1 < a_2 < \cdots < a_n$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a_1$ | $a_2$ | $\cdots$ | $a_{\frac{n}{2}-1}$ | $a_{\frac{n}{2}}$ | $a_{\frac{n}{2}+1}$ | $\cdots$ | $a_{n-1}$ | $a_n$ |

We may compare the file $a$ with $a_{\frac{n}{2}}$. If $a = a_{\frac{n}{2}}$, the job is done by one comparison. If $a < a_{\frac{n}{2}}$, consider the subset $\{a_1, a_2, \ldots, a_{\frac{n}{2}}\}$. If $a > a_{\frac{n}{2}}$, consider the subset $\{a_{\frac{n}{2}+1}, a_{\frac{n}{2}+2}, \ldots, a_n\}$. Then the number of comparisons is at most $x_{\frac{n}{2}} + 1$. We thus obtain a recurrence relation

$$\begin{cases} x_n &= x_{\frac{n}{2}} + 1 \\ x_1 &= 1 \end{cases}$$

Applying the recurrence relation again and again, we obtain

$$x_n = x_{\frac{n}{2}} + 1 = x_{\frac{n}{2^2}} + 2 = x_{\frac{n}{2^3}} + 3 = \cdots = x_{\frac{n}{2^k}} + k = x_1 + k = k + 1.$$

Since $n = 2^k$, we have $k = \log_2 n$. Therefore

$$x_n = \log_2 n + 1.$$

**Example 5.2.** Let $S = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{Z}$, where $n = 2^k$ and $k \geq 1$. How many number of comparisons are needed in worst case to find the minimum in $S$? We assume that the numbers in $S$ are not sorted.

*Solution.* The number of comparisons depends on the method we employed. If all possible pairs of elements in $S$ are compared, then the minimum will be found, and the number of comparisons in worst case is

$$\binom{n}{2} = \frac{n(n-1)}{2} = O(n^2).$$

Of course this is not best possible.

There is another method to find a better solution. Let $x_n$ be the least number of comparisons needed in worst case to find the minimum in $S$. Obviously, $x_1 = 0$ and $x_2 = 1$. For $n = 2^k$ and $k \geq 1$, we may divide $S$ into two subsets

$$\begin{array}{ll} S_1 = \{a_1, a_2, \ldots, a_{\frac{n}{2}}\}, & |S_1| = \frac{n}{2}, \\ S_2 = \{a_{\frac{n}{2}+1}, a_{\frac{n}{2}+2}, \ldots, a_n\}, & |S_2| = \frac{n}{2}. \end{array}$$

It takes $x_{\frac{n}{2}}$ comparisons to find the minimum $m_1$ for $S_1$ and the minimum $m_2$ for $S_2$. Then compare $m_1$ with $m_2$ to determine the minimum in $S$. In this way the total number of comparisons for $S$ in worst case is $2x_{\frac{n}{2}} + 1$. We thus obtain a recurrence relation

$$\begin{cases} x_n &= 2x_{\frac{n}{2}} + 1 \\ x_2 &= 1 \end{cases}$$

Applying the recurrence relation again and again, we have

$$
\begin{aligned}
x_n &= 2\left(2x_{\frac{n}{2^2}}+1\right)+1 = 2^2 x_{\frac{n}{2^2}} + 2 + 1 \\
&= 2^2\left(2x_{\frac{n}{2^3}}+1\right)+2+1 = 2^3 x_{\frac{n}{2^3}} + 2^2 + 2 + 1 \\
&= \cdots = 2^{k-1} x_{\frac{n}{2^{k-1}}} + 2^{k-2} + \cdots + 2 + 1 \\
&= 2^{k-1} + \cdots + 2 + 1 = \frac{2^k - 1}{2-1} \\
&= n - 1 = O(n).
\end{aligned}
$$

We hope that we understand the nature of divide-and-conquer method by the above examples. In order to solve a problem of size $n$, if the size $n$ is large and the problem is complicated, we divide the problem into $a$ smaller subproblems of the same type and of the same size $\lceil \frac{n}{b} \rceil$, where $a, b \in \mathbb{Z}_+$, $1 \le a < n$ and $1 < b < n$. Then we solve the $a$ smaller subproblems and use the results to construct a solution for the original problem of size $n$. We are especially interested in the case where $n = b^k$ and $b = 2$.

**Theorem 5.1** (Divide-and-Conquer Algorithm)**.** *Let $f(n)$ denote the time to solve a problem of size $n$. Assume that $f(n)$ satisfies the following two properties:*

*(a) The time to solve the initial problem of size $n = 1$ is a constant $c \ge 0$.*

*(b) The time to break the given problem of size $n$ into $a$ smaller same type subproblems, together with the time to construct a solution for the original problem by using the solutions for the $a$ subproblems, is a function $h(n)$;*

*Then the time complexity function $f(n)$ is given by the recurrence relation*

$$
\begin{cases} f(1) = c \\ f(n) = af(\frac{n}{b}) + h(n), \qquad n = b^k, k \ge 1 \end{cases}
$$

**Theorem 5.2.** *Let $f : \mathbb{Z}_+ \longrightarrow \mathbb{R}$ be a function satisfying the recurrence relation*

$$
f(n) = af\left(\frac{n}{b}\right) + c, \quad n = b^k,\ k \ge 1 \tag{10}
$$

*where $a, b, c$ are positive integers, $b \geq 2$. Then*

$$f(n) = \begin{cases} f(1) + c\log_b n & \text{for} \quad a = 1 \\ f(1)n^{\log_b a} + c\left(\frac{n^{\log_b a}-1}{a-1}\right) & \text{for} \quad a \neq 1 \end{cases} \tag{11}$$

*Proof.* Applying the recurrence relation, we obtain

$$\begin{aligned}
f(n) &= af(\tfrac{n}{b}) + c \\
af(\tfrac{n}{b}) &= a^2 f(\tfrac{n}{b^2}) + ac \\
a^2 f(\tfrac{n}{b^2}) &= a^3 f(\tfrac{n}{b^3}) + a^2 c \\
&\vdots \\
a^{k-2} f(\tfrac{n}{b^{k-2}}) &= a^{k-1} f(\tfrac{n}{b^{k-1}}) + a^{k-2} c \\
a^{k-1} f(\tfrac{n}{b^{k-1}}) &= a^k f(\tfrac{n}{b^k}) + a^{k-1} c
\end{aligned}$$

Adding both sides of the above $k$ equations and cancelling the like common terms, we have

$$\begin{aligned}
f(n) &= a^k f\left(\frac{n}{b^k}\right) + \left(c + ac + a^2 c + \cdots + a^{k-1} c\right) \\
&= a^k f(1) + c\left(1 + a + a^2 + \cdots + a^{k-1}\right).
\end{aligned}$$

Since $n = b^k$, then $k = \log_b n$. Thus

$$a^k = a^{\log_b n} = \left(b^{\log_b a}\right)^{\log_b n} = \left(b^{\log_b n}\right)^{\log_b a} = n^{\log_b a}.$$

Therefore

$$f(n) = n^{\log_b a} f(1) + c\left(1 + a + a^2 + \cdots + a^{k-1}\right).$$

If $a = 1$, we have

$$f(n) = f(1) + c\log_b n.$$

If $a \neq 1$, we have

$$\begin{aligned}
f(n) &= a^k f(1) + c\left(\frac{a^k - 1}{a - 1}\right) \\
&= f(1)n^{\log_b a} + c\left(\frac{n^{\log_b a} - 1}{a - 1}\right).
\end{aligned}$$

$\square$

## 6 Growth of Functions

Let $f$ and $g$ be functions on the set $\mathbb{P}$ of positive integers. If there exist positive constant $C$ and integer $N$ such that

$$|f(n)| \leq C|g(n)| \quad \text{for all} \quad n \geq N,$$

we say that $f$ is of **big-Oh** of $g$, written as

$$f = O(g).$$

This means that $f$ grows no faster than $g$. We say that $f$ and $g$ have the **same order** if $f = O(g)$ and $g = O(f)$. If $f = O(g)$, but $g \neq O(f)$, then we say that $f$ is of **lower order** than $g$ or $g$ **grows faster** than $f$.

**Example 6.1.** In Example 5.1, the number of comparisons $f(n)$ is a function of integers $n$. In Case I, $f(n) = O(n)$. In Case II, $f(n) = O(\log n)$.

In Example 5.2, the number of comparisons $f(n)$ is a function of positive integers $n$. For Solution I, $f(n) = O(n^2)$. For Solution II, $f(n) = O(n)$.

**Remark.** $f(n) = O(g(n))$ if and only if there exists a constant $C$ such that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \leq C.$$

### Problem Set 5

1. Find an explicit formula for each of the sequences defined by the recurrence relations with initial conditions.

   (a) $x_n = 5x_{n-1} + 3$, $x_1 = 3$.

   (b) $x_n = 3x_{n-1} + 5n$, $x_1 = 5$.

   (c) $x_n = 2x_{n-1} + 15x_{n-2}$, $x_1 = 2$, $x_2 = 4$.

   (d) $x_n = 4x_{n-1} + 5x_{n-2}$, $x_1 = 3$, $x_2 = 5$.

   (e) $x_n = 3x_{n-1} - 2x_{n-2}$, $x_0 = 2$, $x_1 = 4$.

   (f) $x_n = 6x_{n-1} - 9x_{n-2}$, $x_0 = 3$, $x_1 = 9$.

*Solution.* (a) Since $x_n = 5(5x_{n-2}+3)+3 = 5^2x_{n-2}+5\cdot 3+3$, then $x_n = 5^k x_{n-k} + (5^{k-1}+\cdots+5+5^0)\cdot 3$ for $1 \le k \le n-1$. Thus

$$x_n = (5^{n-1}+5^{n-2}+\cdots+5+1)3 = \frac{3(5^n-1)}{5-1}.$$

(b) Let $x_n = A+Bn$. Then $A+Bn = 3(A+B(n-1))+5n$. Thus

$$(2A-3B)+(2B+5)n = 0.$$

Set $2A-3B=0$ and $2B+5=0$; we have $B=-5/2$, $A=-15/4$. Hence the general solution is given by

$$x_n = -\frac{15}{4} - \frac{5n}{2} + 3^n C$$

Applying $x_1 = 5$, we have $C = 15/4$. Therefore

$$x_n = -\frac{15}{4} - \frac{5n}{2} + \frac{15\cdot 3^n}{4}.$$

(c) Set $r^2 = 2r+15$. Then $(r+3)(r-5)=0$. Thus $r_1=-3$, $r_2=5$. Let $x_n = (-3)^n C_1 + 5^n C_2$. Then $C_1 = -1/4$, $C_2 = 1/4$. Thus

$$x_n = \frac{(-1)^{n+1}3^n+5^n}{4}.$$

(d) Set $r^2 = 4r+5$. Then $(r+1)(r-5)=0$. Thus $r_1=-1$, $r=5$. Let $x_n = (-1)^n C_1 + 5^n C_2$. We have $C_1 = 13/3$, $C_2 = 4/15$. Therefore

$$x_n = \frac{13(-1)^n}{3} + \frac{4\cdot 5^n}{15}.$$

(e) Set $r^2 = 3r-2$. Then $r_1=1$, $r_2=2$. Let $x_n = C_1 + 2^n C_2$. Then $C_1=0$, $C_2=2$. Thus $x_n = 2^{n+1}$.

(f) Set $r^2 = 6r-9$. Then $r_1=r_2=3$. Let $x_n = 3^nC_1 + 3^n nC_2$. Then $C_1=3$, $C_2=0$. Therefore $x_n = 3^{n+1}$.

2. Find an explicit formula for each of the sequences defined by the non-homogeneous recurrence relations with initial conditions.

(a) $x_n = 2x_{n-1} + 15x_{n-2} + 2^n, \qquad x_1 = 2,\ x_2 = 4.$

(b) $x_n = 4x_{n-1} + 5x_{n-2} + 3, \qquad x_1 = 3,\ x_2 = 5.$

(c) $x_n = 3x_{n-1} - 2x_{n-2} + 2^n, \qquad x_0 = 2,\ x_1 = 4.$

(d) $x_n = 6x_{n-1} - 9x_{n-2} + 3^{n+2}, \qquad x_0 = 3,\ x_1 = 9.$

*Solution.* (a) Since $r^2 = 2r + 15$, then $r_1 = -3$, $r_2 = 5$. So $r_3 = 2 \neq r_1$, $r_3 = 2 \neq r_2$. Let $x_n = 2^n A$ be a special solution. Then $2^n A = 2 \cdot 2^{n-1} A + 15 \cdot 2^{n-2} A + 2^n$. Thus $A = -4/15$. Therefore the general solution is given by

$$x_n = -4 \cdot 2^n/15 + (-3)^n C_1 + 5^n C_2.$$

Applying the initial conditions $x_1 = 2$, $x_2 = 4$, we have

$$C_1 = -\frac{19}{60}, \quad C_2 = \frac{19}{60}.$$

Hence

$$x_n = -\frac{4 \cdot 2^n}{15} - \frac{(-1)^n 19 \cdot 3^n}{60} + \frac{19 \cdot 5^n}{60}.$$

(b) Set $r^2 = 4r + 5$, then $r_1 = -1$, $r_2 = 5$. We have $r_3 = 1 \neq r_1$, $r_3 = 1 \neq r_2$. Let $x_n = A$ be a special solution. Then $A = 4A + 5A + 3$, i.e., $A = -3/8$. Thus the solution is given by

$$x_n = -\frac{3}{8} + (-1)^n C_1 + 5^n C_2.$$

Applying the initial conditions $x_1 = 3$, $x_2 = 5$, we have

$$C_1 = -\frac{23}{12}, \quad C_2 = \frac{7}{24}.$$

(c) Set $r^2 = 3r - 2$. Then $r_1 = 1$, $r_2 = 2$. Note that $r_3 = 2 = r_2$. Let $x_n = 2^n n A$ be a special solution. Then

$$2^n n A = 3 \cdot 2^{n-1}(n-1)A - 2 \cdot 2^{n-2}(n-2)A + 2^n.$$

Thus $A = 2$. The solution is given by

$$x_n = 2^{n+1} n + C_1 + 2^n C_2.$$

Applying the initial conditions $x_0 = 2$, $x_1 = 4$, we have $C_1 = 4$, $C_2 = -2$. Therefore

$$x_n = 2^{n+1}(n-1) + 4.$$

(d) Set $r^2 = 6r - 9$. Then $r_1 = r_2 = 3$. Thus $r_3 = 3 = r_1 = r_2$. Let $x_n = 3^n n^2 A$ be a special solution. Then

$$3^n n^2 A = 6 \cdot 3^{n-1}(n-1)^2 A - 9 \cdot 3^{n-2}(n-2)^2 A + 3^{n+2}.$$

Thus $A = 9/2$. The solution is given by

$$x_n = \frac{3^{n+2} n^2}{2} + 3^n C_1 + 3^n n C_2.$$

Applying the initial conditions $x_0 = 3$, $x_1 = 9$, we have $C_1 = 3$, $C_2 = -9/2$. Therefore

$$x_n = 3^n \left( \frac{9}{2} n^2 - \frac{9}{2} n + 3 \right).$$

3. Show that if $s_n$ and $t_n$ are solutions for the non-homogeneous linear recurrence relation

$$x_n = a x_{n-1} + b x_{n-2} + f(n), \; n \geq 2,$$

then $x_n = s_n - t_n$ is a solution for the homogeneous linear recurrence relation

$$x_n = a x_{n-1} + b x_{n-2}, \; n \geq 2.$$

*Proof.* Since $x_n = s_n, t_n$ are solutions of the non-homogeneous equations, then for $n \geq 2$,

$$s_n = a s_{n-1} + b s_{n-2} + f(n), \quad t_n = a t_{n-1} + b t_{n-2} + f(n).$$

Thus

$$s_n - t_n = a(s_{n-1} - t_{n-1}) + b(s_{n-2} - t_{n-2}).$$

This means that $x_n = s_n - t_n$ is a solution for the corresponding homogeneous equation. □

4. Let the characteristic equation for the homogeneous linear recurrence relation

$$x_n = a x_{n-1} + b x_{n-2}, \; n \geq 2$$

have two distinct roots $r_1$ and $r_2$. Show that every solution can be written in the form

$$x_n = c_1 r_1^n + c_2 r_2^n$$

for some constants $c_1$ and $c_2$.

*Proof.* Note that any solution $x_n = s_n$ of the recurrence relation is completely determined by the values $x_0$ and $x_1$. Also note that $x_n = c_1 r_1^n + c_2 r_2^n$ satisfies the recurrence relation for any constants $c_1$ and $c_2$. Set

$$\begin{cases} x_0 &= \quad c_1 + c_2 \\ x_1 &= c_1 r_1 + c_2 r_2 \end{cases}$$

By Cramer's rule, we have

$$c_1 = \frac{\begin{vmatrix} x_0 & 1 \\ x_1 & r_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix}} = \frac{x_0 r_2 - x_1}{r_2 - r_1}, \quad c_2 = \frac{\begin{vmatrix} 1 & x_0 \\ r_1 & x_1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix}} = \frac{x_1 - x_0 r_1}{r_2 - r_1}.$$

Then both sequences $s_n$ and $t_n = c_1 r_1^n + c_2 r_2^n$ with above constants $c_1$ and $c_2$ satisfy the same recurrence relation and initial values $x_0$ and $x_1$. Thus $s_n = t_n$. This proves that every solution of the recurrence relation can be written in the form $x_n = c_1 r_1^n + c_2 r_2^n$. □

5. * Let $A_1, A_2, \ldots, A_{n+1}$ be $k \times k$ matrices. Let $C_n$ be the number of ways to evaluate the product $A_1 A_2 \cdots A_{n+1}$ by choosing different orders in which to do the $n$ multiplications.

   (a) Find a recurrence relation with an initial condition for the sequence $C_n$.

   (b) Verify that the sequence $\frac{1}{n+1}\binom{2n}{n}$ satisfies your recurrence relation and conclude that $C_n = \frac{1}{n+1}\binom{2n}{n}$. (The numbers $C_n$ are called **Catalan numbers**.)

   *Solution.* (a) It is clear that $C_0 = C_1 = 1$, $C_2 = 2$. Note that any way to realize the product $A_1 A_2 \cdots A_{n+2}$ must be obtained finally as follows:

$$(\underbrace{A_1 A_2 \cdots A_{k+1}}_{k+1})(\underbrace{A_{k+2} A_{k+3} \cdots A_{n+2}}_{n-k+1}).$$

Thus the sequence $C_n$ satisfies the recurrence relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

(b) Not required.

6. Find a general formula for the recurrence relation

$$x_n = a x_{n-1} + b + cn$$

in terms of $x_0$, where $a, b, c$ are real constants.

*Solution.* Let $x_n = A + Bn$ be a special solution. Then

$$A + Bn = a(A + B(n-1)) + b + cn.$$

Thus

$$A - aA + aB - b + (B - aB - c)n = 0.$$

If $a \neq 1$, we have

$$A = \frac{b - a(b+c)}{(1-a)^2}, \quad B = \frac{c}{1-a}.$$

The general solution is given by

$$x_n = \frac{b - a(b+c)}{(1-a)^2} + \frac{cn}{1-a} + Ca^n.$$

Applying the initial $x_0$, we have $C = x_0 + \frac{a(b+c)-b}{(1-a)^2}$. Hence

$$x_n = \frac{b - a(b+c)}{(1-a)^2} + \frac{cn}{1-a} + \left(x_0 + \frac{a(b+c)-b}{(1-a)^2}\right) a^n.$$

If $a = 1$, then

$$\begin{aligned} x_n &= x_0 + bn + c(n + (n-1) + \cdots + 1) \\ &= x_0 + bn + \frac{n(n+1)c}{2}. \end{aligned}$$

7. Find an explicit formula for each of the sequences defined by the recurrence relations with initial conditions.

(a) $x_n = 5x_{\frac{n}{3}} + 5$, $x_1 = 5$, $n = 3^k$, $k \geq 0$.

(b) $x_n = x_{\lfloor \frac{n}{2} \rfloor} + 3$, $x_1 = 4$, $n \geq 1$.

(c) $x_{2n} = 2x_n + 5 - 7n$, $x_1 = 0$.

*Solution.* (a) $a = 5 \neq 1$, $b = 3$, $c = 5$. Then

$$x_n = \frac{c(an^{\log_b a} - 1)}{a-1} = \frac{5}{4}\left(n^{\log_3 5} - 1\right).$$

(b) $a = 1$, $b = 2$, $c = 3$. Let $2^k \leq n < 2^{k+1}$ for some $k \in \mathbb{Z}_+$. Then

$$\begin{aligned} x_n &= x_{\lfloor \frac{n}{2} \rfloor} + 3 = x_{\lfloor \frac{n}{2^2} \rfloor} + 2 \cdot 3 = x_{\lfloor \frac{n}{2^3} \rfloor} + 3 \cdot 3 \\ &= \cdots = x_{\lfloor \frac{n}{2^k} \rfloor} + k \cdot 3 = x_1 + 3k \\ &= 4 + 3\lfloor \log_2 n \rfloor. \end{aligned}$$

(c) We assume that $n = 2^k$. Then the recurrence relation can be written as

$$x_n = 2x_{\frac{n}{2}} + 5 - 7n/2.$$

Thus

$$\begin{aligned} x_{2^k} &= 2x_{2^{k-1}} + 5 - 7 \cdot 2^{k-1} \\ &= 2\left(2x_{2^{k-2}} + 5 - 7 \cdot 2^{k-2}\right) + 5 - 7 \cdot 2^{k-1} \\ &= 2^2 x_{2^{k-2}} + 5(1+2) - 7 \cdot 2 \cdot 2^{k-1} \\ &= 2^3 x_{2^{k-3}} + 5(1+2+2^2) - 7 \cdot 3 \cdot 2^{k-1} \\ &= 2^k x_{2^0} + 5(1+2+\cdots+2^{k-1}) - 7k2^{k-1} \\ &= 2^k x_1 + 5(2^k - 1) - 7k2^{k-1} \\ &= 5(2^k - 1) - 7k2^{k-1}. \end{aligned}$$

Therefore

$$x_n = 5(n-1) - \frac{7n\log_2 n}{2}.$$

8. Let $f(n)$ be a real sequence defined for $n = 1, b, b^2, \ldots$, and satisfy the recurrence relation

$$f(n) = af\left(\frac{n}{b}\right) + h(n),$$

where $b \geq 2$ is an integer. Show that

$$f(n) = f(1)n^{\log_b a} + \sum_{i=0}^{-1+\log_b n} a^i h\left(\frac{n}{b^i}\right).$$

*Proof.* Let $n = b^k$ for some $k \in \mathbb{Z}_+$. Then

$$\begin{aligned} f(b^k) &= af\left(b^{k-1}\right) + h\left(b^k\right) \\ &= a\left[af\left(b^{k-2}\right) + h\left(b^{k-1}\right)\right] + h\left(b^k\right) \\ &= a^2 f\left(b^{k-2}\right) + ah\left(b^{k-1}\right) + h\left(b^k\right) \\ &= a^k f(1) + \sum_{i=0}^{k-1} a^i h(b^{k-i}). \end{aligned}$$

Thus

$$f(n) = f(1)a^{\log_b n} + \sum_{i=0}^{(\log_b n)-1} a^i h\left(\frac{n}{b^i}\right).$$

$\square$

9. Let $f(n)$ be a real sequence defined for $n = 1, b, b^2, b^3, \ldots$, and satisfy the recurrence relation

$$f(n) = af\left(\frac{n}{b}\right) + a_0 + a_1 n + \cdots + a_k n^k,$$

where $a, b, a_0, a_1, \ldots, a_k$ are real constants, $a > 0$ and $b > 1$. Show that

(a) If $a = b^i$ for some $0 \leq i \leq k$, then

$$f(n) = f(1)n^i + a_i n^i \log_b n + \sum_{j=0, j\neq i}^{k} \frac{b^j a_j}{b^j - b^i}\left(n^j - n^i\right).$$

(b) If $a \neq b^i$ for all $0 \leq i \leq k$, then

$$f(n) = f(1)n^{\log_b a} + \sum_{j=0}^{k} \frac{a_j b^j \left(n^j - n^{\log_b a}\right)}{b^j - a}.$$

*Proof.* Write $h(n) = \sum_{j=0}^{k} a_j n^j$. Then by the previous problem, we have

$$f(n) = f(1)a^{\log_b n} + \sum_{s=0}^{-1+\log_b n} a^s h\left(\frac{n}{b^s}\right).$$

(a) Since $a = b^i$ for some $0 \leq i \leq k$, then

$$a^{\log_b n} = b^{i \log_b n} = b^{\log_b n^i} = n^i;$$

$$\begin{aligned}
\sum_{s=0}^{(\log_b n)-1} a^s h\left(\frac{n}{b^s}\right) &= \sum_{s=0}^{(\log_b n)-1} a^s \sum_{j=0}^{k} a_j \left(\frac{n}{b^s}\right)^j \\
&= \sum_{j=0}^{k} a_j n^j \sum_{s=0}^{(\log_b n)-1} (ab^{-j})^s \\
&= \sum_{j=0, j\neq i}^{k} a_j n^j \cdot \frac{(ab^{-j})^{\log_b n} - 1}{ab^{-j} - 1} \\
&\quad + a_i n^i \log_b n.
\end{aligned}$$

Since $n^j \left(\frac{(ab^{-j})^{\log_b n}-1}{ab^{-j}-1}\right) = n^j \left(\frac{(b^{i-j})^{\log_b n}-1}{b^{i-j}-1}\right) = \frac{(n^i - n^j)b^j}{b^i - b^j}$, then

$$f(n) = f(1)n^i + a_i n^i \log_b n + \sum_{j=0, j\neq i}^{k} \frac{a_j b^j (n^j - n^i)}{b^j - b^i}.$$

(b) Note that

$$a^{\log_b n} = \left(b^{\log_b a}\right)^{\log_b n} = \left(b^{\log_b n}\right)^{\log_b a} = n^{\log_b a}.$$

Since

$$n^j \left(\frac{(ab^{-j})^{\log_b n} - 1}{ab^{-j} - 1}\right) = n^j \left(\frac{n^{\log_b n a} n^{-j} - 1}{ab^{-j} - 1}\right),$$

then

$$f(n) = f(1)n^{\log_b a} + \sum_{j=0}^{k} \frac{a_j b^j \left(n^j - n^{\log_b a}\right)}{b^j - a}.$$

$\square$