CHAPTER 6

Counting Orbits of Group Actions

6.1. Group Action

Let G be a finite group acting on a finite set X, said to be a **group action**, i.e., there is a map

$$G \times X \to X, \quad (g, x) \mapsto gx,$$

satisfying two properties: (i) $ex = x$ for all $x \in X$, where e is the group identity element of G, (ii) $h(gx) = (hg)x$ for all $g, h \in G$ and $x \in X$. Each group element g induces a bijection $g : X \to X$ by $g(x) = gx$, since the map g has the inverse map g^{-1}.

The group action induces an equivalence relation $\sim$ on X, where $x \sim y$ for $x, y \in X$ if and only if $gx = y$ for some g of G. Then X is partitioned into equivalence classes, each of them is called an **orbit** of X under G. For each element $x \in X$, the orbit containing x is also called the **orbit** of x, and it consists of the elements of the form gx, where $g \in G$. So we denote descriptively the orbit of x by

$$Gx = \{y \in X : y = gx,\ g \in G\}.$$

We denote by X/G the set of equivalence classes (orbits) of X under the action of G.

For each element $x \in X$, the **stabilizer** of x is the set

$$\mathrm{Stab}(x) = \{g \in G : gx = x\},$$

which is a subgroup of G. Set $H := \mathrm{Stab}(x)$. The map $Gx \to G/H$, $gx \mapsto gH$, is well-defined and is a bijection. In fact, if $gx = hx$, then $g^{-1}hx = x$, i.e., $g^{-1}h \in H$, thus $gH = g(g^{-1}hH) = hH$; the map is well-defined. The surjectivity is obvious. If $gH = hH$, then $h^{-1}g \in H$, thus $g^{-1}hx = x$, i.e., $hx = gx$; the map is injective. Therefore $|G| = |Gx|\,|H|$, since $|G/H| = |G|/|H|$ and is the number of cosets of H in G. We obtain the following lemma which can be proved directly without use of quotient group.

Lemma 6.1. *Let G be a finite group acting on a finite set X. Then for each element $x \in X$,*

$$|G| = |Gx|\,|\mathrm{Stab}(x)|.$$

Proof. Consider the subset $S_x = \{(g, y) : y = gx\}$ of $G \times X$. Fix an element $y \in Gx$ and an element $g_0 \in G$ such that $g_0x = y$. The map $\{g \in G : gx = y\} \to \mathrm{Stab}(x)$, $g \mapsto g_0^{-1}g$, is a bijection. Thus

$$|S_x| = \sum_{y \in Gx} \sum_{\substack{g \in G \\ gx = y}} 1 = \sum_{y \in Gx} \big|\{g \in G : gx = y\}\big| = \sum_{y \in Gx} |\mathrm{Stab}(x)| = |\mathrm{Stab}(x)|\,|Gx|.$$

The identity follows by $|S_x| = |G|$, since $S_x = \{(g, gx) : g \in G\}$. □

For each group element $g \in G$, the **fixed set** of g is the subset

$$\mathrm{Fix}(g) = \{x \in X : gx = x\}.$$

A **weight function** w on X with values in an abelian group A is just a map $w : X \to A$. Each weigh function w can be extended to a weight function on the power set of X, given by $w(\varnothing) \equiv 0$ and

$$w(S) = \sum_{x \in S} w(x), \quad \varnothing \neq S \subseteq X.$$

There is an **average weight function** W on the power set of X, defined by $W(\varnothing) \equiv 0$ and and

$$W(S) = \frac{1}{|S|} \sum_{x \in S} w(x), \quad \varnothing \neq S \subseteq X.$$

LEMMA 6.2 (Burnside's Lemma). *Let G be a finite group acting on a finite set X. Then the number of orbits of X under the action of G is given by*

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|. \tag{6.1}$$

Moreover, given a weight function $w : X \to A$, where A is an abelian group, we have

$$\sum_{P\in X/G} W(P) = \frac{1}{|G|}\sum_{g\in G} w\big(\mathrm{Fix}(g)\big). \tag{6.2}$$

PROOF. The weight function w can be extended to $w : G\times X \to A$ by $w(g,x) = w(gx)$. Consider the total weight of the subset $S = \{(g,x) : gx = x\} \subseteq G\times H$. On the one hand,

$$\begin{aligned} w(S) &= \sum_{g\in G}\sum_{\substack{x\in X\\ gx=x}} w(gx) \\ &= \sum_{g\in G}\sum_{\substack{x\in X\\ gx=x}} w(x) \\ &= \sum_{g\in G} w\big(\mathrm{Fix}(g)\big). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} w(S) &= \sum_{x\in X}\sum_{\substack{g\in G\\ gx=x}} w(gx) \\ &= \sum_{x\in X}\sum_{\substack{g\in G\\ gx=x}} w(x) \\ &= \sum_{x\in X} w(x)|\mathrm{Stab}(x)| \\ &= \sum_{x\in X} w(x)\cdot\frac{|G|}{|Gx|}. \end{aligned}$$

Since $X = \bigcup_{P\in X/G} P$ is a disjoint union, we see that

$$\begin{aligned} w(S) &= |G|\sum_{x\in X}\frac{w(x)}{|Gx|} \\ &= |G|\sum_{P\in X/G}\sum_{x\in P}\frac{w(x)}{|P|} \\ &= |G|\sum_{P\in X/G} W(P). \end{aligned}$$

The the weighted version of the Birnside identity (6.2) follows immediately. □

6.2. Groups Acting on Sets of Functions

Let G and H be finite groups. Let X be a finite G-set and Y a finite H-set. We denote by $\mathrm{Map}(X,Y)$ the set of all functions from X to Y. Then the product group $G\times H$ acts on $\mathrm{Map}(X,Y)$ in an obvious way:

$$G\times H\times \mathrm{Map}(X,Y) \to \mathrm{Map}(X,Y), \quad (g,h,\phi)\mapsto (g,h)\phi = h\phi g^{-1},$$

where $h\phi g^{-1}(x) = h\phi(g^{-1}x)$ for $x\in X$. In fact, for $(g_1,h_1),(g_2,h_2)\in G\times H$ and $\phi\in\mathrm{Map}(X,Y)$,

$$\begin{aligned} (g_2,h_2)((g_1,h_1)\phi) &= (g_2,h_2)(h_1\phi g_1^{-1}) = h_2h_1\phi g_1^{-1}g_2^{-1} \\ &= h_2h_1\phi(g_2g_1)^{-1} = (g_2g_1,h_2h_1)\phi \\ &= ((g_2,h_2)(g_1,h_1))\phi. \end{aligned}$$

Two functions $\phi_1, \phi_2 \in \mathrm{Map}(X, Y)$ are said to be **equivalent** if they are in the same orbit of $\mathrm{Map}(X, Y)$, i.e.,

$$\phi_1 \sim \phi_2 \quad \text{if and only if} \quad \phi_2 g = h\phi_1$$

for some $g \in G$ and $h \in H$. This means that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi_1} & Y \\ g\downarrow & & \downarrow h \\ X & \xrightarrow{\phi_2} & Y \end{array}$$

is commutative. For each $\phi \in \mathrm{Map}(X, Y)$, we have $\mathrm{Fix}(\phi) = \{(g, h) \in G \times H : \phi g = h\phi\}$.

THEOREM 6.3 (DeBruijin Formula). *Let G, H be finite groups acting on finite sets X, Y respectively. Let $G \times H$ act on $\mathrm{Map}(X, Y)$ as $(g, h) \cdot \phi = h\phi g^{-1}$. Then for each weight function $w : \mathrm{Map}(X, Y) \to A$ (abelian group),*

$$\sum_{P \in \mathrm{Map}(X,Y)/(G\times H)} W(P) = \frac{1}{|G \times H|} \sum_{(g,h)\in G\times H} \sum_{\substack{\phi\in \mathrm{Map}(X,Y) \\ \phi g = h\phi}} w(\phi) \tag{6.3}$$

PROOF. This is a special case of the weighted version (6.2) of the Burnside Lemma with G replaced by $G \times H$ and X replaced by $\mathrm{Map}(X, Y)$. □

EXERCISE 3. Let $|X| = n$, $Y = [k] = \{1, 2, \ldots, k\}$, $n = n_1 + \cdots + n_k$. Consider the set $\mathrm{Map}(X, n_1, \ldots, n_k)$ of functions $f : X \to Y$ such that $|f^{-1}(i)| = n_i$. What is $|\mathrm{Map}(X, n_1, \ldots, n_k)|$? $|\mathrm{Map}(X, n_1, \ldots, n_k)| = \binom{n}{n_1, \ldots, n_k}$. Let G be a group acting on X with $G = \{1, \sigma, \sigma^2, \ldots, \sigma^{n-1}\}$, where $\sigma(x_i) = x_{i+1}$, $x_{n+1} = x_1$. What is $|\mathrm{Map}(X, n_1, \ldots, n_k)^H|$ for each subgroup H of G?

6.3. Pólya's Theorem

We consider a special case of DeBruijin's formula (6.3) with $H = 1$, i.e., there is no group action on Y. Then the group action of G on $\mathrm{Map}(X, Y)$ is given by $g \cdot \phi = \phi g^{-1}$ for $g \in G$ and $\phi \in \mathrm{Map}(X, Y)$. A function $\phi : X \to Y$ is said to be **G-invariant** if $\phi g = \phi$ for all $g \in G$. We denote by $\mathrm{Map}(X, Y)^G$ the set of G-invariant functions.

Let $w : Y \to A$ be a weight function, where A is an arbitrary finite set, not necessarily an abelian group. Then w can be extended into a function $w : \mathrm{Map}(X, Y) \to \mathbb{Z}[A]$, defined by

$$w(\phi) = \prod_{x\in X} w(\phi(x)), \quad \phi \in \mathrm{Map}(X, Y).$$

The product exhibits the information of the map ϕ in terms of weights of the values $\phi(x)$. We shall see that w is constant on each orbit of the group action of G on $\mathrm{Map}(X, Y)$. In fact, given two maps $\phi_1, \phi_2 \in \mathrm{Map}(X, Y)$ in an orbit, i.e., $\phi_2 = \phi_1 g$ for some $g \in G$, we have

$$w(\phi_2) = \prod_{x\in X} w(\phi_1(g(x))) = \prod_{x'\in X} w(\phi_1(x')) = w(\phi_1) \quad (\text{let } x' = g(x)).$$

DEFINITION 6.4. Let G be a finite group acting on an n-set X. The **cycle index** of each group element $g \in G$ of cycle type $1^{\lambda_1(g)} 2^{\lambda_2(g)} \cdots n^{\lambda_n(g)}$ is the monomial

$$Z_g(t_1, t_2, \ldots, t_n) = t_1^{\lambda_1(g)} t_2^{\lambda_2(g)} \cdots t_n^{\lambda_n(g)}.$$

The **cycle index** of G is the polynomial

$$Z_G(t_1, t_2, \ldots, t_n) = \frac{1}{|G|} \sum_{g\in G} Z_g(t_1, t_2, \ldots, t_n). \tag{6.4}$$

THEOREM 6.5 (Pólya's Theorem). *Let G be a finite group acting on a finite set X with $|X| = n$. Let Y be a finite set and $w : Y \to A$ be a weight function. Then*

$$\sum_{P\in \mathrm{Map}(X,Y)/G} W(P) = Z_G\left(\sum_y w(y), \sum_y w(y)^2, \ldots, \sum_y w(y)^n\right). \tag{6.5}$$

In particular, if $w(y) = 1$ *for* $y \in Y$*, then* (6.5) *becomes*

$$|\mathrm{Map}(X,Y)/G| = Z_G\big(|Y|, |Y|, \ldots, |Y|\big). \tag{6.6}$$

Proof. We may assume $X = \{1, 2, \ldots, n\}$. Fix an element $g \in G$. Since g is a permutation of X, we may write g as cycles $(C_1)(C_2)\cdots(C_k)$ with $c_i = |C_i|$, where $C_1, \ldots, C_k$ form a partition of X. Consider maps $\phi : X \to Y$ such that $\phi g = \phi$, which is equivalent to that ϕ is constant on each cycle C_i of g. Let us write $y_i = \phi(C_i) = \phi(x)$, where $x \in C_i$, for $\phi g = \phi$ and $1 \le i \le k$. Notice the bijection

$$\mathrm{Fix}(g) = \{\phi \in \mathrm{Map}(X,Y) : \phi g = \phi\} \to Y^k, \quad \phi \mapsto (\phi(C_1), \ldots, \phi(C_k)).$$

We then have

$$\begin{aligned}
\sum_{\substack{\phi \in \mathrm{Map}(X,Y)\\ \phi g = \phi}} w(\phi) &= \sum_{y_1 \in Y}\sum_{y_2 \in Y}\cdots\sum_{y_k \in Y} w(y_1)^{c_1} w(y_2)^{c_2}\cdots w(y_k)^{c_k}\\
&= \left(\sum_{y_1 \in Y} w(y_1)^{c_1}\right)\left(\sum_{y_2 \in Y} w(y_2)^{c_2}\right)\cdots\left(\sum_{y_k \in Y} w(y_k)^{c_k}\right)\\
&= \left(\sum_{y \in Y} w(y)^{c_1}\right)\left(\sum_{y \in Y} w(y)^{c_2}\right)\cdots\left(\sum_{y \in Y} w(y)^{c_k}\right).
\end{aligned}$$

Let g be of the cycle type $1^{\lambda_1(g)}2^{\lambda_2(g)}\cdots n^{\lambda_n(g)}$. Collecting the like terms in the above product, we see that

$$\begin{aligned}
\sum_{\substack{\phi \in \mathrm{Map}(X,Y)\\ \phi g = \phi}} w(\phi) &= \left(\sum_{y \in Y} w(y)\right)^{\lambda_1(g)}\left(\sum_{y \in Y} w(y)^2\right)^{\lambda_2(g)}\cdots\left(\sum_{y \in Y} w(y)^n\right)^{\lambda_n(g)}\\
&= Z_g\left(\sum_{y \in Y} w(y), \sum_{y \in Y} w(y)^2, \ldots, \sum_{y \in Y} w(y)^n\right).
\end{aligned}$$

Applying the weighted version of the Birnside Lemma (6.2), we see that

$$\begin{aligned}
\sum_{P \in \mathrm{Map}(X,Y)/G} W(P) &= \frac{1}{|G|}\sum_{g \in G}\sum_{\substack{\phi \in \mathrm{Map}(X,Y)\\ \phi g = \phi}} w(\phi)\\
&= Z_G\left(\sum_{y} w(y), \sum_{y} w(y)^2, \ldots, \sum_{y} w(y)^n\right).
\end{aligned}$$

The proof is finished. □

Definition 6.6. A function $\phi : X \to Y$ may be considered as a coloring of X with the set Y of colors. Let $Y = \{y_1, \ldots, y_m\}$. The **indicator** of ϕ is the monomial

$$\mathrm{ind}(\phi) = \prod_{x \in X} \phi(x) = \prod_{i=1}^{m} y_i^{|\phi^{-1}(y_i)|}$$

of the indeterminates $y_1, \ldots, y_m$. The **indicator** (or **patten inventory**) of $\mathrm{Map}(X,Y)^G$ is the multinomial generating function

$$\mathrm{Ind}_G(X; y_1, \ldots, y_m) = \sum_{(k_1, \ldots, k_m) \in \mathbb{N}^m} c_{k_1, \ldots, k_m} y_1^{k_1}\cdots y_m^{k_m},$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$, the coefficient $c_{k_1, \ldots, k_m}$ is the number of G-invariant colorings of X such that the number of elements of X receiving the color y_i is k_i, $1 \le i \le m$.

Corollary 6.7. *Let G be a finite group acting on a finite set X with $|X| = n$, and let $Y = \{y_1, \ldots, y_m\}$. Then the indicator of* $\mathrm{Map}(X,Y)^G$ *is given by*

$$\mathrm{Ind}_G(X; y_1, \ldots, y_m) = Z_G\left(\sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \ldots, \sum_{i=1}^{m} y_i^n\right). \tag{6.7}$$

PROOF. Let $A = Y$ and $w : Y \to A$ be the identity map. The weight function $\mathrm{ind} : \mathrm{Map}(X, Y) \to \mathbb{Z}[Y]$ induced by w is G-invariant, i.e., $\mathrm{Ind}(P) = \mathrm{ind}(\phi)$ for each orbit P of $\mathrm{Map}(X, Y)$ and all $\phi \in P$. If $\phi : X \to Y$ is of type $(k_1, \ldots, k_m)$, where $k_i = |\phi^{-1}(y_i)|$, then $\mathrm{Ind}(P) = \mathrm{ind}(\phi) = y_1^{k_1} \cdots y_m^{k_m}$. Thus

$$\sum_{P \in \mathrm{Map}(X,Y)/G} \mathrm{Ind}(P) = \mathrm{Ind}_G(X; y_1, \ldots, y_m).$$

Now the Pólya cycle index formula (6.5) becomes the generating function formula (6.7). □

6.4. Examples

EXAMPLE 6. *Find the number of ways of coloring the vertices of a square with two colors, black and white.*

Let $X = \{1, 2, 3, 4\}$ denote the four vertices of the square. There are 16 ($= 2^4$) colorings when the vertices of the square are labelled by the numbers 1,2,3,4; see Figure 1. The sixteen colorings can be listed as follows:

		(b,b,w,w)		
		(b,w,b,w)		
	(b,b,b,w)	(b,w,w,b)	(b,w,w,w)	
	(b,b,w,b)	(w,b,b,w)	(w,b,w,w)	
	(b,w,b,b)	(w,b,w,b)	(w,w,b,w)	
(b,b,b,b)	(w,b,b,b)	(w,w,b,b)	(w,w,w,b)	(w,w,w,w)

These colorings can be represented by the following binomial expansion

$$(b + w)^4 = b^4 + 4b^3w + 6b^2w^2 + 4bw^3 + w^4.$$

The coefficient c_{ij} of b^iw^j in the expansion represents the number of colorings with i black vertices and j white vertices. For instance, the coefficient of b^3w means that there are 4 colorings such that 3 vertices are black and one vertex is white; and the coefficient of b^2w^2 means that there are 6 colorings such that two vertices are black and other two vertices are white.

However, when the labels are removed, some colorings are essentially the same. For instance, the following two colorings

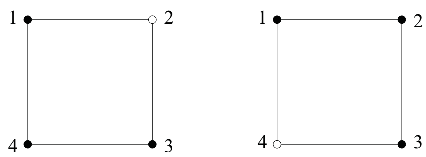


FIGURE 1. Two indistinguishable colorings when the labels are removed.

are indistinguishable when the labels are removed, since the left one can be obtained from the right one by a rotation of 180°. The symmetry group of the square can be identified as a subgroup G of the symmetric group for the set $\{1, 2, 3, 4\}$. The cycle indexes for all $g \in G$ are listed in the following table.

$g \in G$	$\lambda_1(g)$	$\lambda_2(g)$	$\lambda_3(g)$	$\lambda_4(g)$	Z_g
(1)(2)(3)(4)	4	0	0	0	t_1^4
(1)(3)(42)	2	1	0	0	$t_1^2t_2$
(2)(31)(4)	2	1	0	0	$t_1^2t_2$
(21)(43)	0	2	0	0	t_2^2
(31)(42)	0	2	0	0	t_2^2
(32)(41)	0	2	0	0	t_2^2
(4321)	0	0	0	1	t_4
(4123)	0	0	0	1	t_4

Thus the cycle index of G is

$$Z_G(t_1, t_2, t_3, t_4) = \frac{1}{8}\left(t_1^4 + 2t_1^2t_2 + 3t_2^2 + 2t_4\right).$$

Taking account of the symmetry of the square, the indistinguishable colorings of the square by two colors can be listed as follows:

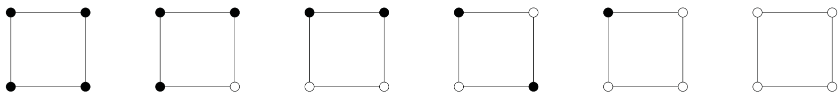


FIGURE 2.

The total number of indistinguishable colorings is 6, which is confirmed by the formula

$$Z_G(2,2,2,2) = \frac{1}{8}\left(2^4 + 2\cdot 2^3 + 3\cdot 2^2 + 2\cdot 2\right) = 6.$$

The generating function of the G-invariant colorings with the two colors b (black) and w (white) is

$$\mathrm{Ind}_G(X; b, w) = b^4 + b^3w + 2b^2w^2 + bw^3 + w^4,$$

and is confirmed by

$$\begin{aligned}
&Z_G(b+w, b^2+w^2, b^3+w^3, b^4+w^4)\\
= \;& \frac{1}{8}\big((b+w)^4 + 2(b+w)^2(b^2+w^2) + 3(b^2+w^2)^2 + 2(b^4+w^4)\big)\\
= \;& \frac{1}{8}\big((b^4 + 4b^3w + 6b^2w^2 + 4bw^3 + w^4) + (2b^4 + 4b^3w + 4b^2w^2 + 4bw^3 + 2w^4)\\
& + (3b^4 + 6b^2w^2 + 3w^4) + (2b^4 + 2w^4)\big)\\
= \;& b^4 + b^3w + 2b^2w^2 + bw^3 + w^4\\
= \;& \mathrm{Ind}_G(X; b, w).
\end{aligned}$$

EXAMPLE 7. Find the number of indistinguishable colorings for the faces of a regular cube with three colors, black (b), red (r), and white (w).

Let $X = \{1,2,3,4,5,6\}$ be the set of six faces of the cube. The symmetric group of the cube can be identified as a subgroup of the symmetric group of X. This subgroup can be obtained by three kinds of rotations, see Figure 3.

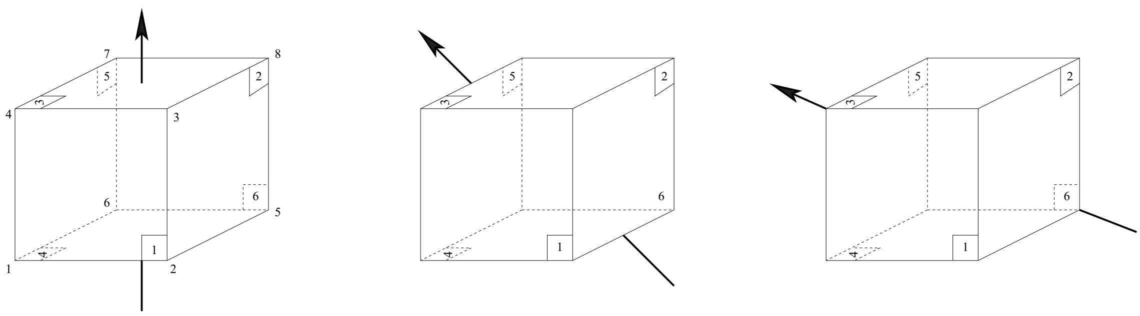


FIGURE 3. Three kinds of rotations of a cube.

$g \in G$	#	$\lambda_1(g)$	$\lambda_2(g)$	$\lambda_3(g)$	$\lambda_4(g)$	$\lambda_5(g)$	$\lambda_6(g)$	Z_g
(1)(2)(3)(4)(5)(6)	1	6	0	0	0	0	0	t_1^6
(3)(4)(6512)	6	2	0	0	1	0	0	$t_1^2t_4$
(3)(4)(52)(61)	3	2	2	0	0	0	0	$t_1^2t_2^2$
(42)(53)(61)	6	0	3	0	0	0	0	t_2^3
(513)(642)	8	0	0	2	0	0	0	t_3^2

There are 3 types of the first kind of rotations, 6 types of the second kind of rotations, and 4 types of the third kind of rotations. Then $|G| = 24$ and the cycle index polynomial for the symmetry group of a cube is

$$\begin{aligned} Z_G(t_1, \ldots, t_6) &= \frac{1}{24}\left(t_1^6 + 3(2t_1^2t_4 + t_1^2t_2^2) + 6t_2^3 + 4 \cdot 2t_3^2\right) \\ &= \frac{1}{24}\left(t_1^6 + 3t_1^2t_2^2 + 6t_1^2t_4 + 6t_2^3 + 8t_3^2\right). \end{aligned}$$

The total number of indistinguishable colorings is given by

$$Z_G(3,3,3,3,3,3) = \frac{1}{24}\left(3^6 + 3 \cdot 3^2 \cdot 3^2 + 6 \cdot 3^2 \cdot 3 + 6 \cdot 3^3 + 8 \cdot 3^2\right) = 57.$$

The generating function for the G-invariant colorings is

$$\begin{aligned} \mathrm{Ind}_G(X; b, r, w) &= (b^6 + r^6 + w^6) \\ &\quad +(b^5r + b^5w + br^5 + r^5w + bw^5 + rw^5) \\ &\quad +2(b^4r^2 + b^4w^2 + b^2r^4 + r^4w^2 + b^2w^4 + r^2w^4) \\ &\quad +2(b^4rw + br^4w + brw^4) \\ &\quad +2(b^3r^3 + b^3w^3 + r^3w^3) \\ &\quad +3(b^3r^2w + b^3rw^2 + b^2r^3w + b^2rw^3 + br^3w + brw^3) \\ &\quad +6b^2r^2w^2. \end{aligned}$$

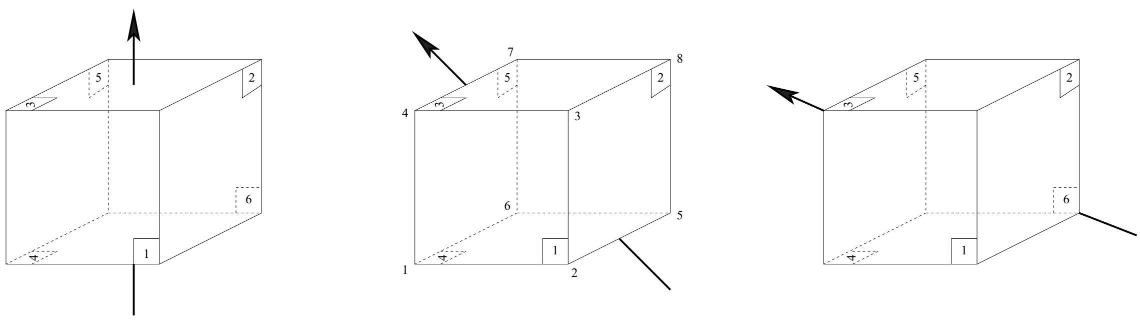


FIGURE 4. Two distinguishable colorings of a regular cube.

EXAMPLE 8. Octahedron:

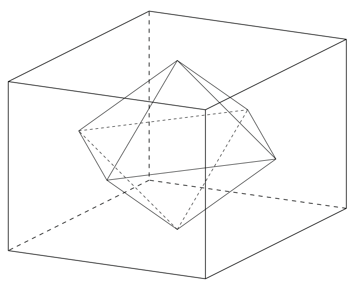


FIGURE 5. Duality of cube and octahedron

$g \in G$	#	$\lambda_1(g)$	$\lambda_2(g)$	$\lambda_3(g)$	$\lambda_4(g)$	Z_g
$(1)(2)(3)\cdots(7)(8)$	1	8	0	0	0	t_1^8
$(8743)(6125)$	6	0	0	0	2	t_4^2
$(51)(62)(73)(84)$	3	0	4	0	0	t_2^4
$(52)(61)(74)(83)$	6	0	4	0	0	t_2^4
$(4)(5)(713)(862)$	8	2	0	2	0	$t_1^2t_3^2$

$$Z_G(t_1,\dots,t_8) = \frac{1}{24}\left(t_1^8 + 9t_2^4 + 6t_4^2 + 8t_1^2t_3^2\right).$$

$$\mathrm{Ind}_G(X;b,w) = b^8 + b^7w + 3b^6w^2 + 3b^5w^3 + 7b^4w^4 + 3b^3w^5 + 3b^2w^7 + w^8.$$

EXAMPLE 9. Tetrahedron: The cycle index polynomials for both vertices and faces are the same.

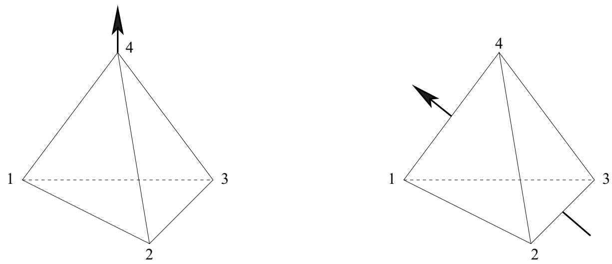


FIGURE 6. Two kinds of rotations of a regular tetrahedron.

$g \in G$	#	$\lambda_1(g)$	$\lambda_2(g)$	$\lambda_3(g)$	$\lambda_4(g)$	Z_g
(1)(2)(3)(4)	1	4	0	0	0	t_1^4
(312)(4)	8	2	1	0	0	t_1t_3
(32)(41)	3	0	0	2	0	t_2^2

$$Z_G(t_1,t_2,t_3,t_4) = \frac{1}{12}\left(t_1^4 + 8t_1t_3 + 3t_2^2\right).$$

EXAMPLE 10. Icosahedron:

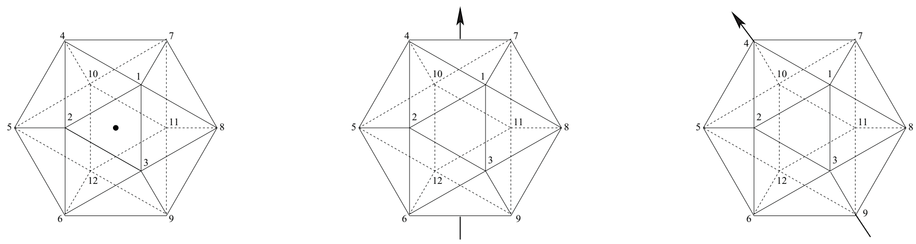


FIGURE 7. Three kinds of rotations of a regular icosahedron.

$g \in G$	#	$\lambda_1(g)$	$\lambda_2(g)$	$\lambda_3(g)$	$\lambda_5(g)$	Z_g
$(1)(2)\cdots(12)$	1	12	0	0	0	t_1^{12}
$(4)(10,5,2,1,7)(12,6,3,8,11)(9)$	24	2	0	0	2	$t_1^2t_5^2$
$(7,4)(8,5)(9,6)(10,1)(11,2)(12,3)$	15	0	6	0	0	t_2^6
$(3,1,2)(8,4,6)(9,6,5)(12,11,10)$	20	0	0	4	0	t_3^4

$$Z_G(t_1,\dots,t_{12}) = \frac{1}{60}\left(t_1^{12} + 24t_1^2t_5^2 + 15t_2^6 + 20t_3^4\right).$$

Dodecahedron

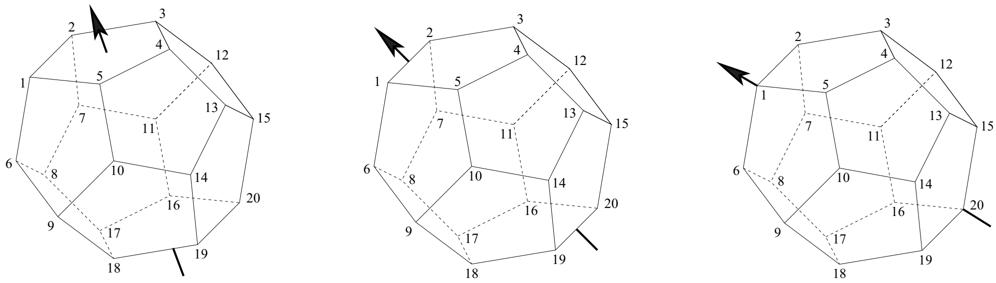


FIGURE 8. Three kinds of rotations of regular dodecahedron.

$g \in G$	#	$\lambda_1(g)$	$\lambda_2(g)$	$\lambda_3(g)$	$\lambda_5(g)$	Z_g
$(1)(2)(3)\cdots(19)(20)$	1	20	0	0	0	t_1^{20}
$(5,4,3,2,1)(20,16,17,18,19)$ $(13,12,7,6,10)(15,11,8,9,14)$	24	0	0	0	4	t_5^4
$(2,1)(6,3)(7,5)(8,4)(11,10)$ $(12,9)(16,14)(17,13)(18,15)(20,19)$	15	0	10	0	0	t_2^{10}
$(1)(6,5,2)(9,4,7)(10,3,8)$ $(17,14,12)(18,13,11)(19,15,16)(20)$	20	2	0	6	0	$t_1^2 t_3^6$

The cycle index

$$Z_G(t_1,\ldots,t_{20}) = \frac{1}{60}\left(t_1^{20} + 20t_1^2t_3^6 + 15t_2^{10} + 24t_5^4\right).$$

EXERCISE 4. Use the Birnside Lemma to prove that the number of round permutations of n objects of type $(n_1,\ldots,n_k)$ is

$$\frac{1}{n}\sum_{d|m}\binom{n/d}{n_1/d,\ldots,n_k/d}\phi(d),$$

where $m = \gcd(n_1,\ldots,n_k)$. (Hint: Consider the cyclic group $G = \{1,\sigma,\sigma^2,\ldots,\sigma^{n-1}\}$ of order n acting on the set $X = \{1,2,\ldots,n\}$, where $\sigma : X \to X$ is defined by $\sigma(j) = j+1$ and $n+1 = 1$.)

PROOF. Let M be a multiset of type $(n; n_1,\ldots,n_k)$. The group G acts naturally on the set $S(M)$ of all permutations of M: for $g \in G$ and $x_1x_2\cdots x_n \in S(M)$,

$$g \cdot x_1x_2\cdots x_n = x_{g(1)}x_{g(2)}\cdots x_{g(n)}.$$

By the Birnside Lemma, the number of round permutations of the type is given by

$$\frac{1}{n}\sum_{j=0}^{n-1}\left|\mathrm{Fix}(\sigma^j)\right|.$$

The problem is to figure out $|\mathrm{Fix}(\sigma^j)|$ for all $0 \le i \le n-1$. Note that for σ^j and $w = x_1x_2\cdots x_n \in S(M)$,

$$\sigma^j \cdot x_1x_2\cdots x_n = x_{j+1}x_{j+2}\cdots x_nx_1x_2\cdots x_j.$$

Let l be a period (not necessarily minimum) of w, that is,

$$w = \underbrace{\underbrace{x_1x_2\cdots x_l}\cdots\underbrace{x_1x_2\cdots x_l}}_{d},$$

then $ld = n$. Let l_i be the number of elements of type i in $x_1x_2\cdots x_l$. Then $l_id = n_i$, that is, $d \mid n_i$. Thus $d \mid m$, where $m = \gcd(n_1,\ldots,n_k)$. Let $S(M,l)$ be the set of all permutations of M with period l. Then

$$|S(M,l)| = \binom{l}{l_1,\ldots,l_k} = \binom{n/d}{n_1/d,\ldots,n_k/d}.$$

First Method: It is clear that if $l \mid j$, then $\sigma^j \cdot w = w$. For a fixed j, we claim that each $w \in \mathrm{Fix}(\sigma^j)$ is counted for some d such that $d \mid m$, $\frac{n}{d} \mid j$, and $\gcd(\frac{di}{n}, d) = 1$. In fact, it is clear that each $w \in \mathrm{Fix}(\sigma^j)$ is counted for some d' such that $d' \mid m$ and $\frac{n}{d'} \mid j$. If $a := \gcd(\frac{d'j}{n}, d') \neq 1$, set $d = \frac{d'}{a}$; obviously, $d \mid m$, $\frac{n}{d} \mid j$, and $\gcd\left(\frac{d'j}{n}, d'\right) = a \gcd\left(\frac{dj}{n}, d\right)$; then $\gcd\left(\frac{dj}{n}, d\right) = 1$. Note that $S\left(M, \frac{n}{d}\right)$ are disjoint for various d such that $d \mid m$, $\frac{n}{d} \mid j$, and $\gcd\left(\frac{dj}{n}, d\right) = 1$. Hence

$$\left|\mathrm{Fix}(\sigma^j)\right| = \sum_{d \mid m,\, \frac{n}{d} \mid j,\, \gcd\left(\frac{dj}{n}, d\right) = 1} \binom{n/d}{n_1/d, \ldots, n_k/d}.$$

It follows that

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n}\left|\mathrm{Fix}(\sigma^j)\right| &= \frac{1}{n} \sum_{\substack{1 \le j \le n,\, d \mid m \\ \frac{n}{d} \mid j, \gcd\left(\frac{dj}{n}, d\right) = 1}} \binom{n/d}{n_1/d, \ldots, n_k/d} \\
&= \frac{1}{n}\sum_{d \mid m} \binom{n/d}{n_1/d, \ldots, n_k/d} \sum_{\substack{1 \le j \le n,\, \frac{n}{d} \mid j \\ \gcd\left(\frac{dj}{n}, d\right) = 1}} 1 \\
&= \frac{1}{n}\sum_{d \mid m} \binom{n/d}{n_1/d, \ldots, n_k/d} \sum_{\substack{1 \le j \le d \\ \gcd(j,d) = 1}} 1 \\
&= \frac{1}{n}\sum_{d \mid m} \binom{n/d}{n_1/d, \ldots, n_k/d} \phi(d).
\end{aligned}$$

Note: One can show that for each fixed j, there exists at most one d such that $d \mid m$, $\frac{n}{d} \mid j$, and $\gcd(\frac{di}{n}, d) = 1$. In fact, for two such d' and d'', let $d = \gcd(d', d'')$, $d' = da'$, and $d'' = da''$. Then $d \mid m$ and $\gcd(a', a'') = 1$. Since $j = (\frac{n}{d'})b' = (\frac{n}{d''})b''$ for some integers b' and b'', we have $(\frac{n}{d})(\frac{b'}{a'}) = (\frac{n}{d})(\frac{b''}{a''})$, which implies that $a'b'' = a''b'$. Since $\gcd(a', a'') = 1$, we have $a' \mid b'$ and $a'' \mid b''$. This means that $\frac{n}{d} \mid j$. Note that $1 = \gcd(\frac{d'j}{n}, d') = a' \gcd(\frac{dj}{n}, d)$, which forces $a' = 1$. Similarly, $a'' = 1$. We thus conclude $d' = d''$.

Second Method: For each j, $1 \le j \le n$, if there is one $w \in \mathrm{Fix}(\sigma^j)$ with period l, then $l \mid j$ and $l \mid n$; so $l \mid \gcd(j, n)$. Since $\frac{n}{l} \mid m$, we have $\frac{n}{\gcd(j,n)} \mid m$. Thus $\mathrm{Fix}(\sigma^j) \neq \emptyset$ if and only if $\frac{n}{\gcd(j,n)} \mid m$. For each positive integer c, note that

$$\#\{j \mid 1 \le j \le n,\, \gcd(j, n) = c\} = \#\left\{j \;\middle|\; 1 \le j \le \frac{n}{c},\, \gcd\left(j, \frac{n}{c}\right) = 1\right\} = \phi\left(\frac{n}{c}\right).$$

We claim that if $\gcd(j, n) = c$ and $\frac{n}{c} \mid m$, then $\mathrm{Fix}(\sigma^j) = S(M, c)$. In fact, it is obvious that $S(M, c) \subset \mathrm{Fix}(\sigma^j)$ because $c \mid j$. For any $w \in \mathrm{Fix}(\sigma^j)$, by the Euclidean Algorithm, there are integers a and b such that $c = aj + bn$, then

$$\sigma^c \cdot w = \sigma^{aj}\sigma^{bn} \cdot w = \sigma^{aj} \cdot w = w;$$

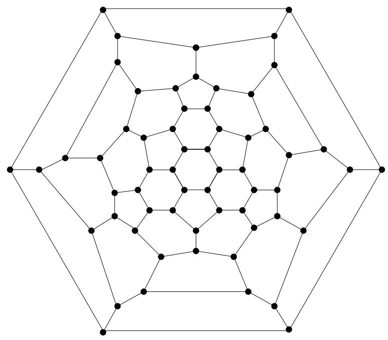
that is, w is of period c; thus $w \in S(M, c)$. Hence $\mathrm{Fix}(\sigma^j) = S(M, c)$.

Now we have

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n}\left|\mathrm{Fix}(\sigma^j)\right| &= \frac{1}{n}\sum_{j=1}^{n} \sum_{\substack{\frac{n}{c} \mid m \\ \gcd(j,n) = c}} \binom{c}{n_1/\frac{n}{c}, \ldots, n_k/\frac{n}{c}} \\
&= \frac{1}{n}\sum_{\frac{n}{c} \mid m} \binom{c}{n_1/\frac{n}{c}, \ldots, n_k/\frac{n}{c}} \sum_{\substack{1 \le j \le n, \\ \gcd(j,n) = c}} 1 \\
&= \frac{1}{n}\sum_{\frac{n}{c} \mid m} \binom{c}{n_1/\frac{n}{c}, \ldots, n_k/\frac{n}{c}} \phi\left(\frac{n}{c}\right) \\
&= \frac{1}{n}\sum_{d \mid m} \binom{n/d}{n_1/d, \ldots, n_k/d} \phi(d).
\end{aligned}$$

$\square$

Exercise 5. Find the number of ways to color the faces of a soccer ball with two colors.



Solution. The soccer has 60 vertices, 90 edges, 12 pentagons, and 20 hexagons. It has three kinds of symmetries: rotations of order 5 around the normal vector at the center of each pentagon, rotations of order 3 around the normal vector at the center of each hexagon, and rotations of order 2 at the center of each edge between two hexagons.

The the cycle index for the group of vertex automorphisms is given as follows:

$g \in G$	#	$\lambda_1(g)$	$\lambda_2(g)$	$\lambda_3(g)$	$\lambda_5(g)$	Z_g
identity	1	60	0	0	0	t_1^{60}
pentagon rotations	24	0	0	0	12	t_5^{12}
hexagon rotations	20	0	0	20	0	t_3^{20}
edge rotations	15	0	30	0	0	t_2^{30}

$$Z_G(t_1, \ldots, t_{60}) = \frac{1}{60}\left(t_1^{60} + 15t_2^{30} + 20t_3^{20} + 24t_5^{12}\right).$$

Exercise 6. Show that for the symmetric group S_n acting on the set of n elements, the cycle index is

$$Z_{S_n}(t_1, \ldots, t_n) = \sum \frac{1}{k_1!k_2!\cdots k_n!}\left(\frac{t_1}{1}\right)^{k_1}\left(\frac{t_2}{2}\right)^{k_2}\cdots\left(\frac{t_n}{n}\right)^{k_n},$$

where the sum is extended over all non-negative integer sequences $(k_1, k_2, \ldots, k_n)$ such that $k_1 + 2k_2 + \cdots + nk_n = n$.

Proof. By definition of cycle index,

$$Z_{S_n}(t_1, \ldots, t_n) = \frac{1}{|S_n|}\sum_{g \in S_n} t_1^{\lambda_1(g)} \cdots t_n^{\lambda_n(g)}.$$

Note that the number of permutations of an n-set of type $1^{k_1}2^{k_2}\cdots n^{k_n}$ (having k_1 cycles of length 1, k_2 cycles of length 2, ..., k_n cycles of length n) is

$$\frac{n!}{1^{k_1}2^{k_2}\cdots n^{k_n}(k_1!)(k_2!)\cdots(k_n!)},$$

where $k_1 + 2k_2 + \cdots + nk_n = n$. Thus

$$\begin{aligned} Z_{S_n}(t_1, \ldots, t_n) &= \frac{1}{n!}\sum \frac{n!}{1^{k_1}2^{k_2}\cdots n^{k_n}(k_1!)(k_2!)\cdots(k_n!)} t_1^{k_1}t_2^{k_2}\cdots t_n^{k_n} \\ &= \sum \frac{t_1^{k_1}t_2^{k_2}\cdots t_n^{k_n}}{1^{k_1}2^{k_2}\cdots n^{k_n}(k_1!)(k_2!)\cdots(k_n!)} \\ &= \sum \frac{1}{k_1!\cdots k_n!}\left(\frac{t_1}{1}\right)^{k_1}\left(\frac{t_2}{2}\right)^{k_2}\cdots\left(\frac{t_n}{n}\right)^{k_n}, \end{aligned}$$

where the sum is extended to nonnegative integer tuples $(k_1, k_2, \ldots, k_n)$ such that $k_1+2k_2+\cdots+nk_n = n$. □