Math2343 Problem Set 7

1. For each of the following problems, determine whether the relation R on the set A is a tree. If it is a tree, find its leaves.
   (a) $A = \{a, b, c, d, e, f\}$, $R = \{(a, b), (c, e), (f, a), (f, c), (f, d)\}$.
   (b) $A = \{u, v, w, x, y, z\}$, $R = \{(u, x), (u, v), (w, v), (x, z), (x, y)\}$.

2. Consider the rooted tree (T, v_0) shown below.

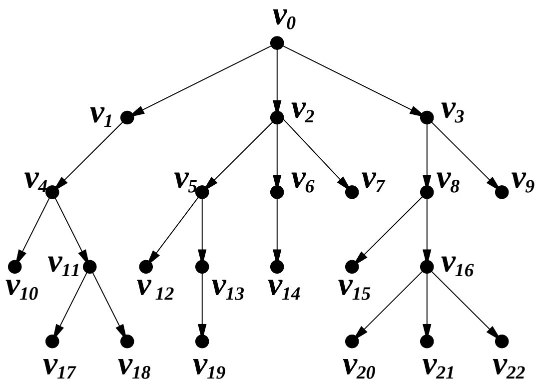

Figure 1: A labelled rooted tree

   (a) List all level-3 vertices.
   (b) List all leaves.
   (c) List all children of the vertex v_2.
   (d) List all descendants of the vertex v_2.
   (e) Find the rooted subtree T_{v_8}.
   (f) Find the height of (T, v_0).
   (g) Find the height of T_{v_8}.

3. Show that the maximum number of vertices in a binary tree of height n is $2^{n+1} - 1$.

4. Let T be a complete m-ary tree.
   (a) If T has exactly three levels. Prove that the number of vertices of T must be $1 + km$, where $2 \leq k \leq m + 1$.
   (b) If T has n vertices of which k are non-leaves and l are leaves. Prove that $n = mk + 1$ and $l = (m - 1)k + 1$.

5. Use Polish notations to construct the trees for the following expressions.
   (a) $(((2 \times 7) + x) \div y) \div (3 - 11)$
   (b) $(3 - (2 - (11 - (9 - 4)))) \div (2 + 3 + (4 + 7))$

6. Show the result of performing the preorder and postorder searches to the tree in Figure 1.

7. (a) Draw a rooted tree whose preorder search produces the string

$$PREODAB.$$

   (b) Draw a rooted tree whose postorder search produces the string

$$POSTWDER.$$

   (c) Draw a binary tree whose inorder search produces the string

$$INORDE.$$

8. Find a minimal spanning tree for the connected graph below by Kruskal's algorithm and Prim's algorithm respectively.

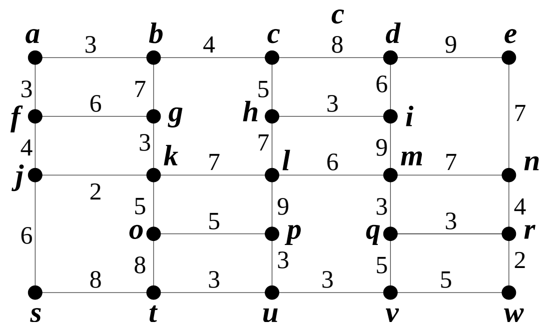

Figure 2: A connected graph

9. Modify Kruskal's and Prim's algorithms so that they will produce a maximal spanning a tree, that is, one with the largest possible sum of the weights.

10. Apply Depth-First Search and Breadth-First Search to find a rooted tree for the graph in Figure 2.

11. Find an Euler circuit or path for the following graph.

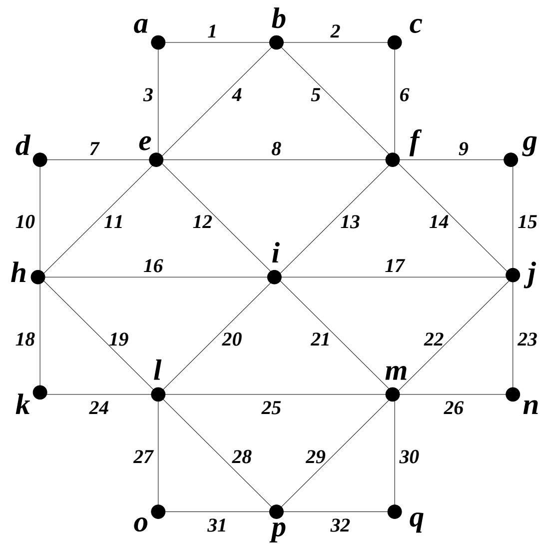

Figure 3: A connected graph

12. Prove that the complete graph K_n with $n \geq 3$ has $(n-1)!$ Hamiltonian cycles.

13. Let G be a graph whose vertex set $V(G) = \{1, 2, \ldots, 15\}$ and two vertices $i + j$ is a multiple of 3. Let R be an equivalence relation on $V(G)$ defined by iRj if and only if $i \equiv j \pmod 7$. Find the quotient graph G/R.

14. Let $G = (V, E)$ be a graph without loops and multiple edges. Show that

$$2|E| \leq |V|^2 - |V|.$$

15. Let $G = (V, E)$ be a graph. Define a relation R on V by uRv if $u = v$ or if there is a path in G from u to v. Show that R is an equivalence relation.

16. Let G be an undirected graph with n vertices. If G is isomorphic to its own complement $\bar{G}$, how many edges must G have? (such a graph is called *self-complementary.*) Find an example of a self-complementary graph on four vertices and one on five vertices.

17. Let $K_{m,n}$ denote the complete bipartite graph with $m, n \geq 2$.
   (1) How many distinct cycles of length 4 are there in $K_{m,n}$?
   (2) How many different paths of length 2 are there in $K_{m,n}$?
   (3) How many different paths of length 3 are there in $K_{m,n}$?

18. Let Q_n be the graph obtained from the n-dimensional unit cube $[0,1]^n$ of which the vertex set and the edge set consist of the vertices of the cube and the edges of the cube. The graph Q_n can be also defined as follows: $V(Q_n)$ is the set of zero-one sequences and two sequences are adjacent if and only if they are differ at only one position.

    (1) For which n the graph Q_n has an Euler circuit or an Euler path?

    (2) For what n the graph Q_n is non-planar? Why?

    (3) When will Q_n have a Hamilton cycle or path?

    (4) Is Q_n bipartite?

19. Find two non-isomorphic spanning trees for the complete bipartite graph $K_{2,3}$. How many non-isomorphic spanning trees are there for $K_{2,3}$?

20. A saturated hydrocarbon is represented by a structural formula in which each C atom has degree 4 and each H has degree 1. Show that the hydrocarbon is acyclic (has no carbon rings in it) if and only if its structural formula is of the form C_nH_{2n+2}.

21. Show that peterson graph is not planar.

22. It is known that one a soccer ball every vertex is of degree 3 and every face is either a pentagon or a hexagon. Find the number of vertices, the number of edges, and the number of faces of a soccer ball respectively not by counting them.