

Week 9-10: Recurrence Relations and Generating Functions

April 15, 2019

1 Some number sequences

An **infinite sequence** (or just a **sequence** for short) is an ordered array

$$a_0, a_1, a_2, \dots, a_n, \dots$$

of countably many real or complex numbers, and is usually abbreviated as $(a_n; n \geq 0)$ or just (a_n) . A sequence (a_n) can be viewed as a function f from the set of nonnegative integers to the set of real or complex numbers, i.e.,

$$f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}, \quad f(n) = a_n, \quad n = 0, 1, 2, \dots$$

We call a sequence (a_n) an **arithmetic sequence** if it is of the form

$$a_0, \quad a_0 + q, \quad a_0 + 2q, \quad \dots, \quad a_0 + nq, \quad \dots$$

The general term of such sequences satisfies the recurrence relation

$$a_n = a_{n-1} + q, \quad n \geq 1.$$

A sequence (a_n) is called a **geometric sequence** if it is of the form

$$a_0, \quad a_0q, \quad a_0q^2, \quad \dots, \quad a_0q^n, \quad \dots$$

The general term of such sequences satisfies the recurrence relation

$$a_n = qa_{n-1}, \quad n \geq 1.$$

The **partial sums** of a sequence (a_n) are the summations:

$$\begin{aligned} s_0 &= a_0, \\ s_1 &= a_0 + a_1, \\ s_2 &= a_0 + a_1 + a_2, \\ &\vdots \\ s_n &= a_0 + a_1 + \cdots + a_n, \\ &\vdots \end{aligned}$$

The partial sums form a new sequence $(s_n; n \geq 0)$.

For an arithmetic sequence $a_n = a_0 + nq$ ($n \geq 0$), we have the partial sum

$$s_n = \sum_{k=0}^n (a_0 + kq) = (n+1)a_0 + \frac{qn(n+1)}{2}.$$

For a geometric sequence $a_n = a_0q^n$ ($n \geq 1$), we have

$$s_n = \sum_{k=0}^n a_0q^k = \begin{cases} \frac{q^{n+1}-1}{q-1}a_0 & \text{if } q \neq 1 \\ (n+1)a_0 & \text{if } q = 1. \end{cases}$$

Example 1.1. Determine the sequence a_n , defined as the number of regions which are created by n mutually overlapping circles in general position on the plane. (By **mutually overlapping** we mean that each pair of two circles intersect in two distinct points. Thus non-intersecting or tangent circles are not allowed. By **general position** we mean that there are no three circles through a common point.)

We easily see that the first few numbers are given as

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 4, \quad a_3 = 8.$$

It seems that we might have $a_4 = 16$. However, by try-and-error we quickly see that $a_4 = 14$.

Assume that there are n circles in general position on a plane. When we take one circle away, say the n th circle, there are $n - 1$ circles in general position on the same plane. By induction hypothesis the $n - 1$ circles divide the plane

into a_{n-1} regions. Note that the n th circle intersects each of the $n - 1$ circles in $2(n - 1)$ distinct points. Let the $2(n - 1)$ points on the n th circle be ordered clockwise as $P_1, P_2, \dots, P_{2(n-1)}$. Then each of the $2(n - 1)$ arcs

$$P_1P_2, \quad P_2P_3, \quad P_3P_4, \quad \dots, \quad P_{2(n-2)+1}P_{2(n-1)}, \quad P_{2(n-1)}P_1$$

intersects one and only one region in the case $n - 1$ circles and separate the region into two regions. There are $2(n - 1)$ more regions produced when the n th circle is drawn. We thus obtain the recurrence relation

$$a_n = a_{n-1} + 2(n - 1), \quad n \geq 2.$$

Repeating the recurrence relation we have

$$\begin{aligned} a_n &= a_{n-1} + 2(n - 1) \\ &= a_{n-2} + 2(n - 1) + 2(n - 2) \\ &= a_{n-3} + 2(n - 1) + 2(n - 2) + 2(n - 3) \\ &\quad \vdots \\ &= a_1 + 2(n - 1) + 2(n - 2) + 2(n - 3) + \dots + 2 \\ &= a_1 + 2 \cdot \frac{(n - 1)n}{2} \\ &= 2 + n(n - 1) \\ &= n^2 - n + 2, \quad n \geq 2. \end{aligned}$$

This formula is also valid for $n = 1$ (since $a_1 = 2$), although it doesn't hold for $n = 0$ (since $a_0 = 1$).

Example 1.2 (Fibonacci Sequence). A pair of newly born rabbits of opposite sexes is placed in an enclosure at the beginning of a year. Baby rabbits need one month to grow mature; they become an adult pair on the first day of the second month. Beginning with the second month the female is pregnant, and gives exactly one birth of one pair of rabbits of opposite sexes on the first day of the third month, and gives exactly one such birth on the first day of each next month. Each new pair also gives such birth to a pair of rabbits on the first day of each month starting from the third month (from its birth). Find the number of pairs of rabbits in the enclosure after one year?

Let f_n denote the number of pairs of rabbits on the first day of the n th month. Some of these pairs are adult and some are babies. Let a_n and b_n denote the number of pairs of adult and baby rabbits respectively on the first day of the n th month. Then the total number of pairs of rabbits on the first day of the n th month is $f_n = a_n + b_n$, $n \geq 1$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
a_n	0	1	1	2	3	5	8	13	21	34	55	89	144
b_n	1	0	1	1	2	3	5	8	13	21	34	55	89
f_n	1	1	2	3	5	8	13	21	34	55	89	144	233

If a pair is adult on the first day of the n th month, then it gives one birth of one pair on the first day of the next month. So $b_{n+1} = a_n$, $n \geq 1$. Note that each adult pair on the first day of the n th month is still an adult pair on first day of the next month and each baby pair on the first day of the n th month becomes an adult pair on the first day of the next month, we have $a_{n+1} = a_n + b_n = f_n$, $n \geq 1$. Thus

$$f_n = a_n + b_n = f_{n-1} + a_{n-1} = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

Let us define $f_0 = 0$. The sequence $f_0, f_1, f_2, f_3, \dots$ satisfying the recurrence relation

$$\begin{cases} f_n = f_{n-1} + f_{n-2}, & n \geq 2 \\ f_0 = 0 \\ f_1 = 1 \end{cases} \quad (1)$$

is known as the **Fibonacci sequence**, and its terms are known as **Fibonacci numbers**.

Example 1.3. The partial sum of Fibonacci sequence is

$$s_n = f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1. \quad (2)$$

This can be verified by induction on n . For $n = 0$, we have $s_0 = f_2 - 1 = 0$. Now for $n \geq 1$, we assume that it is true for $n - 1$, i.e., $s_{n-1} = f_{n+1} - 1$. Then

$$\begin{aligned} s_n &= f_0 + f_1 + \cdots + f_n \\ &= s_{n-1} + f_n \\ &= f_{n+1} - 1 + f_n \quad (\text{by the induction hypothesis}) \\ &= f_{n+2} - 1. \quad (\text{by the Fibonacci recurrence}) \end{aligned}$$

Example 1.4. The Fibonacci number f_n is even if and only if n is a multiple of 3.

Note that $f_1 = f_2 = 1$ is odd and $f_3 = 2$ is even. Assume that f_{3k} is even, f_{3k-2} and f_{3k-1} are odd. Then $f_{3k+1} = f_{3k} + f_{3k-1}$ is odd (*even + odd = odd*), and subsequently, $f_{3k+2} = f_{3k+1} + f_{3k}$ is also odd (*odd + even = odd*). It follows that $f_{3(k+1)} = f_{3k+2} + f_{3k+1}$ is even (*odd + odd = even*).

Theorem 1.1. The general term of the Fibonacci sequence (f_n) is given by

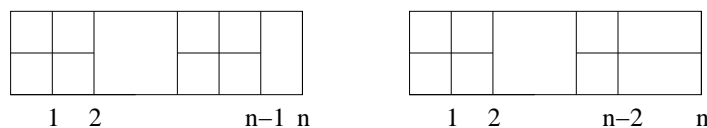
$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n, \quad n \geq 0. \quad (3)$$

Example 1.5. Determine the number h_n of ways to perfectly cover a 2-by- n board with dominoes. (Symmetries are not counted in counting the number of coverings.)

We assume $h_0 = 1$ since a 2-by-0 board is empty and it has exactly one perfect cover, namely, the empty cover. Note that the first few terms can be easily obtained such as

$$h_0 = 1, \quad h_1 = 1, \quad h_2 = 2, \quad h_3 = 3, \quad h_4 = 5.$$

Now for $n \geq 3$, the 2-by- n board can be covered by dominoes in two types:



There are h_{n-1} ways in the first type and h_{n-2} ways in the second type. Thus

$$h_n = h_{n-1} + h_{n-2}, \quad n \geq 2.$$

Therefore the sequence $(h_n; n \geq 0)$ is the Fibonacci sequence $(f_n; n \geq 0)$ with $f_0 = 0$ deleted, i.e.,

$$h_n = f_{n+1}, \quad n \geq 0.$$

Example 1.6. Determine the number b_n of ways to perfectly cover a 1-by- n board by dominoes and monominoes.

$$b_n = b_{n-1} + b_{n-2}, \quad b_0 = b_1 = 1, \quad b_2 = 2.$$

Theorem 1.2. *The Fibonacci number f_n can be written as*

$$f_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}, \quad n \geq 0.$$

Proof. Let $g_0 = 0$ and

$$g_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}, \quad n \geq 1.$$

Note that $k > \lfloor \frac{n-1}{2} \rfloor$ is equivalent to $k > n - k - 1$. Since $\binom{m}{p} = 0$ for any integers m and p such that $p > m$, we may write g_n as

$$g_n = \sum_{k=0}^{n-1} \binom{n-k-1}{k}, \quad n \geq 1.$$

To prove the theorem, it suffices to show that the sequence (g_n) satisfies the Fibonacci recurrence relation with the same initial values. In fact, $g_0 = 0$,

$g_1 = \binom{0}{0} = 1$, and for $n \geq 0$,

$$\begin{aligned}
g_{n+1} + g_n &= \sum_{k=0}^n \binom{n-k}{k} + \sum_{k=0}^{n-1} \binom{n-k-1}{k} \\
&= \binom{n}{0} + \sum_{k=1}^n \binom{n-k}{k} + \sum_{k=1}^n \binom{n-k}{k-1} \\
&= \binom{n}{0} + \sum_{k=1}^n \left[\binom{n-k}{k} + \binom{n-k}{k-1} \right] \\
&= \binom{n}{0} + \sum_{k=1}^n \binom{n-k+1}{k} \quad (\text{By the Pascal formula}) \\
&= \binom{n+1}{0} + \sum_{k=1}^n \binom{n-k+1}{k} + \binom{0}{n+1} \\
&= \sum_{k=0}^{n+1} \binom{(n+2)-k-1}{k} = g_{n+2}.
\end{aligned}$$

We conclude that the sequence (g_n) is the Fibonacci sequence (f_n) . □

2 Linear recurrence relations

Definition 2.1. A sequence $(x_n; n \geq 0)$ of numbers is said to satisfy a **linear recurrence relation of order k** if

$$x_n = \alpha_1(n)x_{n-1} + \alpha_2(n)x_{n-2} + \cdots + \alpha_k(n)x_{n-k} + \beta_n, \quad (4)$$

where $\alpha_k(n) \neq 0$, $n \geq k$, and the coefficients $\alpha_1(n)$, $\alpha_2(n)$, \dots , $\alpha_k(n)$ and β_n are functions of n . The linear recurrence relation (4) is said to be **homogeneous** if $\beta_n = 0$ for all $n \geq k$, and is said to have **constant coefficients** if $\alpha_1(n)$, $\alpha_2(n)$, \dots , $\alpha_k(n)$ are constants. The recurrence relation

$$x_n = \alpha_1(n)x_{n-1} + \alpha_2(n)x_{n-2} + \cdots + \alpha_k(n)x_{n-k}, \quad (5)$$

where $\alpha_k(n) \neq 0$, $n \geq k$, is called the corresponding **homogeneous linear recurrence relation** of (4).

A **solution** of the recurrence relation (4) is a sequence (u_n) satisfying (4). The **general solution** of (4) is a solution

$$x_n = u_n(c_1, c_2, \dots, c_k) \quad (6)$$

with some parameters c_1, c_2, \dots, c_k , provided that for arbitrary initial values u_0, u_1, \dots, u_{k-1} there exist constants c_1, c_2, \dots, c_k such that (6) is the unique sequence satisfying both the recurrence relation (4) and the initial conditions.

Let \mathcal{S}_∞ denote the set of all sequences $(a_n; n \geq 0)$. Then \mathcal{S}_∞ is an infinite-dimensional vector space under the ordinary addition and scalar multiplication of sequences. Let \mathcal{N}_k be the set all solutions of the nonhomogeneous linear recurrence relation (4), and \mathcal{H}_k the set of all solutions of the homogeneous linear recurrence relation (5). We shall see that \mathcal{H}_k is a k -dimensional subspace of \mathcal{S}_∞ , and \mathcal{N}_k a k -dimensional affine subspace of \mathcal{S}_∞ .

Theorem 2.2. (Structure Theorem for Linear Recurrence Relations)

(a) *The solution space \mathcal{H}_k is a k -dimensional subspace of the vector space \mathcal{S}_∞ of sequences. Thus, if $(a_{n,1}), (a_{n,2}), \dots, (a_{n,k})$ are linearly independent solutions of the homogeneous linear recurrence relation (5), then the general solution of (5) is*

$$x_n = c_1 a_{n,1} + c_2 a_{n,2} + \dots + c_k a_{n,k}, \quad n \geq 0,$$

where c_1, c_2, \dots, c_k are arbitrary constants.

(b) *Let (a_n) be a particular solution of the nonhomogeneous linear recurrence relation (4). Then the general solution of (4) is*

$$x_n = a_n + h_n, \quad n \geq 0,$$

where (h_n) is the general solution of the corresponding homogeneous linear recurrence relation (5). In other words, \mathcal{N}_k is a translate of \mathcal{H}_k in \mathcal{S}_∞ , i.e.,

$$\mathcal{N}_k = (a_n) + \mathcal{H}_k.$$

Proof. (a) To show that \mathcal{H}_k is a vector subspace of \mathcal{S}_∞ , we need to show that \mathcal{H}_k is closed under the addition and scalar multiplication of sequences. Let (a_n)

and (b_n) be solutions of (5). Then

$$\begin{aligned} a_n + b_n &= [\alpha_1(n)a_{n-1} + \alpha_2(n)a_{n-2} + \cdots + \alpha_k(n)a_{n-k}] \\ &\quad + [\alpha_1(n)b_{n-1} + \alpha_2(n)b_{n-2} + \cdots + \alpha_k(n)b_{n-k}] \\ &= \alpha_1(n)(a_{n-1} + b_{n-1}) + \alpha_2(n)(a_{n-2} + b_{n-2}) + \\ &\quad \cdots + \alpha_k(n)(a_{n-k} + b_{n-k}), \quad n \geq k. \end{aligned}$$

For scalars c ,

$$\begin{aligned} ca_n &= c[\alpha_1(n)a_{n-1} + \alpha_2(n)a_{n-2} + \cdots + \alpha_k(n)a_{n-k}] \\ &= \alpha_1(n)ca_{n-1} + \alpha_2(n)ca_{n-2} + \cdots + \alpha_k(n)ca_{n-k}, \quad n \geq k. \end{aligned}$$

This means that \mathcal{H}_k is closed under the addition and scalar multiplication of sequences.

To show that \mathcal{H}_k is k -dimensional, consider the projection $\pi : \mathcal{S}_\infty \rightarrow \mathbb{R}^k$, defined by

$$\pi(x_0, x_1, x_2, \dots) = (x_0, x_1, \dots, x_{k-1}).$$

We shall see that the restriction $\pi|_{\mathcal{H}_k} : \mathcal{H}_k \rightarrow \mathbb{R}^k$ is a linear isomorphism. For each $(a_0, a_1, \dots, a_{k-1}) \in \mathbb{R}^k$, define a_n inductively by

$$a_n = \alpha_1(n)a_{n-1} + \alpha_2(n)a_{n-2} + \cdots + \alpha_k(n)a_{n-k}, \quad n \geq k.$$

Obviously, we have $\pi(a_0, a_1, a_2, \dots) = (a_0, a_1, \dots, a_{k-1})$. This means that $\pi|_{\mathcal{H}_k}$ is surjective. Now let $(u_n) \in \mathcal{H}_k$ be such that $\pi(u_0, u_1, u_2, \dots) = (0, 0, \dots, 0)$, i.e.,

$$u_0 = u_1 = \cdots = u_{k-1} = 0.$$

Applying the recurrence relation (5) for $n = k$, we have $u_k = 0$. Again applying (5) for $n = k + 1$, we obtain $u_{k+1} = 0$. Continuing to apply (5), we have $u_n = 0$ for $n \geq k$. Thus (u_n) is the zero sequence. This means that π is injective. We have finished the proof that $\pi|_{\mathcal{H}_k}$ is a linear isomorphism from \mathcal{H}_k onto \mathbb{R}^k . So $\dim \mathcal{H}_k = k$.

(b) For each solution (v_n) of (4), we claim that the sequence $h_n := v_n - u_n$ ($n \geq 0$) is a solution of (5). So

$$v_n = u_n + h_n, \quad n \geq 0.$$

In fact, since (u_n) and (v_n) are solutions of (4), applying the recurrence relation (4), we have

$$\begin{aligned} h_n &= [\alpha_1(n)v_{n-1} + \alpha_2(n)v_{n-2} + \cdots + \alpha_k(n)v_{n-k} + \beta_n] \\ &\quad - [\alpha_1(n)u_{n-1} + \alpha_2(n)u_{n-2} + \cdots + \alpha_k(n)u_{n-k} + \beta_n] \\ &= \alpha_1(n)(v_{n-1} - u_{n-1}) + \alpha_2(n)(v_{n-2} - u_{n-2}) + \cdots + \alpha_k(n)(v_{n-k} - u_{n-k}) \\ &= \alpha_1(n)h_{n-1} + \alpha_2(n)h_{n-2} + \cdots + \alpha_k(n)h_{n-k}, \quad n \geq k. \end{aligned}$$

This means that (h_n) is a solution of (5). Conversely, for any solution (h_n) of (5), we have

$$\begin{aligned} u_n + h_n &= [\alpha_1(n)u_{n-1} + \alpha_2(n)u_{n-2} + \cdots + \alpha_k(n)u_{n-k} + \beta_n] \\ &\quad + [\alpha_1(n)h_{n-1} + \alpha_2(n)h_{n-2} + \cdots + \alpha_k(n)h_{n-k}] \\ &= \alpha_1(n)(u_{n-1} + h_{n-1}) + \alpha_2(n)(u_{n-2} + h_{n-2}) \\ &\quad + \cdots + \alpha_k(n)(u_{n-k} + h_{n-k}) + \beta_n \end{aligned}$$

for $n \geq k$. This means that the sequence $(u_n + h_n)$ is a solution of (4). □

Definition 2.3. The **Wronskian** $W_k(n)$ of k solutions $(u_{n,1}), (u_{n,2}), \dots, (u_{n,k})$ of the homogeneous linear recurrence relation (5) is the determinant

$$W_k(n) = \det \begin{bmatrix} u_{n,1} & u_{n,2} & \cdots & u_{n,k} \\ u_{n+1,1} & u_{n+1,2} & \cdots & u_{n+1,k} \\ \vdots & \vdots & & \vdots \\ u_{n+k-1,1} & u_{n+k-1,2} & \cdots & u_{n+k-1,k} \end{bmatrix}, \quad n \geq 0.$$

The Wronskian of $(u_{n,1}), (u_{n,2}), \dots, (u_{n,k})$ is also an infinite sequence

$$(W_k(n); n \geq 0).$$

Proposition 2.4. *If $W_k(m) = 0$, then $W_k(n) = 0$ for all $n \geq m$.*

Proof. Since $W_k(m) = 0$, there are constants c_1, c_2, \dots, c_k , not all zero, such that for $i = 0, 1, \dots, k-1$,

$$\sum_{j=1}^k c_j u_{m+i,j} = c_1 u_{m+i,1} + c_2 u_{m+i,2} + \cdots + c_k u_{m+i,k} = 0.$$

We claim that $W_k(m+1) = 0$. It is enough to show that for the same coefficients c_1, c_2, \dots, c_k above, the linear relation

$$\sum_{j=1}^k c_j \mathbf{u}_{m+k,j} = c_1 \mathbf{u}_{m+k,1} + c_2 \mathbf{u}_{m+k,2} + \dots + c_k \mathbf{u}_{m+k,k} = 0$$

holds. In fact, applying the recurrence relation (5) to $(\mathbf{u}_{m+k,1}), (\mathbf{u}_{m+k,2}), \dots, (\mathbf{u}_{m+k,k})$ respectively, we have

$$\begin{aligned} \sum_{j=1}^k c_j \mathbf{u}_{m+k,j} &= \sum_{j=1}^k c_j \sum_{i=0}^{k-1} \alpha_i(m+k) \mathbf{u}_{m+i,j} \\ &= \sum_{i=0}^{k-1} \alpha_i(m+k) \sum_{j=1}^k c_j \mathbf{u}_{m+i,j} = 0. \end{aligned}$$

This means that the vectors $(\mathbf{u}_{m+i,1}), (\mathbf{u}_{m+i,2}), \dots, (\mathbf{u}_{m+i,k})$, with $1 \leq i \leq k$, are linearly dependent. So $W_k(m+1) = 0$. Continue the procedure, we see that $W_k(n) = 0$ for $n \geq m$. \square

Theorem 2.5. *The solutions $(u_{n,1}), (u_{n,2}), \dots, (u_{n,k})$ of the homogeneous linear recurrence relation (5) are linearly independent if and only if there is a nonnegative integer n_0 such that the Wronskian*

$$W_k(n_0) \neq 0.$$

In other words, the solutions $(u_{n,1}), (u_{n,2}), \dots, (u_{n,k})$ are linearly dependent if and only if $W_k(n) = 0$ for all $n \geq 0$.

Proof. We show that the sequences $(u_{n,1}), (u_{n,2}), \dots, (u_{n,k})$ are linearly dependent if and only if $W_k(n) = 0$ for all $n \geq 0$. If $(u_{n,1}), (u_{n,2}), \dots, (u_{n,k})$ are linearly dependent, then for $n \geq 0$ the columns of the matrix

$$\begin{bmatrix} u_{n,1} & u_{n,2} & \cdots & u_{n,k} \\ u_{n+1,1} & u_{n+1,2} & \cdots & u_{n+1,k} \\ \vdots & \vdots & & \vdots \\ u_{n+k-1,1} & u_{n+k-1,2} & \cdots & u_{n+k-1,k} \end{bmatrix}$$

are linearly dependent because the columns are part of the sequences $(u_{n,1})$, $(u_{n,2})$, \dots , $(u_{n,k})$ respectively. It follows from linear algebra that the determinant of the matrix is zero, i.e., the Wronskian $W_k(n) = 0$ for all $n \geq 0$.

Conversely, if $W_k(n) = 0$ for all $n \geq 0$, in particular, $W_k(0) = 0$, then there are constants c_1, c_2, \dots, c_k , not all zero, such that

$$\sum_{j=1}^k c_j u_{i,j} = c_1 u_{i,1} + c_2 u_{i,2} + \dots + c_k u_{i,k} = 0, \quad i = 0, 1, \dots, k-1.$$

We claim that $\sum_{j=1}^k c_j u_{n,j} = 0$ for all n by induction. Assume it is true for $n \leq m-1$. Applying the recurrence relation (5) to the sequences $(u_{m,1})$, $(u_{m,2})$, \dots , $(u_{m,k})$, we have

$$\begin{aligned} \sum_{j=1}^k c_j u_{m,j} &= \sum_{j=1}^k c_j \sum_{i=1}^k \alpha_i(k) u_{m-i,j} \\ &= \sum_{i=1}^k \alpha_i(k) \sum_{j=1}^k c_j u_{m-i,j} = 0. \end{aligned}$$

This means that the sequences $(u_{n,1})$, $(u_{n,2})$, \dots , $(u_{n,k})$ are linearly dependent. \square

3 Homogeneous linear recurrence relations with constant coefficients

In this section we only consider linear recurrence relations of the form

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + \dots + \alpha_k x_{n-k}, \quad \alpha_k \neq 0, \quad n \geq k, \quad (7)$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants. We call this kinds of recurrence relations as **homogeneous linear recurrence relations of order k with constant coefficients**. Sometimes it is convenient to write (7) as of the form

$$\alpha_0 x_n + \alpha_1 x_{n-1} + \dots + \alpha_k x_{n-k} = 0, \quad n \geq k \quad (8)$$

where $\alpha_0 \neq 0$ and $\alpha_k \neq 0$. The following polynomial equation

$$\alpha_0 t^k + \alpha_1 t^{k-1} + \dots + \alpha_{k-1} t + \alpha_k = 0, \quad (9)$$

is called the **characteristic equation** associated with the recurrence relation (8), and the polynomial on the left side of (9) is called the **characteristic polynomial**.

Example 3.1. The Fibonacci sequence $(f_n; n \geq 0)$ satisfies the linear recurrence relation

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 2$$

of order 2 with $\alpha_1 = \alpha_2 = 1$ in (7).

Example 3.2. The geometric sequence $(x_n; n \geq 0)$, where $x_n = q^n$, satisfies the linear recurrence relation

$$x_n = qx_{n-1}, \quad n \geq 1$$

of order 1 with $\alpha_1 = q$ in (7).

It is quite heuristic that solutions of the first order homogeneous linear recurrence relations are geometric sequences. This hints that the recurrence relation (7) may have solutions of the form $x_n = q^n$. The following theorem confirms the speculation.

Theorem 3.1. (a) *For any number $q \neq 0$, the geometric sequence*

$$x_n = q^n$$

is a solution of the k th order homogeneous linear recurrence relation (8) with constant coefficients if and only if the number q is a root of the characteristic equation (9).

(b) If the characteristic equation (9) has k distinct roots q_1, q_2, \dots, q_k , then the general solution of (8) is

$$x_n = c_1q_1^n + c_2q_2^n + \dots + c_kq_k^n, \quad n \geq 0. \quad (10)$$

Proof. (a) Put $x_n = q^n$ into the recurrence relation (8). We obtain

$$\alpha_0q^n + \alpha_1q^{n-1} + \dots + \alpha_kq^{n-k} = 0. \quad (11)$$

Since $q \neq 0$, dividing both sides of (11) by q^{n-k} , we have

$$\alpha_0q^k + \alpha_1q^{k-1} + \dots + \alpha_{k-1}q + \alpha_k = 0 \quad (12)$$

This means that (11) and (12) are equivalent. This finishes the proof of Part (a).

(b) Since q_1, q_2, \dots, q_k are roots of the characteristic equation (9), $x_n = q_i^n$ are solutions of the homogeneous linear recurrence relation (8) for all i ($1 \leq i \leq k$). Since the solution space of (8) is a vector space, the linear combination

$$x_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_k q_k^n, \quad n \geq 0$$

are also solutions (8). Now given arbitrary values for x_0, x_1, \dots, x_{k-1} , the sequence (x_n) is uniquely determined by the recurrence relation (8). Set

$$c_1 q_1^i + c_2 q_2^i + \cdots + c_k q_k^i = x_i, \quad 0 \leq i \leq k-1.$$

The coefficients c_1, c_2, \dots, c_k are uniquely determined by Cramer's rule as follows:

$$c_i = \frac{\det A_i}{\det A}, \quad 1 \leq i \leq k$$

where A is the **Vandermonde matrix**

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_k \\ q_1^2 & q_2^2 & \cdots & q_k^2 \\ \vdots & \vdots & & \vdots \\ q_1^{k-1} & q_2^{k-1} & \cdots & q_k^{k-1} \end{bmatrix},$$

and A_i is the matrix obtained from A by replacing its i th column by the column $[x_0, x_1, \dots, x_{k-1}]^T$. The determinant of A is given by

$$\det A = \prod_{1 \leq i < j \leq k} (q_j - q_i) \neq 0.$$

This finishes the proof of Part (b). □

Example 3.3. Find the sequence (x_n) satisfying the recurrence relation

$$x_n = 2x_{n-1} + x_{n-2} - 2x_{n-3}, \quad n \geq 3$$

and the initial conditions $x_0 = 1$, $x_1 = 2$, and $x_2 = 0$.

Proof. The **falling factorial polynomial** of degree i is the polynomial

$$[t]_i := t(t-1)\cdots(t-i+1).$$

Let q be a root of the characteristic polynomial

$$P(t) = \alpha_0 t^k + \alpha_1 t^{k-1} + \cdots + \alpha_{k-1} t + \alpha_k = (t-q)^m Q(t)$$

with multiplicity m . Then q is a root of the i th derivative $P^{(i)}(t)$ of $P(t)$, where $0 \leq i \leq m-1$. We claim that $x_n = [n]_i q^n$ is a solution of the recurrence relation for each $i = 0, 1, \dots, m-1$. In fact,

$$\begin{aligned} \text{LHS of (8)} &= \alpha_0 [n]_i q^n + \alpha_1 [n-1]_i q^{n-1} + \cdots + \alpha_k [n-k]_i q^{n-k} \\ &= \frac{d^i}{dt^i} \Big|_{t=q} q^i (\alpha_0 t^n + \alpha_1 t^{n-1} + \cdots + \alpha_k t^{n-k}) \\ &= q^i \cdot \frac{d^i}{dt^i} \Big|_{t=q} (\alpha_0 t^k + \alpha_1 t^{k-1} + \cdots + \alpha_{k-1} t + \alpha_k) t^{n-k} \\ &= q^i \cdot \frac{d^i}{dt^i} \Big|_{t=q} (t-q)^m R(t), \quad \text{where } R(t) = Q(t)t^{n-k}. \end{aligned}$$

Applying the Leibniz rule,

$$\begin{aligned} \text{LHS of (8)} &= q^i \sum_{j=0}^i \binom{i}{j} \left(\frac{d^j}{dt^j} \Big|_{t=q} (t-q)^m \right) \left(\frac{d^{i-j}}{dt^{i-j}} \Big|_{t=q} R(t) \right) \\ &= q^i \sum_{j=0}^i \binom{i}{j} [m]_j (t-q)^{m-j} \Big|_{t=q} \left(\frac{d^{i-j}}{dt^{i-j}} \Big|_{t=q} R(t) \right) = 0. \end{aligned}$$

So $x_n = [n]_i q^n$ is a solution. Note that

$$[t]_i = a_0 + a_1 t + \cdots + a_i t^i, \quad (i = 0, 1, \dots, m-1)$$

for some integers a_0, a_1, \dots, a_i . It follows that there some integers b_0, b_1, \dots, b_i such that

$$t^i = b_0 [t]_0 + b_1 [t]_1 + \cdots + b_i [t]_i, \quad i = 0, 1, \dots, m-1.$$

We see that

$$x_n = n^i q^n = b_0 [n]_0 q^n + b_1 [n]_1 q^n + \cdots + b_i [n]_i q^n$$

is a solution, where $0 \leq i \leq m-1$. Next we show that $q^n, nq^n, \dots, n^{m-1}q^n$ are linearly independent. Set

$$c_0q^n + c_1nq^n + \dots + c_{m-1}n^{m-1}q^n = 0, \quad \forall n \geq 0.$$

Since $q \neq 0$, we have

$$c_0 + c_1n + \dots + c_{m-1}n^{m-1} = 0 \quad \forall n \geq 0.$$

Consider c_0, c_1, \dots, c_{m-1} as variables and let $n = 1, 2, \dots, m$. We have a system of linear equations about c_0, c_1, \dots, c_{m-1} . The coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 1^2 & \dots & 1^{m-1} \\ 1 & 2 & 2^2 & \dots & 2^{m-1} \\ 1 & 3 & 3^2 & \dots & 3^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & m & m^2 & \dots & m^{m-1} \end{bmatrix}$$

and $\det A = \prod_{1 \leq i < j \leq m} (j - i) \neq 0$. Hence $c_0 = c_1 = \dots = c_{m-1} = 0$. \square

Proof of General Linear Independence of Solutions: Consider the linear combination

$$\sum_{j=1}^s (c_{j,1}q_j^n + c_{j,2}nq_j^n + \dots + c_{j,m_j}n^{m_j-1}q_j^n) = 0 \quad (13)$$

with coefficients $c_{j,1}, c_{j,2}, \dots, c_{j,m_j}$, where $j = 1, \dots, s$. We assume the roots q_1, \dots, q_s are already arranged in increasing order of their modulus as $|q_1| \leq |q_2| \leq \dots \leq |q_s|$, and the roots of equal modulus are already arranged in increasing order of their multiplicities. Let q_r, \dots, q_s be all roots of largest modulus with the same largest multiplicity $m = m_r = \dots = m_s$. We may have $r = 1$. Assume $r > 1$. Then

$$|q_1| \leq \dots \leq |q_{r-1}| \leq |q_r| = \dots = |q_s|.$$

We have either $|q_{r-1}| < |q_r|$ or $|q_{r-1}| = |q_r|$. If $|q_{r-1}| = |q_r|$, we must have $m_{r-1} < m_r$. Let $\omega_j = q_j/q_s$ for $j = 1, \dots, s$. Then $\omega_r, \dots, \omega_s$ are distinct roots of unity. Divide both sides of (13) by $n^{m-1}q_s^n$ and move the lower order terms

to the right, we obtain

$$\sum_{j=r}^s c_{j,m} \omega_j^n = \varepsilon(n) \rightarrow 0 \quad (n \rightarrow \infty).$$

Let $A(n)$ be the $(s - r + 1) \times (s - r + 1)$ matrix

$$A(n) = \begin{bmatrix} \omega_r^n & \omega_{r+1}^n & \cdots & \omega_s^n \\ \omega_r^{n+1} & \omega_{r+1}^{n+1} & \cdots & \omega_s^{n+1} \\ \vdots & \vdots & & \vdots \\ \omega_r^{n+s-r} & \omega_{r+1}^{n+s-r} & \cdots & \omega_s^{n+s-r} \end{bmatrix},$$

and $A_j(n)$ the matrix obtained from $A(n)$ by replacing its j th column with

$$[\varepsilon(n), \varepsilon(n+1), \dots, \varepsilon(n+s-r)]^T.$$

Then $\det A(n) = \omega_r^n \omega_{r+1}^n \cdots \omega_s^n \prod_{r \leq i < j \leq s} (\omega_j - \omega_i) \neq 0$. By Cramer's rule, we have

$$c_{j,m} = \frac{\det A_j(n)}{\det A(n)} \rightarrow 0 \quad (n \rightarrow \infty), \quad j = r, \dots, s.$$

It follows that $c_{j,m} = 0$ for $j = r, \dots, s$. Likewise, applying the same method to other nonzero coefficients with highest order, we see that all coefficients in (13) are zero. \square

4 Matric representation of linear recurrence relations

Let us write $\mathbf{x}_0 = [x_0, x_1, \dots, x_{k-1}]^T$, $\mathbf{x}_1 = [x_k, x_{k+1}, \dots, x_{2k-1}]^T$, \dots ,

$$\mathbf{x}_n = [x_{nk}, x_{nk+1}, \dots, x_{nk+k-1}]^T, \quad n \geq 0.$$

Assume $x_m = a_1 x_{m-1} + a_2 x_{m-2} + \cdots + a_k x_{m-k}$, $m \geq k$. We see that

$$\mathbf{x}_n = A \mathbf{x}_{n-1}, \quad n \geq 1.$$

Then

$$\mathbf{x}_n = A^n \mathbf{x}_0, \quad n \geq 0.$$

Example 4.1. Find the sequence (x_n) satisfying the recurrence relation

$$x_n = 2x_{n-1} + x_{n-2} - 2x_{n-3}, \quad n \geq 3$$

and the initial conditions $x_0 = 1$, $x_1 = 2$, and $x_2 = 0$.

The recurrence relation can be repeated as

$$\begin{cases} x_3 = 2x_2 + x_1 - 2x_0 = -2x_0 + x_1 + 2x_2 \\ x_4 = 2x_3 + x_2 - 2x_1 = -4x_0 + 5x_2 \\ x_5 = 2x_4 + x_3 - 2x_2 = -10x_0 + x_1 + 10x_2 \end{cases}$$

Thus

$$\begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ -4 & 0 & 5 \\ -10 & 1 & 10 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}, \quad \text{i.e., } \mathbf{x}_1 = A\mathbf{x}_0$$

It follows that $\mathbf{x}_n = A\mathbf{x}_{n-1} = A^2\mathbf{x}_{n-2} = \cdots = A^n\mathbf{x}_0$.

$$\mathbf{x}_n = \begin{bmatrix} x_{3n} \\ x_{3n+1} \\ x_{3n+2} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ -4 & 0 & 5 \\ -10 & 1 & 10 \end{bmatrix}^n \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}, \quad n \geq 0.$$

5 Nonhomogeneous linear recurrence relations with constant coefficients

Theorem 5.1. *Given a nonhomogeneous linear recurrence relation of the first order*

$$x_n = \alpha x_{n-1} + \beta_n, \quad n \geq 1. \quad (14)$$

(a) *Let $\beta_n = cq^n$ be an exponential function of n . Then (14) has a particular solution of the following form.*

- *If $q \neq \alpha$, then $x_n = Aq^n$.*
- *If $q = \alpha$, then $x_n = Anq^n$.*

(b) *Let $\beta_n = \sum_{i=0}^k b_i n^i$ be a polynomial function of n with degree k .*

- *If $\alpha \neq 1$, then (14) has a particular solution of the form*

$$x_n = A_0 + A_1 n + A_2 n^2 + \cdots + A_k n^k,$$

where the coefficients A_0, A_1, \dots, A_k are recursively determined as

$$A_k = \frac{b_k}{1 - \alpha},$$

$$A_i = \frac{1}{1 - \alpha} \left[b_i + \alpha \sum_{j=i+1}^k (-1)^{j-i} \binom{j}{i} A_j \right], \quad 0 \leq i \leq k - 1.$$

• If $\alpha = 1$, then the solution of (14) is given by

$$x_n = x_0 + \sum_{i=1}^n \beta_i.$$

Proof. (a) We may assume $q \neq 0$; otherwise the recurrence (14) is homogeneous.

For the case $q \neq \alpha$, put $x_n = Aq^n$ in (14); we have

$$Aq^n = \alpha Aq^{n-1} + cq^n.$$

The coefficient A is determined as $A = cq/(q - \alpha)$.

For the case $q = \alpha$, put $x_n = Anq^n$ in (14); we have

$$Anq^n = \alpha A(n - 1)q^{n-1} + cq^n.$$

Since $q = \alpha$, then $\alpha Aq^{n-1} = cq^n$. The coefficient A is determined as $A = cq/\alpha$.

(b) For the case $\alpha \neq 1$, put $x_n = \sum_{j=0}^k A_j n^j$ in (14); we obtain

$$\sum_{j=0}^k A_j n^j = \alpha \sum_{j=0}^k A_j (n - 1)^j + \sum_{j=0}^k b_j n^j.$$

Then

$$\sum_{j=0}^k A_j n^j = \alpha \sum_{j=0}^k A_j \sum_{i=0}^j \binom{j}{i} n^i (-1)^{j-i} + \sum_{j=0}^k b_j n^j.$$

$$\sum_{i=0}^k A_i n^i = \alpha \sum_{i=0}^k n^i \sum_{j=i}^k (-1)^{j-i} \binom{j}{i} A_j + \sum_{i=0}^k b_i n^i.$$

$$\sum_{i=0}^k \left[A_i - b_i - \alpha \sum_{j=i}^k (-1)^{j-i} \binom{j}{i} A_j \right] n^i = 0.$$

The coefficients A_0, A_1, \dots, A_k are determined recursively as

$$A_k = \frac{b_k}{1 - \alpha},$$

$$A_i = \frac{1}{1 - \alpha} \left[b_i + \alpha \sum_{j=i+1}^k (-1)^{j-i} \binom{j}{i} A_j \right], \quad 0 \leq i \leq k - 1.$$

As for the case $\alpha = 1$, iterate the recurrence relation (14); we have

$$\begin{aligned} x_n &= x_{n-1} + \beta_n = x_{n-2} + \beta_{n-1} + \beta_n \\ &= x_{n-1} + \beta_{n-2} + \beta_{n-1} + \beta_n = \dots \\ &= x_0 + \beta_1 + \beta_2 + \dots + \beta_n. \end{aligned}$$

□

Example 5.1. Solve the difference equation

$$\begin{cases} x_n = x_{n-1} + 3n^2 - 5n^3, & n \geq 1 \\ x_0 = 2. \end{cases}$$

Solution.

$$\begin{aligned} x_n &= x_0 + \sum_{i=1}^n b_i = 2 + \sum_{i=1}^n (3i^2 - 5i^3) \\ &= 2 + 3 \sum_{i=1}^n i^2 - 5 \sum_{i=1}^n i^3 \\ &= 2 + 3 \times \frac{n(n+1)(2n+1)}{6} - 5 \times \left(\frac{n(n+1)}{2} \right)^2. \end{aligned}$$

We have applied the following identities

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

Example 5.2. Solve the equation

$$\begin{cases} x_n = 3x_{n-1} - 4n, & n \geq 1 \\ x_0 = 2. \end{cases}$$

Solution. Note that $x_n = 3^n c$ is the general solution of the corresponding homogeneous linear recurrence relation. Let $x_n = An + B$ be a particular solution. Then

$$An + B = 3[A(n-1) + B] - 4n$$

Comparing the coefficients of n^0 and n , it follows that $A = 2$ and $B = 3$. Thus the general solution is given by

$$x_n = 2n + 3 + 3^n c.$$

The initial condition $x_0 = 2$ implies that $c = -1$. Therefore the solution is

$$x_n = -3^n + 2n + 3.$$

Theorem 5.2. *Given a nonhomogeneous linear recurrence relation of the second order*

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + cq^n. \quad (15)$$

Let q_1 and q_2 be solutions of its characteristic equation

$$x^2 - \alpha_1 x - \alpha_2 = 0.$$

Then (15) has a particular solution of the following forms, where A is a constant to be determined.

- (a) *If $q \neq q_1, q \neq q_2$, then $x_n = Aq^n$.*
- (b) *If $q = q_1, q_1 \neq q_2$, then $x_n = Anq^n$.*
- (c) *If $q = q_1 = q_2$, then $x_n = An^2q^n$.*

Proof. The homogeneous linear recurrence relation corresponding to (15) is

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2}, \quad n \geq 2. \quad (16)$$

We may assume $q \neq 0$. Otherwise (15) is homogeneous.

(a) Put $x_n = Aq^n$ into (15); we have

$$Aq^n = \alpha_1 Aq^{n-1} + \alpha_2 Aq^{n-2} + cq^n.$$

Then

$$A(q^2 - \alpha_1 q - \alpha_2) = cq^2.$$

Since q is not a root of the characteristic equation $x^2 = \alpha_1 x + \alpha_2$, that is, $q^2 - \alpha_1 q - \alpha_2 \neq 0$, the coefficient A is determined as

$$A = \frac{cq^2}{q^2 - \alpha_1 q - \alpha_2}.$$

(b) Since $q = q_1 \neq q_2$, then $x_n = q^n$ is a solution of (16) but $x_n = nq^n$ is not, that is,

$$q^2 - \alpha_1 q - \alpha_2 = 0 \quad \text{and} \quad nq^n \neq \alpha_1(n-1)q^{n-1} + \alpha_2(n-2)q^{n-2}.$$

It follows that

$$\begin{aligned} nq^2 - \alpha_1(n-1)q - \alpha_2(n-2) &= n(q^2 - \alpha_1 q - \alpha_2) + \alpha_1 q + 2\alpha_2 \\ &= \alpha_1 q + 2\alpha_2 \neq 0. \end{aligned}$$

Put $x_n = Anq^n$ into (15); we have

$$Anq^n = \alpha_1 A(n-1)q^{n-1} + \alpha_2 A(n-2)q^{n-2} + cq^n.$$

Then

$$A[nq^2 - \alpha_1(n-1)q - \alpha_2(n-2)] = cq^2.$$

Since $\alpha_1 q + 2\alpha_2 \neq 0$, the coefficient A is determined as

$$A = \frac{cq^2}{\alpha_1 q + 2\alpha_2}.$$

(c) Since $q = q_1 = q_2$, then both $x_n = q^n$ and $x_n = nq^n$ are solutions of (16), but $x_n = n^2 q^n$ is not. It then follows that

$$q^2 - \alpha_1 q - \alpha_2 = 0, \quad \alpha_1 q + 2\alpha_2 = 0, \quad \text{and}$$

$$\begin{aligned} n^2 q^2 - \alpha_1(n-1)^2 q - \alpha_2(n-2)^2 &= n^2(q^2 - \alpha_1 q - \alpha_2) + 2n(\alpha_1 q + 2\alpha_2) - \alpha_1 q - \\ &= -\alpha_1 q - 4\alpha_2 \neq 0. \end{aligned}$$

Put $x_n = An^2q^n$ into (15); we have

$$Aq^{n-2} [n^2q^2 - \alpha_1(n-1)^2q - \alpha_2(n-2)^2] = cq^n.$$

The coefficient A is determined as

$$A = -\frac{cq^2}{\alpha_1q + 4\alpha_2}.$$

□

Example 5.3. Solve the equation

$$\begin{cases} x_n = 10x_{n-1} - 25x_{n-2} + 5^{n+1}, & n \geq 2 \\ x_0 = 5 \\ x_1 = 15. \end{cases}$$

Put $x_n = An^2 \times 5^n$ into the recurrence relation. We have

$$An^2 \times 5^n = 10A(n-1)^2 \times 5^{n-1} - 25A(n-2)^2 \times 5^{n-2} + 5^{n+1}.$$

Dividing both sides we further have

$$An^2 = 2A(n-1)^2 - A(n-2)^2 + 5.$$

Thus $A = 5/2$. The general solution is given by

$$x_n = \frac{5}{2}n^25^n + c_15^n + c_2n5^n.$$

Applying the initial conditions $x_0 = 5$ and $x_1 = 15$, we have $c_1 = 5$ and $c_2 = -9/2$. Hence

$$x_n = \left(\frac{5}{2}n^2 - \frac{9}{2}n + 5 \right) 5^n.$$

Theorem 5.3. Given a nonhomogeneous linear recurrence relation of the second order

$$x_n = \alpha_1x_{n-1} + \alpha_2x_{n-2} + \beta_n, \quad n \geq 2, \quad (17)$$

where β_n is a polynomial function of n with degree k .

(a) If $\alpha_1 + \alpha_2 \neq 1$, then (17) has a particular solution of the form

$$x_n = A_0 + A_1n + \cdots + A_kn^k,$$

where A_0, A_1, \dots, A_k are constants to be determined. If $k \leq 2$, then a particular solution has the form

$$x_n = A_0 + A_1n + A_2n^2.$$

(b) If $\alpha_1 + \alpha_2 = 1$, then (17) can be reduced to a first order recurrence relation

$$y_n = (\alpha_1 - 1)y_{n-1} + \beta_n, \quad n \geq 2,$$

where $y_n = x_n - x_{n-1}$ for $n \geq 1$.

Proof. (a) Let $\beta_n = \sum_{j=0}^k b_jn^j$. Put $x_n = \sum_{j=0}^k A_jn^j$ into the recurrence relation (17); we obtain

$$\begin{aligned} \sum_{j=0}^k A_jn^j &= \alpha_1 \sum_{j=0}^k A_j(n-1)^j + \alpha_2 \sum_{j=0}^k A_j(n-2)^j + \sum_{j=0}^k b_jn^j \\ &= \alpha_1 \sum_{j=0}^k A_j \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} n^i \\ &\quad + \alpha_2 \sum_{j=0}^k A_j \sum_{i=0}^j (-2)^{j-i} \binom{j}{i} n^i + \sum_{j=0}^k b_jn^j \\ &= \alpha_1 \sum_{i=0}^k n^i \sum_{j=i}^k (-1)^{j-i} \binom{j}{i} A_j \\ &\quad + \alpha_2 \sum_{i=0}^k n^i \sum_{j=i}^k (-2)^{j-i} \binom{j}{i} A_j + \sum_{j=0}^k b_jn^j. \end{aligned}$$

Collecting the coefficients of n^i , we have

$$\sum_{i=0}^k \left[A_i - \alpha_1 \sum_{j=i}^k (-1)^{j-i} \binom{j}{i} A_j - \alpha_2 \sum_{j=i}^k (-2)^{j-i} \binom{j}{i} A_j - \sum_{i=0}^k b_i \right] n^i = 0.$$

Since $\alpha_1 + \alpha_2 \neq 1$, the coefficients A_0, A_1, \dots, A_k are determined as

$$A_k = \frac{b_k}{1 - \alpha_1 - \alpha_2},$$

$$A_i = \frac{1}{1 - \alpha_1 - \alpha_2} \left[b_i + \sum_{j=i+1}^k (-1)^{j-i} \binom{j}{i} (\alpha_1 + 2^{j-i}\alpha_2) A_j \right],$$

where $0 \leq i \leq k - 1$.

(b) The recurrence relation (17) becomes

$$x_n = \alpha_1 x_{n-1} + (1 - \alpha_1)x_{n-2} + \beta_n, \quad n \geq 2.$$

Set $y_n = x_n - x_{n-1}$ for $n \geq 1$; the recurrence (17) reduces to a first order recurrence relation. \square

Example 5.4. Solve the following recurrence relation

$$\begin{cases} x_n = 6x_{n-1} - 9x_{n-2} + 8n^2 - 24n \\ x_0 = 5 \\ x_1 = 5. \end{cases}$$

Solution. Put $x_n = A_0 + A_1n + A_2n^2$ into the recurrence relation; we obtain

$$A_0 + A_1n + A_2n^2 = 6[A_0 + A_1(n-1) + A_2(n-1)^2] - 9[A_0 + A_1(n-2) + A_2(n-2)^2] + 8n^2 - 24n$$

Collecting the coefficients of n^2 , n , and the constant, we have

$$(4A_2 - 8)n^2 + (4A_1 - 24A_2 + 24)n + (4A_0 - 12A_1 + 30A_2) = 0.$$

We conclude that $A_2 = 2$, $A_1 = 6$, and $A_0 = 3$. So $x_n = 2n^2 + 6n + 3$ is a particular solution. Then the general solution of the recurrence is

$$x_n = 2n^2 + 6n + 3 + 3^n c_1 + 3^n n c_2.$$

Applying the initial condition $x_0 = x_1 = 5$, we have $c_1 = 2$, $c_2 = -4$. The sequence is finally obtained as

$$x_n = 2n^2 + 6n + 3 + 2 \times 3^n - 4n \times 3^n.$$

6 Generating functions

The (ordinary) **generating function** of an infinite sequence

$$a_0, a_1, a_2, \dots, a_n, \dots$$

is the infinite series

$$A(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots .$$

A finite sequence

$$a_0, a_1, a_2, \dots, a_n$$

can be regarded as the infinite sequence

$$a_0, a_1, a_2, \dots, a_n, 0, 0, \dots$$

and its generating function

$$A(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

is a polynomial.

Example 6.1. The generating function of the constant infinite sequence

$$1, 1, \dots, 1, \dots$$

is the function

$$A(x) = 1 + x + x^2 + \dots + x^n + \dots = \frac{1}{1-x}.$$

Example 6.2. For any positive integer n , the generating function for the binomial coefficients

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}, 0, \dots$$

is the function

$$\sum_{k=0}^n \binom{n}{k} x^k = (1+x)^n.$$

Example 6.3. For any real number α , the generating function for the infinite sequence of binomial coefficients

$$\binom{\alpha}{0}, \binom{\alpha}{1}, \binom{\alpha}{2}, \dots, \binom{\alpha}{n}, \dots$$

is the function

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = (1+x)^\alpha.$$

Example 6.4. Let k be a positive integer and let

$$a_0, a_1, a_2, \dots, a_n, \dots$$

be the infinite sequence whose general term a_n is the number of nonnegative integral solutions of the equation

$$x_1 + x_2 + \dots + x_k = n.$$

Then the generating function of the sequence (a_n) is

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} \left(\sum_{i_1+\dots+i_k=n} 1 \right) x^n = \sum_{n=0}^{\infty} \sum_{i_1+\dots+i_k=n} x^{i_1+\dots+i_k} \\ &= \left(\sum_{i_1=0}^{\infty} x^{i_1} \right) \cdots \left(\sum_{i_k=0}^{\infty} x^{i_k} \right) = \frac{1}{(1-x)^k} \\ &= \sum_{n=0}^{\infty} (-1)^n \binom{-k}{n} x^n = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n. \end{aligned}$$

Example 6.5. Let a_n be the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = n,$$

where $0 \leq x_1 \leq 3$, $0 \leq x_2 \leq 2$, $x_3 \geq 2$, and $3 \leq x_4 \leq 5$. The generating function of the sequence (a_n) is

$$\begin{aligned} A(x) &= (1+x+x^2+x^3) (1+x+x^2) (x^2+x^3+\dots) (x^3+x^4+x^5) \\ &= \frac{x^5 (1+x+x^2+x^3) (1+x+x^2)^2}{1-x}. \end{aligned}$$

Example 6.6. Determine the generating function for the number of n -combinations of apples, bananas, oranges, and pears where in each n -combination the number of apples is even, the number of bananas is odd, the number of oranges is between 0 and 4, and the number of pears is at least two.

The required generating function is

$$\begin{aligned} A(x) &= \left(\sum_{i=0}^{\infty} x^{2i} \right) \left(\sum_{i=0}^{\infty} x^{2i+1} \right) \left(\sum_{i=0}^4 x^i \right) \left(\sum_{i=2}^{\infty} x^i \right) \\ &= \frac{x^3(1-x^5)}{(1-x^2)^2(1-x)^2}. \end{aligned}$$

Example 6.7. Determine the number a_n of bags with n pieces of fruit (apples, bananas, oranges, and pears) such that the number of apples is even, the number of bananas is a multiple of 5, the number of oranges is at most 4, and the number of pears is either one or zero.

The generating function of the sequence (a_n) is

$$\begin{aligned} A(x) &= \left(\sum_{i=0}^{\infty} x^{2i} \right) \left(\sum_{i=0}^{\infty} x^{5i} \right) \left(\sum_{i=0}^4 x^i \right) \left(\sum_{i=0}^1 x^i \right) \\ &= \frac{(1+x+x^2+x^3+x^4)(1+x)}{(1-x^2)(1-x^5)} \\ &= \frac{(1+x)(1-x^5)/(1-x)}{(1+x)(1-x)(1-x^5)} \\ &= \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (-1)^n \binom{-2}{n} x^n \\ &= \sum_{n=0}^{\infty} \binom{n+1}{n} x^n = \sum_{n=0}^{\infty} (n+1)x^n. \end{aligned}$$

Thus $a_n = n + 1$.

Example 6.8. Find a formula for the number $a_{n,k}$ of integer solutions (i_1, i_2, \dots, i_k) of the equation

$$x_1 + x_2 + \dots + x_k = n$$

such that i_1, i_2, \dots, i_k are nonnegative odd numbers.

The generating function of the sequence (a_n) is

$$\begin{aligned}
A(x) &= \left(\sum_{i=0}^{\infty} x^{2i+1} \right) \cdots \left(\sum_{i=0}^{\infty} x^{2i+1} \right) = \frac{x^k}{(1-x^2)^k} \\
&= x^k \sum_{i=0}^{\infty} \binom{i+k-1}{i} x^{2i} = \sum_{i=0}^{\infty} \binom{i+k-1}{i} x^{2i+k} \\
&= \begin{cases} \sum_{j=r}^{\infty} \binom{j+r-1}{j-r} x^{2j} & \text{for } k = 2r \\ \sum_{j=r}^{\infty} \binom{j+r}{j-r} x^{2j+1} & \text{for } k = 2r + 1. \end{cases}
\end{aligned}$$

We then conclude that $a_{2s,2r} = \binom{s+r-1}{s-r}$, $a_{2s+1,2r+1} = \binom{s+r}{s-r}$, and $a_{n,k} = 0$ otherwise. We may combine the three cases into

$$a_{n,k} = \begin{cases} \binom{\lfloor \frac{n}{2} \rfloor + \lceil \frac{k}{2} \rceil - 1}{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{k}{2} \rfloor} & \text{if } n - k = \text{even,} \\ 0 & \text{if } n - k = \text{odd.} \end{cases}$$

Example 6.9. Let a_n denote the number of nonnegative integral solutions of the equation

$$2x_1 + 3x_2 + 4x_3 + 5x_4 = n.$$

Then the generating function of the sequence (a_n) is

$$\begin{aligned}
A(x) &= \sum_{n=0}^{\infty} \left(\sum_{\substack{i,j,k,l \geq 0 \\ 2i+3j+4k+5l=0}} 1 \right) x^n \\
&= \left(\sum_{i=0}^{\infty} x^{2i} \right) \left(\sum_{j=0}^{\infty} x^{3j} \right) \left(\sum_{k=0}^{\infty} x^{4k} \right) \left(\sum_{l=0}^{\infty} x^{5l} \right) \\
&= \frac{1}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}.
\end{aligned}$$

Theorem 6.1. Let s_n be the number of nonnegative integral solutions of the equation

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = n.$$

Then the generating function of the sequence (s_n) is

$$A(x) = \frac{1}{(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_k})}.$$

7 Recurrence and generating functions

Since

$$\frac{1}{(1 - x)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k, \quad |x| < 1;$$

then

$$\frac{1}{(1 - ax)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-ax)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} a^k x^k, \quad |x| < \frac{1}{|a|}.$$

Example 7.1. Determine the generating function of the sequence

$$0, 1, 2^2, \dots, n^2, \dots$$

Since $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, then

$$\frac{1}{(1-x)^2} = \frac{d}{dx} \left(\frac{1}{1-x} \right) = \sum_{k=0}^{\infty} \frac{d}{dx} (x^k) = \sum_{k=0}^{\infty} kx^{k-1}.$$

Thus $\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} kx^k$. Taking the derivative with respect to x we have

$$\frac{1+x}{(1-x)^3} = \sum_{k=0}^{\infty} k^2 x^{k-1}.$$

Therefore the desired generating function is

$$A(x) = \frac{x(1+x)}{(1-x)^3}.$$

Example 7.2. Solve the recurrence relation

$$\begin{cases} a_n = 5a_{n-1} - 6a_{n-2}, & n \geq 2 \\ a_0 = 1 \\ a_1 = -2 \end{cases}$$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Applying the recurrence relation, we have

$$\begin{aligned} A(x) &= a_0 + a_1 x + \sum_{n=2}^{\infty} (5a_{n-1} - 6a_{n-2}) x^n \\ &= a_0 + a_1 x - 5xa_0 + 5xA(x) - 6x^2A(x). \end{aligned}$$

Applying the initial values and collecting the coefficient functions of $A(x)$, we further have

$$(1 - 5x + 6x^2) A(x) = 1 - 7x.$$

Thus the function $g(x)$ is solved as

$$A(x) = \frac{1 - 7x}{1 - 5x + 6x^2}.$$

Observing that $1 - 5x + 6x^2 = (1 - 2x)(1 - 3x)$ and applying partial fraction,

$$\frac{1 - 7x}{1 - 5x + 6x^2} = \frac{A}{1 - 2x} + \frac{B}{1 - 3x}.$$

The constants A and B can be determined by

$$A(1 - 3x) + B(1 - 2x) = 1 - 7x.$$

Then

$$\begin{cases} A + B = 1 \\ -3A - 2B = -7 \end{cases}$$

Thus $A = 5$, $B = -4$. Hence

$$A(x) = \frac{1 - 7x}{1 - 5x + 6x^2} = \frac{5}{1 - 2x} - \frac{4}{1 - 3x}.$$

Since

$$\frac{1}{1 - 2x} = \sum_{n=0}^{\infty} 2^n x^n \quad \text{and} \quad \frac{1}{1 - 3x} = \sum_{n=0}^{\infty} 3^n x^n$$

We obtain the sequence

$$a_n = 5 \times 2^n - 4 \times 3^n, \quad n \geq 0.$$

Theorem 7.1. Let $(a_n; n \geq 0)$ be a sequence satisfying the homogeneous linear recurrence relation of order k with constant coefficients, i.e.,

$$a_n = \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \cdots + \alpha_k a_{n-k}, \quad (18)$$

where $\alpha_k \neq 0$, $n \geq k$. Then its generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is a rational function of the form

$$A(x) = \frac{P(x)}{Q(x)}, \quad (19)$$

where $Q(x)$ is a polynomial of degree k with a nonzero constant term and $P(x)$ is a polynomial of degree strictly less than k .

Conversely, given such polynomials $P(x)$ and $Q(x)$, there exists a unique sequence (a_n) satisfying the linear homogeneous recurrence relation (18), and its generating function is the rational function in (19).

Proof. The generating function $A(x)$ of the sequence (a_n) can be written as

$$\begin{aligned} A(x) &= \sum_{i=0}^{k-1} a_i x^i + \sum_{n=k}^{\infty} a_n x^n = \sum_{i=0}^{k-1} a_i x^i + \sum_{n=k}^{\infty} \left(\sum_{i=1}^k \alpha_i a_{n-i} \right) x^n \\ &= \sum_{i=0}^{k-1} a_i x^i + \sum_{i=1}^k \alpha_i \sum_{n=k}^{\infty} a_{n-i} x^n = \sum_{i=0}^{k-1} a_i x^i + \sum_{i=1}^k \alpha_i \sum_{n=k-i}^{\infty} a_n x^{n+i} \\ &= \sum_{i=0}^{k-1} a_i x^i + \alpha_k x^k \sum_{n=0}^{\infty} a_n x^n + \sum_{i=1}^{k-1} \alpha_i x^i \left(\sum_{n=0}^{\infty} a_n x^n - \sum_{j=0}^{k-i-1} a_j x^j \right) \\ &= \sum_{i=0}^{k-1} a_i x^i + A(x) \sum_{i=1}^k \alpha_i x^i - \sum_{i=1}^{k-1} \alpha_i x^i \sum_{j=0}^{k-i-1} a_j x^j \\ &= A(x) \sum_{i=1}^k \alpha_i x^i + \sum_{i=0}^{k-1} a_i x^i - \sum_{l=1}^{k-1} x^l \sum_{i=1}^l \alpha_i a_{l-i}. \end{aligned}$$

Then

$$\begin{aligned} A(x) \left(1 - \sum_{i=1}^k \alpha_i x^i \right) &= \sum_{i=0}^{k-1} a_i x^i - \sum_{l=1}^{k-1} x^l \sum_{i=1}^l \alpha_i a_{l-i} \\ &= a_0 + \sum_{l=1}^{k-1} \left(a_l - \sum_{i=1}^l \alpha_i a_{l-i} \right) x^l. \end{aligned}$$

Thus

$$\begin{aligned} P(x) &= a_0 + \sum_{l=1}^{k-1} \left(a_l - \sum_{i=1}^l \alpha_i a_{l-i} \right) x^l, \\ Q(x) &= 1 - \sum_{i=1}^k \alpha_i x^i. \end{aligned}$$

Conversely, let (a_n) be the sequence whose generating function is $A(x) = P(x)/Q(x)$. Write

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \quad P(x) = \sum_{i=0}^k b_i x^i, \quad Q(x) = 1 - \sum_{i=1}^k \alpha_i x^i.$$

Then $A(x) = \frac{P(x)}{Q(x)}$ is equivalent to

$$\left(1 - \sum_{i=1}^k \alpha_i x^i \right) \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{i=0}^k b_i x^i.$$

The polynomial $Q(x)$ can be viewed as an infinite series with $\alpha_i = 0$ for $i > k$.

Thus

$$\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} \left(\sum_{i=1}^n \alpha_i a_{n-i} \right) x^n = \sum_{i=0}^k b_i x^i.$$

Equating the coefficients of x^n , we have the recurrence relation

$$a_n = \sum_{i=1}^k \alpha_i a_{n-i}, \quad n \geq k.$$

□

Proposition 7.2 (Partial Fractions). (a) *If $P(x)$ is a polynomial of degree less than k , then*

$$\frac{P(x)}{(1-ax)^k} = \frac{A_1}{1-ax} + \frac{A_2}{(1-ax)^2} + \cdots + \frac{A_k}{(1-ax)^k},$$

where A_1, A_2, \dots, A_k are constants to be determined.

(b) *If $P(x)$ is a polynomial of degree less than $p + q + r$, then*

$$\frac{P(x)}{(1-ax)^p(1-bx)^q(1-cx)^r} = \frac{A_1(x)}{(1-ax)^p} + \frac{A_2(x)}{(1-bx)^q} + \frac{A_3(x)}{(1-cx)^r},$$

where $A_1(x)$, $A_2(x)$, and $A_3(x)$ are polynomials of degree $q + r$, $p + r$, and $p + q$, respectively.

8 A geometry example

A polygon P in \mathbb{R}^2 is called **convex** if the segment joining any two points in P is also contained in P . Let C_n denote the number of ways to divide a labeled convex polygon with $n + 2$ sides into triangles. The first a few such numbers are $C_1 = 1$, $C_2 = 2$, $C_3 = 5$.

We first establish a recurrence relation for C_{n+1} in terms of C_0, C_1, \dots, C_n . Let $P_{v_1 v_2 \dots v_{n+3}}$ denote a convex $(n + 3)$ -polygon with vertices v_1, v_2, \dots, v_{n+3} . In each triangular decomposition of $P_{v_1 v_2 \dots v_{n+3}}$ into triangles, the segment $v_1 v_{n+3}$ is one side of a triangle Δ in the decomposition; the third vertex of the triangle Δ is one of the vertices v_2, v_3, \dots, v_{n+2} . Let v_{k+2} be the third vertex of Δ other than v_1 and v_{n+3} ($0 \leq k \leq n$); see Figure 1 below. Then we have a convex $(k + 2)$ -

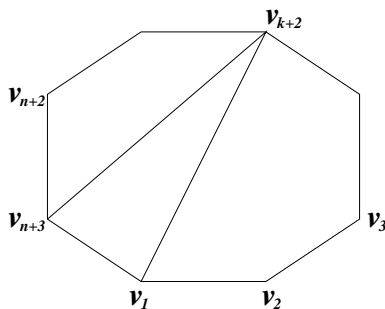


Figure 1: v_{k+2} is the third vertex of the triangle with the side $v_1 v_{n+3}$

polygon $P_{v_1 v_2 \dots v_{k+2}}$ and another convex $(n - k + 2)$ -polygon $P_{v_{k+2} v_{k+3} \dots v_{n+3}}$. Then

by induction there are C_k ways to divide $P_{v_1 v_2 \dots v_{k+2}}$ into triangles, and C_{n-k} ways to divide $P_{v_{k+2} v_{k+3} \dots v_{n+3}}$ into triangles. We thus have the recurrence relation

$$C_{n+1} = \sum_{k=0}^n C_k C_{n-k} \quad \text{with} \quad C_0 = 1.$$

Consider the generating function $F(x) = \sum_{n=0}^{\infty} C_n x^n$. Then

$$\begin{aligned} F(x)F(x) &= \left(\sum_{n=0}^{\infty} C_n x^n \right) \left(\sum_{n=0}^{\infty} C_n x^n \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n C_k C_{n-k} \right) x^n \\ &= \sum_{n=0}^{\infty} C_{n+1} x^n = \frac{1}{x} \sum_{n=1}^{\infty} C_n x^n \\ &= \frac{F(x)}{x} - \frac{1}{x}. \end{aligned}$$

We thus obtain the equation

$$xF(x)^2 - F(x) + 1 = 0.$$

Solving for $F(x)$, we obtain

$$F(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

Note that

$$\begin{aligned} \sqrt{1 - 4x} &= 1 + \sum_{n=1}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} 4^n x^n \\ &= 1 + \sum_{n=1}^{\infty} a_n x^n, \end{aligned}$$

where

$$\begin{aligned}
a_n &= (-1)^n \left[\frac{1}{2} \left(\frac{1}{2} - 1 \right) \cdots \left(\frac{1}{2} - n + 1 \right) / n! \right] \cdot 2^n \cdot 2^n \\
&= (-1)^n \cdot \frac{(-1)(-3)(-5) \cdots (-2\overline{n-1} + 1)}{n!} \cdot 2^n \\
&= -\frac{1 \cdot 3 \cdot 5 \cdots (2\overline{n-1} - 1)}{n!} \cdot 2^n \\
&= -2 \cdot \frac{(2(n-1))!}{n!(n-1)!}.
\end{aligned}$$

Then

$$\begin{aligned}
\sqrt{1-4x} &= 1 - 2 \sum_{n=1}^{\infty} \frac{(2(n-1))!}{n!(n-1)!} x^n \\
&= 1 - 2 \sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!} x^{n+1}.
\end{aligned}$$

We conclude that

$$\begin{aligned}
F(x) &= \frac{1 - \sqrt{1-4x}}{2x} \\
&= \sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!} x^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n.
\end{aligned}$$

Hence the sequence (C_n) is given by the binomial coefficients:

$$C_n = \frac{1}{n+1} \binom{2n}{n}, \quad n \geq 0.$$

The sequence (C_n) is known as the **Catalan sequence** and the numbers C_n as the **Catalan numbers**.

Example 8.1. Let C_n be the number of ways to evaluate a matrix product

$$A_1 A_2 \cdots A_{n+1}, \quad n \geq 0$$

by adding various parentheses. For instance, $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, and $C_3 = 5$. In general the formula is given by

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Note that each way of evaluating the matrix product $A_1 A_2 \cdots A_{n+2}$ will be finished by multiplying of two matrices at the end. There are exactly $n+1$ ways of multiplying the two matrices at the end:

$$A_1 A_2 \cdots A_{n+2} = (A_1 \cdots A_{k+1})(A_{k+2} \cdots A_{n+2}), \quad 0 \leq k \leq n.$$

This yields the recurrence relation

$$C_{n+1} = \sum_{k=0}^n C_k C_{n-k}.$$

Thus $C_n = \frac{1}{n+1} \binom{2n}{n}$, $n \geq 0$.

9 Exponential generating functions

The ordinary generating function method is a powerful algebraic tool for finding unknown sequences, especially when the sequences are certain binomial coefficients or the order is immaterial. However, when the sequences are not binomial type or the order is material in defining the sequences, we may need to consider a different type of generating functions. For example, the sequence $a_n = n!$ is the counting of the number of permutations of n distinct objects; its ordinary generating function

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n! x^n$$

cannot be easily figure out as a closed known expression. However, the generating function

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

is obvious.

The **exponential generating function** of a sequence $(a_n; n \geq 0)$ is the infinite series

$$E(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

Example 9.1. The exponential generating function of the sequence

$$P(n, 0), P(n, 1), \dots, P(n, n), 0, \dots$$

is given by

$$\begin{aligned} E(x) &= \sum_{k=0}^n \frac{P(n, k)}{k!} x^k \\ &= \sum_{k=0}^n \binom{n}{k} x^k \\ &= (1 + x)^n. \end{aligned}$$

Example 9.2. The exponential generating function of the constant sequence $(a_n = 1; n \geq 0)$ is

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

The exponential generating function of the geometric sequence $(a_n = a^n; n \geq 0)$ is

$$E(x) = \sum_{n=0}^{\infty} \frac{a^n x^n}{n!} = e^{ax}.$$

Theorem 9.1. Let $M = \{n_1\alpha_1, n_2\alpha_2, \dots, n_k\alpha_k\}$ be a multiset over the set $S = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$ with n_1 many α_1 's, n_2 many α_2 's, \dots , n_k many α_k 's. Let a_n be the number of n -permutations of the multiset M . Then the exponential generating function of the sequence $(a_n; n \geq 0)$ is given by

$$E(x) = \left(\sum_{i=0}^{n_1} \frac{x^i}{i!} \right) \left(\sum_{i=0}^{n_2} \frac{x^i}{i!} \right) \cdots \left(\sum_{i=0}^{n_k} \frac{x^i}{i!} \right). \quad (20)$$

Proof. Note that $a_n = 0$ for $n > n_1 + \dots + n_k$. Thus $E(x)$ is a polynomial. The right side of (20) can be expanded to the form

$$\sum_{i_1, i_2, \dots, i_k=0}^{n_1, n_2, \dots, n_k} \frac{x^{i_1+i_2+\dots+i_k}}{i_1!i_2!\dots i_k!} = \sum_{n=0}^{n_1+n_2+\dots+n_k} \frac{x^n}{n!} \sum_{\substack{i_1+i_2+\dots+i_k=n \\ 0 \leq i_1 \leq n_1, \dots, 0 \leq i_k \leq n_k}} \frac{n!}{i_1!i_2!\dots i_k!}.$$

Note that the number of permutation of M with exactly i_1 α_1 's, i_2 α_2 's, \dots , and i_k α_k 's such that

$$i_1 + i_2 + \dots + i_k = n$$

is the multinomial coefficient

$$\binom{n}{i_1, i_2, \dots, i_k} = \frac{n!}{i_1!i_2!\dots i_k!}.$$

It turns out that the sequence (a_n) is given by

$$a_n = \sum_{\substack{i_1+i_2+\dots+i_k=n \\ 0 \leq i_1 \leq n_1, \dots, 0 \leq i_k \leq n_k}} \binom{n}{i_1, i_2, \dots, i_k}, \quad n \geq 0.$$

We proved the closed form of the required exponential generating function. \square

Example 9.3. Determine the number of ways to color the squares of a 1-by- n chessboard using the colors, red, white, and blue, if an even number of squares are colored red.

Let a_n denote the number of ways of such colorings and set $a_0 = 1$. Each such coloring can be considered as a permutation of three objects r (for red), w (for white), and b (for blue) with repetition allowed, and the element r appears even

number of times. The exponential generating function of the sequence (a_n) is

$$\begin{aligned}
 E(x) &= \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right) \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right)^2 \\
 &= \frac{e^x + e^{-x}}{2} e^{2x} = \frac{1}{2} (e^{3x} + e^x) \\
 &= \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{3^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \\
 &= \frac{1}{2} \sum_{n=0}^{\infty} (3^n + 1) \cdot \frac{x^n}{n!}.
 \end{aligned}$$

Thus the sequence is given by

$$a_n = \frac{3^n + 1}{2}, \quad n \geq 0.$$

Example 9.4. Determine the number a_n of n digit (under base 10) numbers with each digit odd where the digit 1 and 3 occur an even number of times.

Assume $a_0 = 1$. The number a_n equals the number of n -permutations of the multiset $M = \{\infty 1, \infty 3, \infty 5, \infty 7, \infty 9\}$, in which 1 and 3 occur an even number of times. The exponential generating function of the sequence a_n is

$$\begin{aligned}
 E(x) &= \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right)^2 \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right)^3 \\
 &= \left(\frac{e^x + e^{-x}}{2} \right)^2 e^{3x} \\
 &= \frac{1}{4} (e^{5x} + 2e^{3x} + e^x) \\
 &= \frac{1}{4} \left(\sum_{n=0}^{\infty} \frac{5^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{3^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \\
 &= \sum_{n=0}^{\infty} \left(\frac{5^n + 2 \times 3^n + 1}{4} \right) \frac{x^n}{n!}.
 \end{aligned}$$

Thus

$$a_n = \frac{5^n + 2 \times 3^n + 1}{4}, \quad n \geq 0.$$

Example 9.5. Determine the number of ways to color the squares of a 1-by- n board with the colors, red, blue, and white, where the number of red squares is even and there is at least one blue square.

The exponential generating function for the sequence is

$$\begin{aligned} E(x) &= \left(\sum_{i=0}^{\infty} \frac{x^{2i}}{(2i)!} \right) \left(\sum_{i=0}^{\infty} \frac{x^i}{i!} \right) \left(\sum_{i=1}^{\infty} \frac{x^i}{i!} \right) \\ &= \frac{e^x + e^{-x}}{2} e^x (e^x - 1) \\ &= \frac{1}{2} (e^{3x} - e^{2x} + e^x - 1) \\ &= -\frac{1}{2} + \sum_{n=0}^{\infty} \frac{3^n - 2^n + 1}{2} \cdot \frac{x^n}{n!} \end{aligned}$$

Thus

$$a_n = \frac{3^n - 2^n + 1}{2}, \quad n \geq 1$$

and

$$a_0 = 0.$$

10 Combinatorial interpretations

Theorem 10.1. *The combinatorial interpretations of ordinary generating functions:*

- (a) *The number of ways of placing n indistinguishable balls into m distinguishable boxes is the coefficient of x^n in*

$$(1 + x + x^2 + \cdots)^m = \left(\sum_{k=0}^{\infty} x^k \right)^m = \frac{1}{(1-x)^m}.$$

- (b) *The number of ways of placing n indistinguishable balls into m distinguishable boxes with at most r_k balls in the k th box is the coefficient of x^n in the expression*

$$\prod_{k=1}^m (1 + x + x^2 + \cdots + x^{r_k}).$$

(c) The number of ways of placing n indistinguishable balls into m distinguishable boxes with at least s_k balls in the k th box is the coefficient of x^n in the expression

$$\prod_{k=1}^m x^{s_k} (1 + x + x^2 + \cdots) = \frac{x^{s_1 + \cdots + s_m}}{(1 - x)^m}.$$

(d) The number of ways of placing n indistinguishable balls into m distinguishable boxes, such that the number of balls held in the k th box is allowed in a subset $C_k \subseteq \mathbb{Z}_{\geq 0}$ ($1 \leq k \leq m$), is the coefficient of x^n in the expression

$$\left(\sum_{k \in C_1} x^k \right) \left(\sum_{k \in C_2} x^k \right) \cdots \left(\sum_{k \in C_m} x^k \right).$$

Proof. Let us consider the situation of placing infinitely many indistinguishable balls into m distinguishable boxes. We use the symbol x to represent a ball and x^k to represent indistinguishable k balls. In the course of placing the balls into m boxes, each box contains either none of balls ($x^0 = 1$), or one ball (x), or two balls (x^2), or three balls (x^3), etc.; that is, each box contains

$$1 + x + x^2 + x^3 + \cdots$$

balls, where the plus sign “+” means “or”.

Box 1	Box 2	Box 3	...	Box m
1	1	1	1	1
x	x	x	x	x
x^2	x^2	x^2	x^2	x^2
x^3	x^3	x^3	x^3	x^3
\vdots	\vdots	\vdots	\vdots	\vdots

If we use multiplication to represent “and”, then all possible distributions of balls in the m boxes in the course of placing are contained in the expression

$$(1 + x + x^2 + \cdots)^m. \tag{21}$$

Thus, when there are n distinguishable balls to be placed into the m distinguishable boxes, the number of distributions of the n balls in the m boxes will be the coefficient of x^n in the expression (21). Indeed, to compute the coefficient of x^n in (21), we select x^{n_1} from column 1, x^{n_2} from column 2, \dots , x^{n_m} from column m in the above table such that $x^{n_1}x^{n_2}\dots x^{n_m} = x^n$, and do this in all possible ways. The number of such ways is clearly the number of nonnegative integer solutions of the equation

$$y_1 + y_2 + \dots + y_m = n,$$

and is given by $\binom{m+n-1}{n}$.

The situation for other cases are similar. □

Theorem 10.2. *The combinatorial interpretation of exponential generating functions:*

- (a) *The number of ways of selecting an ordered n objects with repetition allowed from an m -set is the coefficient of $\frac{x^n}{n!}$ in the expression*

$$\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^m = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)^m = e^{mx}. \quad (22)$$

- (b) *The number of ways of selecting an ordered n objects from an m -set such that the number of k th objects is allowed in a subset $C_k \subseteq \mathbb{Z}_{\geq 0}$, is the coefficient of $\frac{x^n}{n!}$ in the expression*

$$\left(\sum_{k \in C_1} \frac{x^k}{k!}\right) \left(\sum_{k \in C_2} \frac{x^k}{k!}\right) \dots \left(\sum_{k \in C_m} \frac{x^k}{k!}\right). \quad (23)$$

Example 10.1. Find the number of 4-permutations of the multiset $M = \{a, a, b, b, b, c\}$.

There are 5 sub-multisets of M having cardinality 4.

$\{a, a, b, b\}$	$\{a, a, b, c\}$	$\{a, b, b, b\}$	$\{a, b, b, c\}$	$\{b, b, b, c\}$
$4!/2!2!0!$	$4!/2!1!1!$	$4!/1!3!0!$	$4!/1!2!1!$	$4!/0!3!1!$

Then the answer is

$$\frac{4!}{2!2!0!} + \frac{4!}{2!1!1!} + \frac{4!}{1!3!0!} + \frac{4!}{1!2!1!} + \frac{4!}{0!3!1!} = 38,$$

which is the coefficient of $\frac{x^4}{4!}$ in the expansion of

$$E(x) = \left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right) \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right) \left(1 + \frac{x}{1!}\right).$$

In fact, the coefficient of $x^4/4!$ is

$$a_4 = \sum_{\substack{i+j+k=4 \\ i \leq 2, j \leq 3, k \leq 1}} \frac{4!}{i!j!k!} = \frac{4!}{2!2!0!} + \frac{4!}{2!1!1!} + \frac{4!}{1!3!0!} + \frac{4!}{1!2!1!} + \frac{4!}{0!3!1!}.$$