Week 4-5: Generating Permutations and Combinations

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1 Generating Permutations

We have learned that there are $n!$ permutations of $\{1, 2, \ldots, n\}$. It is important in many instances to generate a list of such permutations. For example, for the permutation 3142 of $\{1, 2, 3, 4\}$, we may insert 5 in 3142 to generate five permutations of $\{1, 2, 3, 4, 5\}$ as follows:

$$53142, \quad 35142, \quad 31542, \quad 31452, \quad 31425.$$

If we have a complete list of permutations for $\{1, 2, \ldots, n-1\}$, then we can obtain a complete list of permutations for $\{1, 2, \ldots, n\}$ by inserting the number n in n ways to each permutation of the list for $\{1, 2, \ldots, n-1\}$.

For $n = 1$, the list is just

$$1$$

For $n = 2$, the list is

$$\begin{array}{ccc} & 1 & \mathbf{2} \\ \mathbf{2} & 1 & \end{array} \quad \Rightarrow \quad \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}$$

For $n = 3$, the list is

$$\begin{array}{ccccc} & 1 & & 2 & \mathbf{3} \\ & 1 & \mathbf{3} & 2 & \\ \mathbf{3} & 1 & & 2 & \\ \mathbf{3} & 2 & & 1 & \\ & 2 & \mathbf{3} & 1 & \\ & 2 & & 1 & \mathbf{3} \end{array} \quad \Rightarrow \quad \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{array}$$

To generate a complete list of permutations for the set $\{1, 2, \ldots, n\}$, we assign a **direction** to each integer $k \in \{1, 2, \ldots, n\}$ by writing an arrow above it pointing to the left or to the right:

$$\overleftarrow{k} \quad \text{or} \quad \overrightarrow{k}\ .$$

We consider permutations of $\{1, 2, \ldots, n\}$ in which each integer is given a direction; such permutations are called **directed permutations**. An integer k in a directed permutation is called **mobile** if its arrow points to a smaller integer adjacent to it. For example, for $\overrightarrow{3}\,\overrightarrow{2}\,\overleftarrow{5}\,\overrightarrow{4}\,\overrightarrow{6}\,\overrightarrow{1}$, the integers 3, 5, and 6 are mobile. It follows that 1 can never be mobile since there is no integer in $\{1, 2, \ldots, n\}$ smaller than 1. The integer n is mobile, except two cases:

(i) n is the first integer and its arrow points to the left, i.e., $\overleftarrow{n}\ \cdots$;

(ii) n is the last integer and its arrow points to the right, i.e., $\cdots\ \overrightarrow{n}$.

For $n = 4$, we have the list

1 2 3 **4**		1 2 3 4
1 2 **4** 3		1 2 4 3
1 **4** 2 3		1 4 2 3
4 1 2 3		4 1 2 3
4 1 3 2		4 1 3 2
1 **4** 3 2		1 4 3 2
1 3 **4** 2		1 3 4 2
1 3 2 **4**		1 3 2 4
3 1 2 **4**		3 1 2 4
3 1 **4** 2		3 1 4 2
3 **4** 1 2		3 4 1 2
4 3 1 2	$\Rightarrow$	4 3 1 2
4 3 2 1		4 3 2 1
3 **4** 2 1		3 4 2 1
3 2 **4** 1		3 2 4 1
3 2 1 **4**		3 2 1 4
2 3 1 **4**		2 3 1 4
2 3 **4** 1		2 3 4 1
2 **4** 3 1		2 4 3 1
4 2 3 1		4 2 3 1
4 2 1 3		4 2 1 3
2 **4** 1 3		2 4 1 3
2 1 **4** 3		2 1 4 3
2 1 3 **4**		2 1 3 4

Algorithm 1.1. Algorithm for Generating Permutations of $\{1, 2, \ldots, n\}$:

Step 0. Begin with $\overleftarrow{1}\,\overleftarrow{2}\cdots\overleftarrow{n}$.

Step 1. Find the largest mobile integer m.

Step 2. Switch m and the adjacent integer its arrow points to.

Step 3. Switch the directions for all integers $p > m$.

Step 4. Write down the resulting permutation with directions and return to Step 1.

Step 5. Stop if there is no mobile integer.

For example, for $n = 2$, we have $\overleftarrow{1}\,\overleftarrow{2}$ and $\overleftarrow{2}\,\overleftarrow{1}$. For $n = 3$, we have

$$\overleftarrow{1}\ \overleftarrow{2}\ \overleftarrow{\mathbf{3}},\quad \overleftarrow{1}\ \overleftarrow{\mathbf{3}}\ \overleftarrow{2},\quad \overleftarrow{3}\ \overleftarrow{1}\ \overleftarrow{\mathbf{2}},\quad \overrightarrow{\mathbf{3}}\ \overleftarrow{2}\ \overleftarrow{1},\quad \overleftarrow{2}\ \overrightarrow{\mathbf{3}}\ \overleftarrow{1},\quad \overleftarrow{2}\ \overleftarrow{1}\ \overrightarrow{3}\ .$$

For $n = 4$, the algorithm produces the list

$$\begin{array}{cccccccccccccc}
\overleftarrow{1} & \overleftarrow{2} & \overleftarrow{3} & \overleftarrow{\mathbf{4}} & \quad & \overleftarrow{3} & \overleftarrow{1} & \overleftarrow{2} & \overleftarrow{\mathbf{4}} & \quad & \overleftarrow{2} & \overrightarrow{3} & \overleftarrow{1} & \overleftarrow{\mathbf{4}} \\
\overleftarrow{1} & \overleftarrow{2} & \overleftarrow{\mathbf{4}} & \overleftarrow{3} & & \overleftarrow{3} & \overleftarrow{1} & \overleftarrow{\mathbf{4}} & \overleftarrow{2} & & \overleftarrow{2} & \overrightarrow{3} & \overleftarrow{\mathbf{4}} & \overleftarrow{1} \\
\overleftarrow{1} & \overleftarrow{\mathbf{4}} & \overleftarrow{2} & \overleftarrow{3} & & \overleftarrow{3} & \overleftarrow{\mathbf{4}} & \overleftarrow{1} & \overleftarrow{2} & & \overleftarrow{2} & \overleftarrow{\mathbf{4}} & \overrightarrow{3} & \overleftarrow{1} \\
\overleftarrow{4} & \overleftarrow{1} & \overleftarrow{2} & \overleftarrow{\mathbf{3}} & & \overleftarrow{4} & \overleftarrow{3} & \overleftarrow{1} & \overleftarrow{\mathbf{2}} & & \overleftarrow{4} & \overleftarrow{2} & \overrightarrow{\mathbf{3}} & \overleftarrow{1} \\
\overrightarrow{\mathbf{4}} & \overleftarrow{1} & \overleftarrow{3} & \overleftarrow{2} & & \overrightarrow{\mathbf{4}} & \overrightarrow{3} & \overleftarrow{2} & \overleftarrow{1} & & \overrightarrow{\mathbf{4}} & \overleftarrow{2} & \overleftarrow{1} & \overrightarrow{3} \\
\overleftarrow{1} & \overrightarrow{\mathbf{4}} & \overleftarrow{3} & \overleftarrow{2} & & \overrightarrow{3} & \overrightarrow{\mathbf{4}} & \overleftarrow{2} & \overleftarrow{1} & & \overleftarrow{2} & \overrightarrow{\mathbf{4}} & \overleftarrow{1} & \overrightarrow{3} \\
\overleftarrow{1} & \overleftarrow{3} & \overrightarrow{\mathbf{4}} & \overleftarrow{2} & & \overrightarrow{3} & \overleftarrow{2} & \overrightarrow{\mathbf{4}} & \overleftarrow{1} & & \overleftarrow{2} & \overleftarrow{1} & \overrightarrow{\mathbf{4}} & \overrightarrow{3} \\
\overleftarrow{1} & \overleftarrow{\mathbf{3}} & \overleftarrow{2} & \overrightarrow{4} & & \overrightarrow{\mathbf{3}} & \overleftarrow{2} & \overleftarrow{1} & \overrightarrow{4} & & \overleftarrow{2} & \overleftarrow{1} & \overrightarrow{3} & \overrightarrow{4}
\end{array}$$

Proof. Observe that when n is *not* the largest mobile the direction of n must be either like

$$\overleftarrow{n}\ \cdots \quad \text{or} \quad \cdots\ \overrightarrow{n}$$

in the permutation. When the largest mobile m (with $m < n$) is switched with its target integer to produce a new permutation, the direction of n will be changed simultaneously, and the permutation with direction becomes

$$\overrightarrow{n}\ \cdots \quad \text{or} \quad \cdots\ \overleftarrow{n}\ .$$

Now n is the largest mobile. Switching n with its target integer for $n-1$ times to produce $n-1$ more permutations, we obtain exactly n new permutations (including the one before switching n). Each member k ($2 \le k \le n$) moves $k-1$ times from right to left, then $k-1$ times from left to right, and goes in this way from one side to the other for $(k-1)!$ times (which is an even number when $k \ge 3$); the total number of moves of k is $(k-1)!(k-1)$. Thus the total number of moves is

$$\sum_{k=1}^{n} (k-1)!(k-1),$$

which is $n! - 1$ by induction on n. The algorithm stops at the permutation

$$\overleftarrow{2}\ \overleftarrow{1}\ \overrightarrow{3}\ \overrightarrow{4}\ \cdots\ \overrightarrow{n}\ .$$

$\square$

2 Inversions of Permutations

Let $u_1u_2\ldots u_n$ be a permutation of $S = \{1, 2, \ldots, n\}$. We can view $u_1u_2\ldots u_n$ as a bijection $\pi : S \to S$ defined by

$$\pi(1) = u_1,\ \pi(2) = u_2, \quad \ldots, \quad \pi(n) = u_n.$$

If $u_i > u_j$ for some $i < j$, the ordered pair (u_i, u_j) is called an **inversion** of π. The number of inversions of π is denoted by $\mathrm{inv}(\pi)$. For example, the permutation 3241765 of $\{1, 2, \ldots, 7\}$ has the inversions:

$$(2, 1),\ (3, 1),\ (4, 1),\ (3, 2),\ (6, 5),\ (7, 5),\ (7, 6).$$

For $k \in \{1, 2, \ldots, n\}$ and $u_j = k$, let a_k be the number of integers that precede k in the permutation $u_1u_2\ldots u_n$ but greater than k, i.e.,

$$\begin{aligned} a_k &= \#\{u_i : i < j,\ u_i > u_j = k\} \\ &= \#\{\pi(i) : i < j,\ \pi(i) > \pi(j) = k\}. \end{aligned}$$

It measures how much k is out of order by counting the number of integers larger than k but located before k. The tuple $(a_1, a_2, \ldots, a_n)$ is called the **inversion sequence** (or **inversion table**) of the permutation $\pi = u_1u_2\ldots u_n$. The sum

$$\mathrm{inv}(\pi) := a_1 + a_2 + \cdots + a_n$$

measures the total **disorder** of a permutation.

Example 2.1. The inversion sequence of the permutation 3241765 of $\{1, 2, \ldots, 7\}$ is $(3, 1, 0, 0, 2, 1, 0)$.

It is clear that for any permutation π of $\{1, 2, \ldots, n\}$, the inversion sequence $(a_1, a_2, \ldots, a_n)$ of π satisfies

$$0 \le a_1 \le n-1, \quad 0 \le a_2 \le n-2, \quad \ldots, \quad 0 \le a_{n-1} \le 1, \quad a_n = 0. \tag{1}$$

It is easy to see that the number of sequences $(a_1, a_2, \ldots, a_n)$ satisfying (1) equals

$$n \cdot (n-1) \cdots 2 \cdot 1 = n!.$$

This suggests that the inversion sequences may be characterized by (1).

Theorem 2.1. *Let* $(a_1, a_2, \ldots, a_n)$ *be an integer sequence satisfying*

$$0 \le a_1 \le n-1, \quad 0 \le a_2 \le n-2, \quad \ldots, \quad 0 \le a_{n-1} \le 1, \quad a_n = 0.$$

Then there is a unique permutation π *of* $\{1, 2, \ldots, n\}$ *whose inversion sequence is* $(a_1, a_2, \ldots, a_n)$.

Proof. We give two algorithms to uniquely construct the permutation whose inversion sequence is $(a_1, a_2, \ldots, a_n)$.

Algorithm I. Construction of a Permutation from Its Inversion Sequence:

Step 0. Write down n.

Step 1. If $a_{n-1} = 0$, place $n-1$ before n; if $a_{n-1} = 1$, place $n-1$ after n.

Step 2. If $a_{n-2} = 0$, place $n-2$ before the two members n and $n-1$; if $a_{n-2} = 1$, place $n-2$ between n and $n-1$; if $a_{n-2} = 2$, place $n-2$ after both n and $n-1$.

$\vdots$

Step k. If $a_{n-k} = 0$, place $n-k$ to the left of the first position; if $a_{n-k} = 1$, place $n-k$ to the right of the 1st existing number; if $a_{n-k} = 2$, place $n-k$ to the right of the 2nd existing number; $\ldots$; if $a_{n-k} = k$, place $n-k$ to the right of the last existing number. In general, insert $n-k$ to the right of the a_{n-k}-th existing number.

$\vdots$

Step $n-1$. If $a_1 = 0$, place 1 before all existing numbers; otherwise, place 1 to the right of the a_1th existing number.

For example, for the inversion sequence $(a_1, a_2, \ldots, a_8) = (4, 6, 1, 0, 3, 1, 1, 0)$, its permutation can be constructed by Algorithm I as follows:

8	Write down 8.
87	Since $a_7 = 1$, insert 7 to the right of the first number 8.
867	Since $a_6 = 1$, insert 6 to the right of the first number 8.
8675	Since $a_5 = 3$, insert 5 to the right of the third number 7.
48675	Since $a_4 = 0$, insert 4 to the left of the first number 8.
438675	Since $a_3 = 1$, insert 3 to the right of the first number 4.
4386752	Since $a_2 = 6$, insert 2 to the right of the sixth number 5.
43861752	Since $a_1 = 4$, insert 1 to the right of the fifth number 6.

Algorithm II. Construction of a Permutation from Its Inversion Sequence:

Step 0. Mark down n empty spaces $\square\square\square\cdots\square\square\square$.

Step 1. Put 1 into the (a_1+1)-th empty space from left.

Step 2. Put 2 into the (a_2+1)-th empty space from left.

$\vdots$

Step k. Put k into the (a_k+1)-th empty space from left.

$\vdots$

Step n. Put n into the (a_n+1)-th empty space (the last empty box) from left.

For example, the permutation for the inversion sequence

$$(a_1, a_2, \dots, a_8) = (4, 6, 1, 0, 3, 1, 1, 0)$$

can be constructed by Algorithm II as follows:

□□□□□□□□	Mark down 8 empty spaces.
□□□□1□□□	Since $a_1 = 4$, put 1 into the 5th empty space.
□□□□1□□2	Since $a_2 = 6$, put 2 into the 7th empty space.
□3□□1□□2	Since $a_3 = 1$, put 3 into the 2nd empty space.
43□□1□□2	Since $a_4 = 0$, put 4 into the 1st empty space.
43□□1□52	Since $a_5 = 3$, put 5 into the 4th empty space.
43□61□52	Since $a_6 = 1$, put 6 into the 2nd empty space.
43□61752	Since $a_7 = 1$, put 7 into the 2nd empty space.
43861752	Since $a_8 = 0$, put 8 into the 1st empty space.

□

3 Generating Combinations

Let S be an n-set. For convenience of generating combinations of S, we take S to be the set

$$S = \{x_{n-1}, x_{n-2}, \ldots, x_2, x_1, x_0\}.$$

Each subset A of S can be identified as a function $\chi_A : S \to \{0, 1\}$, called the **characteristic function** of A, defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A. \end{cases}$$

In practice, χ_A is represented by a 0-1 sequence or a base 2 numeral. For example, for $S = \{x_7, x_6, x_5, x_4, x_3, x_2, x_1, x_0\}$,

$\varnothing$	00000000
$\{x_7, x_5, x_2, x_1\}$	10100110
$\{x_6, x_5, x_3, x_1, x_0\}$	01101011
$\{x_7, x_6, x_5, x_4, x_3, x_2, x_1, x_0\}$	11111111

Algorithm 3.1. The algorithm for Generating Combinations of $\{x_{n-1}, \ldots, x_1, x_0\}$:

Step 0. Begin with $a_{n-1} \cdots a_1 a_0 = 0 \cdots 00$.

Step 1. If $a_{n-1} \cdots a_1 a_0 = 1 \cdots 11$, stop.

Step 2. If $a_{n-1} \cdots a_1 a_0 \neq 1 \cdots 11$, find the smallest integer j such that $a_j = 0$.

Step 3. Change $a_j, a_{j-1}, \ldots, a_0$ (either from 0 to 1 or from 1 to 0), write down $a_{n-1} \cdots a_1 a_0$, and return to Sept 1.

For $n = 4$, the algorithm produces the list

0000	0100	1000	1100
0001	0101	1001	1101
0010	0110	1010	1110
0011	0111	1011	1111

The **unit n-cube** Q_n is a graph whose vertex set is the set of all 0-1 sequences of length n, and two sequences are adjacent if they differ in only one place. A **Gray code of order** n is a path of Q_n that visits every vertex of

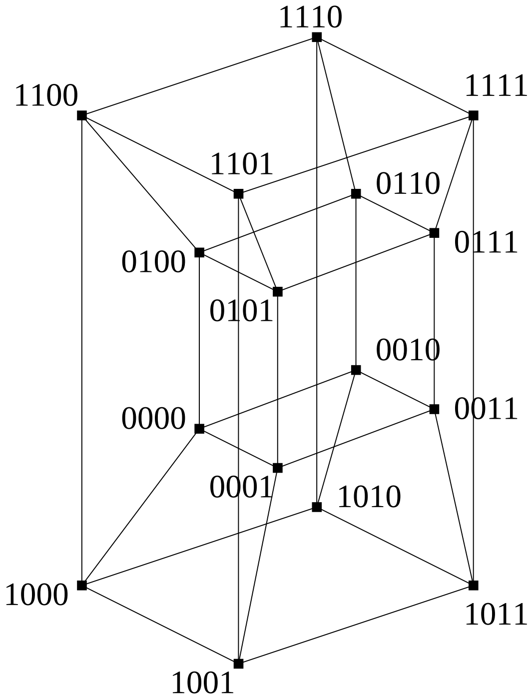

Figure 1: 4-dimensional cube.

Q_n exactly once, i.e., a Hamilton path of Q_n. For example,

$$000 \to 001 \to 101 \to 100 \to 110 \to 010 \to 011 \to 111$$

is a Gray code of order 3. It is obvious that this Gray code can not be a part of any Hamilton cycle since 000 and 111 are not adjacent. A **cyclic Gray code of order** n is a Hamilton cycle of Q_n. For example, the closed path

$$000 \to 001 \to 011 \to 010 \to 110 \to 111 \to 101 \to 100 \to 000$$

is a cyclic Gray code of order 3.

For $n = 1$, we have the Gray code $0 \to 1$.

For $n = 2$, we use $0 \to 1$ to produce the path $00 \to 01$ by adding a 0 in the front, and use $1 \to 0$ to produce $11 \to 10$ by adding a 1 in the front, then combine the two paths to produce the Gray code

$$00 \to 01 \to 11 \to 10.$$

For $n = 3$, we use the Gray code $00 \to 01 \to 11 \to 10$ (of order 2) to produce the path $000 \to 001 \to 011 \to 010$ by adding 0 in the front, and use the Gray code $10 \to 11 \to 01 \to 00$ (the reverse of $00 \to 01 \to 11 \to 10$) to produce the path $110 \to 111 \to 101 \to 100$ by adding 1 in the front. Combine the two paths to produce the Gray code of order 3

$$000 \to 001 \to 011 \to 010 \to 110 \to 111 \to 101 \to 100.$$

The Gray codes obtained in this way are called **reflected Gray codes**.

Algorithm 3.2. Algorithm of generating a Reflected Gray Code of order n:

Step 0. Begin with $a_n a_{n-1} \cdots a_2 a_1 = 00 \cdots 0$.

Step 1. If $a_n a_{n-1} \cdots a_2 a_1 = 10 \cdots 00$, stop.

Step 2. If $a_n + a_{n-1} + \cdots + a_2 + a_1 = \mathit{even}$, then change a_1 (either from 0 to 1 or from 1 to 0), i.e., set $a_1 := 1 - a_1$.

Step 3. If $a_n + a_{n-1} \cdots + a_2 + a_1 = \mathit{odd}$, find the smallest index j such that $a_j = 1$ and change a_{j+1} (either from 0 to 1 or from 1 to 0), i.e., set $a_{j+1} := 1 - a_{j+1}$.

Step 4. Write down $a_n a_{n-1} \cdots a_2 a_1$ and return to Step 1.

We note that if $a_n a_{n-1} \cdots a_1 \neq 10 \cdots 0$ and $a_n + a_{n-1} + \cdots + a_1 = \mathit{odd}$, then $j \leq n-1$ so that $j+1 \leq n$ and a_{j+1} is defined.

Proof. We proceed by induction on n. For $n = 1$, it is obviously true. For $n = 2$, we have $00 \to 01 \to 11 \to 10$. Let $n \geq 3$ and assume that it is true for $1, 2, \ldots, n-1$.

(1) When the algorithm is applied, by the induction hypothesis the first resulted 2^{n-1} words form the reflected Gray code of order $n-1$ with a 0 attached to the head of each word; the 2^{n-1}-th word is $010 \cdots 0$.

(2) Continuing the algorithm, we have

$$010 \cdots 0 \to 110 \cdots 0.$$

Now for each word of the form $11b_{n-2} \cdots b_1$, the parity of $11b_{n-2} \cdots b_1$ is the same as the parity of $b_{n-2} \cdots b_1$. Continuing the algorithm, the next 2^{n-2} words (including $110 \cdots 0$) form a reflected Gray code (of order $n-2$) with a 11 attached at the beginning; the last word is $1110 \cdots 0$.

(3) Continuing the algorithm, we have

$$1110 \cdots 0 \to 1010 \cdots 0 \to \cdots$$

The next 2^{n-3} words (including $1010 \cdots 0$) form a reflected Gray code (of order $n-3$) with a 101 attached at the beginning; the last word is $10110 \cdots 0$.

(4) $10110\cdots 0 \to 10010\cdots 0 \to$; there are 2^{n-4} words with 1001 attached at the beginning.

$\vdots$

$(n-2)$ Continuing the algorithm, we have

$$10\cdots 01100 \to 10\cdots 00100 \to .$$

The next 2^2 words (including $10\cdots 0100$) form a reflected Gray code (of order 2) with $10\cdots 01$ attached at the beginning; the last word is $10\cdots 0110$.

$(n-1)$ $10\cdots 0110 \to 10\cdots 0010 \to 10\cdots 0011$; there are 2^1 words (of order 1), $10\cdots 0010 \to 10\cdots 0011$, with $10\cdots 001$ attached at the beginning.

(n) $10\cdots 0011 \to 10\cdots 0001$. The algorithm produces 1 word $10\cdots 0001$.

$(n+1)$ Finally, the algorithm ends at $10\cdots 0001 \to 10\cdots 0000$.

Notice that all words produced from Step (k) to Step $(n+1)$ inclusive are distinct from the words produced in Step $(k-1)$, where $2 \le k \le n+1$. Thus the words produced by the algorithm are distinct and the total number of words is

$$2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2^1 + 2^0 + 1 = 2^n.$$

This implies that the sequence of the words produced forms a Gray code of order n.

Next we show that the words resulted in the second half of the algorithm are the words obtained from the reversing of the Gray codes of order $n-1$ with 1 attached in the front. Consider two consecutive words of length n in the second half of the algorithm, say,

$$1\,a_{n-1}a_{n-2}\cdots a_2a_1 \to 1\,b_{n-1}b_{n-2}\cdots b_2b_1.$$

We need to show that

$$b_{n-1}b_{n-2}\cdots b_2b_1 \to a_{n-1}a_{n-2}\cdots a_2a_1$$

by the same algorithm for $n-1$.

Note that $1\,a_{n-1}a_{n-2}\cdots a_2a_1$ and $1\,b_{n-1}b_{n-2}\cdots b_2b_1$ have opposite parity. It follows that $a_{n-1}a_{n-2}\cdots a_2a_1$ and $b_{n-1}b_{n-2}\cdots b_2b_1$ have opposite parity. We have two cases.

(1) $b_{n-1}b_{n-2}\cdots b_2b_1$ is even. Then $a_{n-1}a_{n-2}\cdots a_2a_1$ is odd. It follows that $1\,a_{n-1}a_{n-2}\cdots a_2a_1$ is even. By definition of the algorithm, we have $b_{n-1}=a_{n-1}$, $b_{n-2}=a_{n-2}$, ..., $b_2=a_2$, and $b_1=1-a_1$; i.e., $a_{n-1}=b_{n-1}$, ..., $a_2=b_2$, and $a_1=1-b_1$. This means that $b_{n-1}b_{n-2}\cdots b_2b_1 \to a_{n-1}a_{n-2}\cdots a_2a_1$.

(2) $b_{n-1}b_{n-2}\cdots b_2b_1$ is odd. Then $a_{n-1}a_{n-2}\cdots a_2a_1$ is even. It follows that $1\,a_{n-1}a_{n-2}\cdots a_2a_1$ is odd. Note that not all of $a_1,\ldots,a_{n-1}$ are zero, since it is not stopped yet. Since $1\,a_{n-1}a_{n-2}\cdots a_2a_1 \to 1\,b_{n-1}b_{n-2}\cdots b_2b_1$, then by definition of the algorithm, there exists an index j $(1\le j\le n-2)$ such that $a_1=\cdots=a_{j-1}=b_1=\cdots=b_{j-1}=0$, $a_j=b_j=1$, and $b_{j+1}=1-a_{j+1}$, $b_{j+2}=a_{j+2}$, ..., $b_{n-1}=a_{n-1}$. Note that $a_{j+1}=1-b_{j+1}$. We see that $b_{n-1}b_{n-2}\cdots b_2b_1 \to a_{n-1}a_{n-2}\cdots a_2a_1$ by definition of the algorithm. $\square$

4 Generating r-Combinations

Let $S=\{1,2,\ldots,n\}$. When an r-combination or an r-subset $A=\{a_1,a_2,\ldots,a_r\}$ of S is given, we always assume that $a_1<a_2<\cdots<a_r$. For two r-combinations $A=\{a_1,a_2,\ldots,a_r\}$ and $B=\{b_1,b_2,\ldots,b_r\}$ of S, if there is an integer k $(1\le k\le r)$ such that

$$a_1=b_1,\quad a_2=b_2,\quad \ldots,\quad a_{k-1}=b_{k-1},\quad a_k<b_k,$$

we say that A **precedes** B **in the lexicographic order**, written $A<B$. Then the set $P_r(S)$ of all r-subsets of S is linearly ordered by the lexicographic order. For simplicity, we write an r-combination $\{a_1,a_2,\ldots,a_r\}$ as an r-permutation

$$a_1a_2\cdots a_r \quad \text{with} \quad a_1<a_2<\cdots<a_r.$$

Theorem 4.1. *Let $a_1a_2\cdots a_r$ be an r-combination of $\{1,2,\ldots,n\}$. The first r-combination in lexicographic order is $12\cdots r$, and the last r-combination in lexicographic order is*

$$(n-r+1)\cdots(n-1)n.$$

If $a_1a_2\cdots a_k\cdots a_r\neq(n-r+1)\cdots(n-1)n$ and k is the largest index such that $a_k\neq n-r+k$, then the successor of $a_1a_2\cdots a_r$ is

$$a_1a_2\cdots a_{k-1}(a_k+1)(a_k+2)\cdots(a_k+r-k+1).$$

Proof. Since $a_i \le (n-r+i)$ for all $1 \le i \le r$, then $a_k \ne n-r+k$ implies $a_k < n-r+k$. Hence $a_k + r - k + 1 < n+1$. $\square$

Algorithm 4.2. Algorithm for Generating r-Combinations of $\{1, 2, \ldots, n\}$ in Lexicographic Order:

Step 0. Begin with the r-combination $a_1 a_2 \cdots a_r = 12 \cdots r$.
Step 1. If $a_1 a_2 \cdots a_r = (n-r+1) \cdots (n-1)n$, stop.
Step 2. If $a_1 a_2 \cdots a_r \ne (n-r+1) \cdots (n-1)n$, find the largest k such that $a_k < n-r+k$.
Step 3. Change $a_1 a_2 \cdots a_r$ to

$$a_1 \cdots a_{k-1}(a_k+1)(a_k+2) \cdots (a_k+r-k+1),$$

write down $a_1 a_2 \cdots a_r$, and return back to Step 1.

Example 4.1. The collection of all 4-combinations of $\{1, 2, 3, 4, 5, 6\}$ are listed by the algorithm:

1234	1245	1345	1456	2356
1235	1246	1346	2345	2456
1236	1256	1356	2346	3456

Theorem 4.3. *Let $a_1 a_2 \cdots a_r$ be an r-combination of $\{1, 2, \ldots, n\}$. Then the number of r-combinations up to the place $a_1 a_2 \cdots a_r$ in lexicographic order equals*

$$\binom{n}{r} - \binom{n-a_1}{r} - \binom{n-a_2}{r-1} - \cdots - \binom{n-a_{r-1}}{2} - \binom{n-a_r}{1}.$$

Proof. The r-combinations $b_1 b_2 \cdots b_r$ after $a_1 a_2 \cdots a_r$ can be classified into r kinds:

(1) $b_1 > a_1$; there are $\binom{n-a_1}{r}$ such r-combinations.
(2) $b_1 = a_1$, $b_2 > a_2$; there are $\binom{n-a_2}{r-1}$ such r-combinations.
(3) $b_1 = a_1$, $b_2 = a_2$, $b_3 > a_3$; there are $\binom{n-a_3}{r-2}$ such r-combinations.
$\vdots$
$(r-1)$ $b_1 = a_1$, …, $b_{r-2} = a_{r-2}$, $b_{r-1} > a_{r-1}$; there are $\binom{n-a_{r-1}}{2}$ such r-combinations.
(r) $b_1 = a_1, \ldots, b_{r-1} = a_{r-1}$, $b_r > a_r$; there are $\binom{n-a_r}{1}$ such r-combinations.

Since the number of r-combinations of $\{1, 2, \ldots, n\}$ is $\binom{n}{r}$, the conclusion follows immediately. $\square$

Example 4.2. The 3-combinations of $\{1, 2, 3, 4, 5\}$ are as follows:

$$123,\ 124,\ 125,\ 134,\ 135,\ 145,\ 234,\ 235,\ 245,\ 345$$

The 3-permutations of $\{1, 2, 3, 4, 5\}$ can be obtained by making 3! permutations for each 3-combination:

123	124	125	134	135	145	234	235	245	345
132	142	152	143	153	154	243	253	254	354
213	214	215	314	315	415	324	325	425	435
231	241	251	341	351	451	342	352	452	453
312	412	512	413	513	514	423	523	524	534
321	421	521	431	531	541	432	532	542	543

5 Partially Ordered Sets

Definition 5.1. A **(binary) relation** on a set X is a subset $R \subseteq X \times X$. Two members x and y of X are said to satisfy the relation R, written xRy, if $(x, y) \in R$. A relation R on X is said to be

(1) **reflexive** if xRx for all $x \in X$;

(2) **irreflexive** if $x\bar{R}x$ (also write $x\not{R}x$) for all $x \in X$, where $\bar{R} = (X \times X) \smallsetminus R$.

(3) **symmetric** provided that if xRy then yRx;

(4) **antisymmetric** provided that if $x \neq y$ and xRy then $y\bar{R}x$ (equivalently, if xRy and yRx then $x = y$);

(5) **transitive** provided that if xRy and yRz then xRz;

(6) **complete** provided that if $x, y \in X$ and $x \neq y$ then either xRy or yRx.

Example 5.1. (1) The relation of **set containment**, $\subseteq$, is a reflexive and transitive relation on the power set $P(X)$ of all subsets of X.

(2) The relation of **divisibility**, denoted $|$, is a reflexive and transitive relation on the set $\mathbb{Z}$ of integers.

A **partial order** $\leq$ on a set X is a reflexive, antisymmetric, and transitive relation, that is,

(P1) $x \leq x$ for all x,

(P2) $x \leq y$ and $y \leq x$ then $x = y$,

(P3) if $x \leq y$ and $y \leq z$ then $x \leq z$.

A **strict partial order** on a set X is an irreflexive and transitive relation, that is,

(SP1) $x \not< x$ for all x, and

(SP2) if $x < y$ and $y < z$ then $x < z$.

If a relation R is a partial order, which is usually denoted by $\leq$, then the relation $<$, defined by $a < b$ iff $a \leq b$ but $a \neq b$, is a strict partial order. Conversely, for a strict partial order $<$ on a set X, the relation $\leq$ defined by $a \leq b$ iff $a < b$ or $a = b$ is a partial order. A set X with a partial order $\leq$ is called a **partially ordered set** (or **poset** for short), denoted $(X, \leq)$.

A **linear order** on a set X is a partial order $\leq$ such that either $a \leq b$ or $b \leq a$ for any two members a and b of X, i.e., a complete partial order. A **strict linear order** is an irreflexive, transitive, and complete relation. A **preference relation** is a relation which is reflexive and transitive. An **equivalence relation** is a reflexive, symmetric, and transitive relation.

Example 5.2. Let $S = \{1, 2, 3, 4\}$. Then relation "larger than" in its ordinary meaning is the relation

$$\text{“} > \text{”} = \big\{(2,1), (3,1), (4,1), (3,2), (4,2), (4,3)\big\},$$

and it is a strict linear order relation. The relation "less than or equal to" in its ordinary meaning is the relation

$$\text{“} \leq \text{”} = \big\{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\big\},$$

and is a linear order relation.

Example 5.3. Let $\mathbb{C}$ denote the set of complex numbers. Consider the relation R on $\mathbb{C}$ given by zRw if $|z| = |w|$, where $z = a+ib$ and $|z| = \sqrt{a^2+b^2}$. Then R is an equivalence relation on $\mathbb{C}$.

The relation L on $\mathbb{C}$, defined by zLw (where $z = a+ib$ and $w = c+id$) if $a < c$ or $a = c$ but $b \le d$, is a linear order on $\mathbb{C}$.

Example 5.4. Let V be a vector space over $\mathbb{R}$. A partial order $\preceq$ on V is said to be **compatible with the addition and scalar multiplication** provided that

(i) $u \preceq u$ for all $u \in V$;

(ii) if $u \preceq v$ then $cu \preceq cv$ for all $c \ge 0$;

(iii) if $u_1 \preceq v_1$ and $u_2 \preceq v_2$ then $u_1 + u_2 \preceq v_1 + v_2$.

Properties (i) and (iii) imply the **translation preserving property**

(iv) if $u \preceq v$ then $u + w \preceq v + w$ for all $w \in V$.

A **convex cone** of V is a nonempty subset $C \subseteq V$ satisfying

(a) if $u \in C$ and $c \ge 0$ then $cu \in C$;

(b) if $u, v \in C$ then $u + v \in C$.

A convex cone C is said to be **strongly convex** if it further satisfies

(c) if $u \in C$ and $u \ne 0$, then $-u \notin C$, i.e., if $u, -u \in C$ then $u = 0$.

Strongly convex cone does not contain any 1-dimensional subspace.

Theorem 5.2. *Let C be a strongly convex cone of V. Then the binary relation $\preceq$ on V, defined by*

$$u \preceq v \quad \textit{if} \quad v - u \in C,$$

is a partial order compatible with the addition and scalar multiplication.

Conversely, let $\preceq$ be a partial order on V compatible with the addition and scalar multiplication. Then the subset

$$C := \{v \in V \mid 0 \preceq v\}$$

is a strongly convex cone of V.

Proof. The relation $\preceq$ is antisymmetric and transitive, since $u \preceq v$ and $v \preceq u$ imply $u = v$, and $u \preceq v$ and $v \preceq w$ imply $u \preceq w$. Clearly, the relation $\preceq$ is compatible with the addition and scalar multiplication. In fact, (i) $u \preceq u$ for all $u \in V$; (ii) if $u \preceq v$, then $cu \preceq cv$ for all $c \geq 0$; (iii) if $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1 + u_2 \preceq v_1 + v_2$.

In fact, (i) $u \in C$ implies $cu \in C$ for $c \geq 0$; (ii) $u, v \in C$ imply $u + v \in C$; (iii) if $u, -u \in C$, i.e., $0 \preceq u$ and $0 \preceq -u$, then $u = u + 0 \preceq u - u = 0$, thus $u = 0$. $\square$

Let H be a subspace of V. Then the relation $\sim$ on V, defined by $u \sim v$ if $v - u \in H$, is an equivalence relation on V.

Let $\leq_1$ and $\leq_2$ be two partial orders on a set X. The poset $(X, \leq_2)$ is called an **extension** of the poset $(X, \leq_1)$ if, whenever $a \leq_1 b$, then $a \leq_2 b$. In particular, an extension of a partial order has more compatible pairs. We show that every finite poset has a **linear extension**, that is, an extension which is a linearly ordered set.

Theorem 5.3. *Let $(X, \leq)$ be a finite partially ordered set. Then there is a linear order $\preceq$ such that $(X, \preceq)$ is an extension of $(X, \leq)$.*

Proof. We need to show that the elements of X can be listed in some order $x_1, x_2, \ldots, x_n$ in such a way that if $x_i \leq x_j$ then x_i comes before x_j in this list, i.e., $i \leq j$. The following algorithm does the job.

Algorithm 5.4. Algorithm for a Linear Extension of an n-Poset:

Step 1. Choose a minimal element x_1 from X (with respect to the ordering $\leq$; if such elements are not unique, choose any one).

Step 2. Delete x_1 from X; choose a minimal element x_2 from $X \smallsetminus \{x_1\}$.

Step 3. Delete x_2 from $X \smallsetminus \{x_1\}$; choose a minimal element x_3 from $X \smallsetminus \{x_1, x_2\}$.

$\vdots$

Step n. Delete x_{n-1} from $X \smallsetminus \{x_1, \ldots, x_{n-2}\}$ and choose the only element x_n in $X \smallsetminus \{x_1, \ldots, x_{n-1}\}$.

$\square$

Let R be an equivalence relation on a set X. For each element $x \in X$, we call the set

$$[x] = \{y \in X : xRy\}$$

an **equivalence class** of R and x a **representative** of the class $[x]$.

Theorem 5.5. *Let R be an equivalence relation on a set X. Then for any $x, y \in X$, the following statements are logically equivalent:*

1) $[x] \cap [y] \neq \emptyset$;

2) $[x] = [y]$;

3) xRy.

A collection $\mathcal{P} = \{A_1, A_2, \ldots, A_k\}$ of nonempty subsets of a set X is called a **partition** of X if $A_i \cap A_j = \emptyset$ for $i \neq j$ and $X = \bigcup_{i=1}^{k} A_i$. We will show below that if R is an equivalence relation on a set X, then the collection

$$\mathcal{P}_R = \big\{[x] : x \in X\big\}$$

is a partition of X. Conversely, if $\mathcal{P} = \{A_1, A_2, \ldots, A_k\}$ is a partition of X, then the relation

$$R_{\mathcal{P}} = \bigcup_{i=1}^{k} A_i \times A_i$$

is an equivalence relation on X.

Theorem 5.6. *Let R be an equivalence relation on a set X, and let $\mathcal{P} = \{A_1, A_2, \ldots, A_k\}$ be a partition of X. Then*

(a) *$\mathcal{P}_R$ is a partition of X.*

(b) *$R_{\mathcal{P}}$ is an equivalence relation on X.*

(c) *$R_{\mathcal{P}_R} = R$, $\mathcal{P}_{R_{\mathcal{P}}} = \mathcal{P}$.*

Proof. Parts (a) and (b) are trivial. We only prove Part (c).

Let $(x, y) \in R_{\mathcal{P}_R}$. Then, by definition of equivalence relation induced from the partition $\mathcal{P}_R$, there exists a part $A \in \mathcal{P}_R$ such that $x, y \in A$. Note that the parts in $\mathcal{P}_R$ are the equivalence classes $[x]$ of the equivalence relation R; in particular A is an equivalence class of R. Since A contains both x and y, it follows that $A = [x] = [y]$. So $(x, y) \in R$. Hence $R_{\mathcal{P}_R} \subseteq R$.

Let $(x, y) \in R$. Then $[x] = [y]$ is a part of $\mathcal{P}_R$. Thus $[x] \times [y] \subseteq R_{\mathcal{P}_R}$. Since $(x, y) \in [x] \times [y]$, we see that $(x, y) \in R_{\mathcal{P}_R}$. Therefore $R \subseteq R_{\mathcal{P}_R}$.

$A \in \mathcal{P}_{R_\mathcal{P}} \Leftrightarrow A$ is an equivalence class of the relation $R_\mathcal{P} \Leftrightarrow A$ is a part of the partition $\mathcal{P}$. $\square$

Example 5.5. Let V be a vector space over $\mathbb{R}$. An equivalence relation $\sim$ on V is said to be **translation preserving** if $u \sim v$ implies $u + w \sim v + w$ for all $w \in V$.

A **subspace** of V is a nonempty subset $H \subset V$ such that if $u, v \in H$ then $au + bv \in H$ for all $a, b \in \mathbb{R}$.

Given a subspace $H \subset V$. The relation $\sim$ on V, defined by $u \sim v$ if $v - u \in H$, is an order preserving equivalence relation on V.