Homework 4

Chapter 7 p.246 (3rd edition) 7, 12, 19, 22, 24, 25b,e, 26, 30, 32, 37

1. Let a_n equal the number of different ways in which the squares of 1-by-n chessboard can be colored, using the colors red, white, and blue so that no two squares colored red are adjacent. Find and verify a recurrence relation that a_n satisfies. Then find a formula for a_n.

2. Solve the recurrence relation $a_n = 8a_{n-1} - 2a_{n-2}$, $(n \geq 2)$, with initial values $a_0 = -1$ and $a_1 = 0$.

3. Solve the nonhomogeneous recurrence relation $a_n = 3a_{n-1} - 2$ with $a_0 = 1$.

4. Solve the nonhomogeneous recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 3n + 1$ with $a_0 = 1$ and $a_1 = 2$.

5. Let M be the multiset $\{\infty \cdot e_1, \infty \cdot e_2, \infty \cdot e_3, \infty \cdot e_4\}$. Determine the generating function for the sequence $(a_n; n \geq 0)$, where a_n is the number of n-combinations of M with the additional restrictions:
   (a) Each e_i occurs an odd number of times.
   (b) Each e_i occurs a multiple-of-3 number of times.
   (c) The element e_1 does not occur, and e_2 occurs at most once.
   (d) The element e_1 occurs $1, 3$, or 11 times, and the element e_2 occurs $2, 4$, or 5 times.
   (e) Each e_i occurs at least 10 times.

6. Solve the following recurrence relations by using the method of generating functions.
   (a) $a_n = a_{n-1} + a_{n-2}$, $(n \geq 2)$; $a_0 = 1$, $a_1 = 3$.
   (b) $a_n = 3a_{n-2} - 2a_{n-3}$, $n \geq 3$; $a + 0 = 1$, $a_1 = 1$, $a_2 = 0$.

7. Solve the nonhomogeneous recurrence relation
$$a_n = 4a_{n-1} + 4^n, \quad n \geq 1; \quad a_0 = 3.$$

8. Determine the generating function for the number a_n of the bags of fruit of apples, oranges, bananas, and pears in which there are an even number of apples, at most two oranges, a multiple of three number of bananas, and at most one pear. Then find the formula for a_n from the generating function.

9. Let $a_n = \binom{n}{2}$, $n \geq 0$. Determine the generating function of $(a_n; n \geq 0)$.

10. Let M be the multiset $\{\infty \cdot e_1, \infty \cdot e_2, \ldots, \infty \cdot e_k\}$. determine the exponential generating function for the sequence $(a_n; n \geq 0)$, where $a_0 = 1$ and for $n \geq 1$:
    (a) a_n equals the number of n-permutations of M in which each object occurs an odd number of times.
    (b) a_n equals the number of n-permutations of M in which each object occurs at least four times.
    (c) a_n equals the number of n-permutations of M in which $_1$ occurs at least once, e_2 occurs at least twice, $\ldots$, e_k occurs at least k times.
    (d) a_n equals the number of n-permutations of M in which $_1$ occurs at most once, e_2 occurs at most twice, $\ldots$, e_k occurs at most k times.

1. Let q be a root of the characteristic polynomial of the recurrence relation

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + \cdots + \alpha_k x_{n-k}, \quad n \geq k, \tag{1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants and $\alpha_k \neq 0$.

(a) If the multiplicity of the root q is m, show that $x_n = n^i q^n$, where $0 \leq i \leq m-1$, is a solution of the recurrence relation.

(b) Prove that the solutions q^n, nq^n, …, $n^{m-1}q^n$ are linearly independent solutions.

2. In the recurrence relation (1), let $Y_n = [y_{n,0}, y_{n,1}, \ldots, y_{n,k-1}]^T$, where $y_{n,i} = x_{kn+i}$. Show that the recurrence relation (1) can be changed into the following matrix recurrence relation of order 1:

$$Y_n = AY_{n-1}.$$

Find possible relation between the roots of the characteristic polynomial of (1) and the eigenvalues of the matrix A.

Chapter 8, pp.290: 2, 6, 7, 12, 15, 19, 25, 29

1. Prove that the number of 2-by-n arrays

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \end{bmatrix}$$

that can be made from the numbers $1, 2, \ldots, 2n$ so that

$$x_{11} < x_{12} < \cdots < x_{1n}, \qquad x_{21} < x_{22} < \cdots < x_{2n},$$

and

$$x_{11} < x_{21}, \quad x_{12} < x_{22}, \quad \ldots, \quad x_{1n} < x_{2n},$$

equals the nth Catalan number C_n.

2. Let m and n be the non-negative integers with $m \leq n$. There are $m+n$ people in line to get into a theater for which admission is 5 dollars. Of the $m+n$ people, n have a 5 dollar single coin m have a 10 dollar bill. The box office opens with an empty cash register. Show that the number of ways the people can line up so that change is available when needed is

$$\frac{n-m+1}{n+1}\binom{m+n}{m}.$$

3. Let $(a_n; n \geq 0)$ be defined by $a_n = 2n^2 - n + 3$. Determine the difference table of $(a_n; \geq 0)$; and find a formula for $\sum_{k=0}^{n} a_k$.

4. Show that the Stirling numbers of the second kind satisfy the relation:

   (a) $S(n,1) = 1$ for $n \geq 1$;

   (b) $S(n,2) = 2^{n-1} - 1$ for $n \geq 2$;

   (c) $S(n, n-1) = \binom{n}{2}$ for $n \geq 1$;

   (d) $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$.

5. The number of partitions of a set of n elements into k distinguishable boxes (some of which may be empty) is k^n. By counting in a different way prove that

$$k^n = \sum_{i=1}^{n} \binom{k}{i} i! S(n,i).$$

6. Show that the Stirling numbers of the first kind satisfy

   (a) $S(n, 1) = (n-1)!$, $n \geq 1$.

   (b) $S(n, n-1) = \binom{n}{2}$, $n \geq 1$.

7. Let $a_1, a_2, \ldots, a_m$ be distinct positive integers, and let $q_n = q_n(a_1, a_2, \ldots, a_m)$ be equal to the number of partitions of n in which all parts are taken from $a_1, a_2, \ldots, a_m$. Define $q_0 = 1$. Show that the generating function for $q_1, q_2, \ldots, q_n, \ldots$ is

$$\prod_{k=1}^{m} \frac{1}{(1 - x^{t_k})}.$$

8. Evaluation $h_{k-1}^{(k)}$, the number of regions into which k-dimensional spaces is partitioned by $k-1$ hyperplanes in general position.