

# Week 1

## 1 What is combinatorics?

The question of *what is combinatorics* is similar to the question of *what is mathematics*. If we say that mathematics is about the study of numbers and figures, then combinatorics is about the counting (enumeration) of objects, including discrete elements of a finite set, pieces in a continuum geometric shape, and measuring structures of spaces, etc.

Examples of combinatorial problems:

(1) Finding the number of games that  $n$  teams would play if each team played with every other team exactly once.

(2) Constructing a magic square.

(3) Attempting to trace through a network without removing your pencil from the paper and without tracing any part of the network more than once.

(4) Counting the number of poker hands which are full houses.

Historically, combinatorics has its roots in mathematical recreations and games. Many problems that were studied in the past, either for amusement or for aesthetic appeal, are today of great importance in pure and applied sciences. Now combinatorics is an important branch of mathematics, and its influence continues to expand. Part of the reason for the tremendous growth of combinatorics since the sixties has been the major impact that computers have had and continue to have in our society. Another reason for the recent growth of combinatorics is its applicability to disciplines that had previously had little serious contact with mathematics. It is often found that the ideas and techniques of combinatorics are being used not only in the traditional areas of mathematical application, namely, the physical sciences, but also in the social sciences, the biological sciences, information theory, and recently emerged data science, etc.

*Combinatorics is concerned with arrangements of the objects of a set into patterns satisfying specified rules.*

Combinatorial problems can be classified into following categories:

*Existence of the arrangement.*

*Enumeration of the arrangements.*

*Classification of the arrangements.*

*Study of a known arrangement.*

*Construction of an optimal arrangement.*

In other words, combinatorics is concerned with the existence, enumeration, analysis, and optimization of discrete structures.

There are very few general methods in combinatorics that can apply to solve large number of combinatorial problems. The typical general methods in combinatorics: Induction; inclusion-exclusion principle, pigeonhole principle; bijective counting; methods of recurrence relations and generating function; Burnside's theorem and Pólya counting; and Möbius inversion formula.

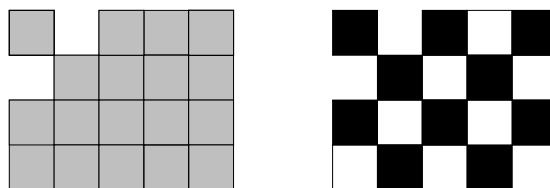
## 2 Examples

### 2.1 Perfect Cover of Chessboards

It is obvious that the chessboard can be covered by 32 dominoes so that no two dominoes overlap. Such a cover is called a **perfect cover**. However, the chessboard with two diagonal corners removed cannot be perfectly covered by 31 dominoes since

$$31BW \neq 32B + 30W.$$

The following board cannot be perfectly covered by dominoes.



A ***b*-omino** is a  $(1, b)$ -board or  $(b, 1)$ -board, where  $b \geq 2$ . A **perfect cover** of an  $m$ -by- $n$  board by  $b$ -ominoes is an arrangement of  $b$ -ominoes on the board so that no two  $b$ -ominoes overlap. When does an  $m \times n$ -board have a perfect cover by  $b$ -ominoes? The answer is given by the following theorem.

**Theorem 2.1.** *An  $m$ -by- $n$ -board has a perfect cover by  $b$ -ominoes iff  $b$  divides either  $m$  or  $n$ .*

*Proof.* “ $\Leftarrow$ :” The condition is obviously sufficient.

“ $\Rightarrow$ :” Let  $m$  and  $n$  be divided by  $b$  to have remainders  $r$  and  $s$  respectively:

$$\begin{aligned} m &= pb + r, & 0 \leq r < b, \\ n &= qb + s, & 0 \leq s < b. \end{aligned}$$

Without loss of generality we may assume  $r \leq s$ . We claim that  $r = 0$ . Let the  $m \times n$ -board  $B$  be cyclicly colored (or labeled) by  $b$  colors as follows:

1	2	3	$\dots$	$b-1$	$b$
$b$	1	2	$\dots$	$b-2$	$b-1$
$b-1$	$b$	1	$\dots$	$b-3$	$b-2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
3	4	5	$\dots$	1	2
2	3	4	$\dots$	$b$	1

For example, for  $m = 10$ ,  $n = 15$ , and  $b = 4$ , the 10-by-15 board can be colored as

1	2	3	4	1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2	3	4	1	2
3	4	1	2	3	4	1	2	3	4	1	2	3	4	1
2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2	3	4	1	2
3	4	1	2	3	4	1	2	3	4	1	2	3	4	1
2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2	3	4	1	2

Note that each  $b$ -omino of a perfect covering covers exactly one square of each of the  $b$  colors, no matter how the  $b$ -omino is placed. It follows that there must be the same number of squares of each color of the board. Let the  $m$ -by- $n$  board be divided into three parts as follows:

Obviously, the boards  $B_1, B_2$  are perfectly covered by  $b$ -ominoes. Note that the number of squares of each of  $b$  colors is constant in each of the three boards  $B, B_1$ , and  $B_2$ . It follows that the number of squares of each of the  $b$  colors in the

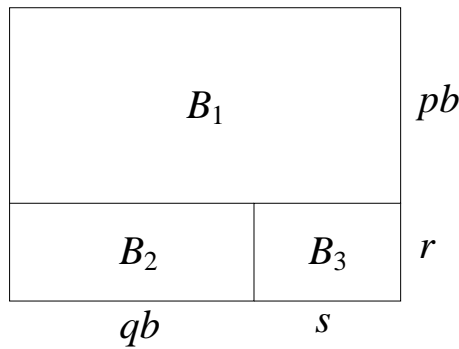


Figure 1: The  $m \times n$ -board is cut into three boards.

$r$ -by- $s$  board  $B_3$  is also constant. Since  $r \leq s$ , the number of squares of the color  $i$  in the left-upper corner of  $B_3$  is  $r$ , and there are  $rb$  squares in the board  $B_3$ . We thus have  $rs = rb$ . If  $r > 0$ , then  $s = b$ , a contradiction. Hence,  $r = 0$ . This means that  $b \mid m$ .  $\square$

## 2.2 Cutting a Cube

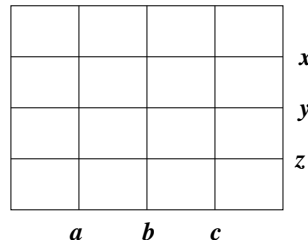
What is the minimal number of cuts to cut a cube of side length 3 into 27 small cubes of unit side length? Geometrically, it is easy to see that each cut must parallel to a face of the cube. It is also easy to see that 6 cuts are enough to do the job. The problem is whether 6 is minimal or not. The answer is yes. (The middle cube has 6 faces which need at least 6 cuts.)

A problem related to the cube-cutting problem is the following: Consider a 4-by-4 chessboard that is perfectly covered by 8 dominoes. A **fault-line** for a perfect cover of a board is either a horizontal line or a vertical line that does not cut any domino in the cover. Do there always exist a fault-line?

**Theorem 2.2.** *Every perfect cover has a fault-line.*

We only prove the theorem for 4-by-4 board. Suppose the  $4 \times 4$ -board has a perfect cover which has no fault-line. Let  $a, b, c, x, y, z$  denote the number of dominoes cut by the three vertical lines and the three horizontal lines respectively. Since there is no fault-line, the numbers  $a, b, c, x, y, z$  are positive. By try-and-error, we see that  $a, b, c, x, y, z \geq 2$ . It is clear that no domino can be cut by more than two lines. Then there are at least

$$a + b + c + x + y + z \geq 2 \cdot 6 = 12$$



dominoes in the perfect cover. However, the number of dominoes must be 8. This is a contradiction.

### 2.3 Magic Squares

A **magic square** of order  $n$  is an  $n \times n$ -array constructed out of the integers  $1, 2, \dots, n^2$  in such a way that the sum of the integers in each row, in each column, and in each of the two diagonals is the same number  $s$ . The number  $s$  is called the **magic sum** of the magic square. For example,

$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}, \quad \begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}, \quad \begin{bmatrix} 7 & 12 & 1 & 14 \\ 2 & 13 & 8 & 11 \\ 16 & 3 & 10 & 5 \\ 9 & 6 & 15 & 4 \end{bmatrix}.$$

The first matrix is a magic square of order 3 with magic sum 15. The last two matrices are magic squares of order 4 with magic sum 34.

Necessary condition:

$$1 + 2 + \dots + n^2 = \frac{n^2(n^2 + 1)}{2}.$$

$$s = \frac{n(n^2 + 1)}{2}.$$

De La Loubère's Method: This is only for constructing magic square of odd order  $n$ . The integer 1 is placed in the middle square of the top row. The successive integers are then placed in their natural order along a diagonal line which slopes upwards and to the right, with the following modifications:

- 1) When the top row is reached, the next integer is put in the bottom row as if it came immediately above the top row.

- 2) When the rightmost column is reached, the next integer is put in the leftmost column as if it immediately succeeded the rightmost column.
- 3) When a square is reached which has already been filled or when the top rightmost square is reached, the next integer is placed in the square immediately below the last square which was filled.

For instance, the magic squares of order 5 and 7 are constructed as follows:

$$\begin{bmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{bmatrix}, \quad \begin{bmatrix} 30 & 39 & 48 & 1 & 10 & 19 & 28 \\ 38 & 47 & 7 & 9 & 18 & 27 & 29 \\ 46 & 6 & 8 & 17 & 26 & 35 & 37 \\ 5 & 14 & 16 & 25 & 34 & 36 & 45 \\ 13 & 15 & 24 & 33 & 42 & 44 & 4 \\ 21 & 23 & 32 & 41 & 43 & 3 & 12 \\ 22 & 31 & 40 & 49 & 2 & 11 & 20 \end{bmatrix}$$

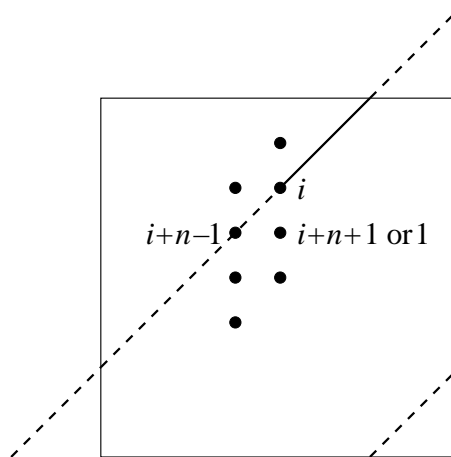


Figure 2: Every row and every column have the same sum.

Let  $n = 2k + 1$ . Let  $a_{i,j}$  be the  $(i, j)$ -entry of the constructed magic matrix. Initially,  $a_{1,k+1} = 1$ . The entries satisfy the relations:

- (a) if  $n \nmid a_{i,j}$ , then  $a_{i-1,j+1} = a_{i,j} + 1$ ;
- (b) if  $n \mid a_{i,j}$ , then  $a_{i-1,j+1} = a_{i,j} - n + 1$ ,
- (c) if  $n \mid a_{i,j}$  and  $a_{ij} < n^2$ , then  $a_{i+1,j} = a_{i,j} + 1$ .

We have only  $n$  numbers divisible by  $n$ , namely,  $1n, 2n, \dots, nn(=n^2)$ . It is easy to see from (a)–(b) that

$$\text{if } a_{i,j} = mn, \text{ then } a_{i+2,j-1} = (m+1)n, \quad (1)$$

where the row and column indices are understood to be modulo  $n$ .

We immediately see that each column has exactly one entry divisible by  $n$ . Since  $a_{1,k+1} = 1$ , we have  $a_{0,k+2} = 2, a_{-1,k+3} = 3, \dots, a_{-(n-2),k+n} = n$ , namely,

$$a_{n,k+2} = 2, \quad a_{n-1,k+3} = 3, \quad \dots, \quad a_{2,k} = n.$$

Based on  $a_{2,k} = n$ , we obtain by the recurrence (1)

$$a_{2,k} = n, \quad a_{4,k-1} = 2n, \quad a_{6,k-2} = 3n, \quad \dots, \quad a_{2k,1} = kn. \quad (2)$$

Based on  $a_{2k,1} = kn$ , we have  $a_{2(k+1),0} = a_{1,n} = (k+1)n$ . Thus

$$\begin{aligned} a_{1,n} &= (k+1)n, \quad a_{3,n-1} = (k+2)n, \quad a_{5,n-2} = (k+3)n, \\ \dots\dots\dots, \quad a_{2k+1,n-k} &= a_{n,k+1} = (k + \overline{k+1})n = n^2. \end{aligned} \quad (3)$$

We have located all multiples of  $n$  from 1 to  $n^2$ . It is clear that each row and each column contains exactly one multiple of  $n$ .

Now consider two columns next to each other, say, the  $j$ th column and the  $(j+1)$ th column. There exist exactly  $n-1$  entries of  $a_{i,j+1}$  which are exactly one larger than the corresponding entries  $a_{i+1,j}$ , and there exists exactly one entry of the  $(j+1)$ th column which is exactly  $n-1$  less than the corresponding entry of the  $j$ th column. It follows that the two columns have the same sum. Likewise, any two rows next to each other also have the same sum.

Since  $a_{2k,1} = kn$ , we have  $a_{2k+1,1} = a_{n,1} = kn+1$ . We obtain

$$a_{n-\ell+1,\ell} = kn + \ell, \quad \text{where } \ell = 1, \dots, n.$$

The rising diagonal sum is confirmed as

$$\begin{aligned} a_{n,1} + a_{n-1,2} + \dots + a_{1,n} &= (kn+1) + \dots + (kn+n) \\ &= \frac{n(2kn+n+1)}{2} = \frac{n(n^2+1)}{2}. \end{aligned}$$

It needs more work to figure out the downing diagonal sum. To do this we separate  $k$  in even and odd cases. For the even case, let  $k = 2p$  with  $p$  a positive

integer, then  $n = 4p + 1$ . We gather the entries which are multiples of  $n$  of the magic matrix into four groups (I), (II), (III), and (IV).

Consider the first group, consisting the entries of odd multiples of  $n$  in (2) as

$$(I) \begin{cases} a_{2,2p} &= (4p + 1) \\ a_{6,2p-2} &= 3(4p + 1) \\ &\vdots \\ a_{4p-6,4} &= (2p - 3)(4p + 1) \\ a_{4p-2,2} &= (2p - 1)(4p + 1) \end{cases}$$

which can be written as  $a_{4r+2,2p-2r} = (2r + 1)(4p + 1)$  with  $0 \leq r < p$ . Then

$$a_{4r+2+s,2p-2r-s} = (2r + 1)(4p + 1) - s,$$

where  $0 \leq r < p$ ,  $0 \leq s < n$ . Setting  $4r + 2 + s = 2p - 2r - s$ , we have  $s = p - 3r - 1$ ,  $0 \leq r < p$ . Then  $i = j = p + r + 1$ . Note that if  $s$  is negative, we must have  $a_{4r+2+s,2p-2r-s} = 2r(4p + 1) - s$ . Thus we obtain

$$\begin{aligned} a_{p+r+1,p+r+1} &= \begin{cases} (2r + 1)(4p + 1) - (p - 3r - 1) & \text{if } p \geq 3r + 1 \\ 2r(4p + 1) - (p - 3r - 1) & \text{if } p < 3r + 1 \end{cases} \\ &= \begin{cases} (2n + 3)r + (3n + 5)/4 & \text{if } p \geq 3r + 1 \\ (2n + 3)r + (3n + 5)/4 - n & \text{if } p < 3r + 1 \end{cases} \end{aligned} \quad (4)$$

where  $0 \leq r < p$ . The second group consists of the entries of even multiples of  $n$  in (2) as

$$(II) \begin{cases} a_{4,2p-1} &= 2(4p + 1) \\ a_{8,2p-3} &= 4(4p + 1) \\ &\vdots \\ a_{4p-4,3} &= (2p - 3)(4p + 1) \\ a_{4p,1} &= 2p(4p + 1) \end{cases}$$

which be written as

$$a_{4r+4,2p-1-2r} = 2(r + 1)(4p + 1), \quad 0 \leq r < p.$$

By the recurrence (1),

$$a_{4r+4+s,2p-1-2r-s} = 2(r + 1)(4p + 1) - s,$$



where  $0 \leq r < p$  and  $0 \leq s < n$ . Setting  $4r + 4 + s = 2p - 1 - 2r - s \pmod{n}$ , since  $2p - 1 - 2r - s$  will be non-positive, we should have

$$4r + 4 + s = 2p - 1 - 2r - s + (4p + 1).$$

Then  $s = 3p - 3r - 2$ , and  $i = j = 3p + r + 2$ , where  $0 \leq r < p$ . We obtain

$$\begin{aligned} a_{3p+r+2,3p+r+2} &= 2(r+1)(4p+1) - (3p-3r-2) \\ &= (2n+3)r + (5n+11)/4, \quad 0 \leq r < p. \end{aligned} \quad (5)$$

The third group consists of the entries of even multiples of  $n$  in (3) as

$$(III) \begin{cases} a_{1,4p+1} &= (2p+1)(4p+1) \\ a_{5,4p-1} &= (2p+3)(4p+1) \\ &\vdots \\ a_{4p-3,2p+3} &= (4p-1)(4p+1) \\ a_{4p+1,2p+1} &= (2p+\overline{2p+1})(4p+1) \end{cases}$$

which can be written as

$$a_{4r+1,4p-2r+1} = (2p+2r+1)(4p+1), \quad 0 \leq r \leq p.$$

By the recurrence relation (1)

$$a_{4r+1+s,4p-2r+1-s} = (2p+2r+1)(4p+1) - s,$$

where  $0 \leq r \leq p$  and  $0 \leq s < n$ . Setting  $4r+1+s = 4p-2r+1-s$ , we have  $s = 2p-3r$ ; consequently,  $i = j = 2p+r+1$ , where  $0 \leq r \leq p$ . We obtain

$$\begin{aligned} a_{2p+r+1,2p+r+1} &= \begin{cases} (2p+2r+1)(4p+1) - (2p-3r) & \text{if } 3r \leq 2p \\ (2p+2r)(4p+1) - (2p-3r) & \text{if } 3r > 2p \end{cases} \\ &= \begin{cases} (2n+3)r + (n^2+1)/2 & \text{if } 3r \leq 2p \\ (2n+3)r + (n^2+1)/2 - n & \text{if } 3r > 2p \end{cases} \end{aligned} \quad (6)$$

where  $0 \leq r \leq p$ . The last group consists of entries of the odd multiples of  $n$  in (3) as

$$(IV) \begin{cases} a_{3,4p} &= 2(p+1)(4p+1) \\ a_{7,4p-2} &= 2(p+2)(4p+1) \\ &\vdots \\ a_{4p-1,2p+2} &= 2(p+p)(4p+1) \end{cases}$$

which can be written as

$$a_{4r+3,4p-2r} = 2(p+r+1)(4p+1), \quad 0 \leq r < p.$$

By the recurrence relation,

$$a_{4r+3+s,4p-2r-s} = 2(p+r+1)(4p+1) - s,$$

where  $0 \leq r < p$  and  $0 \leq s < n$ . Setting  $4r+3+s = 4p-2r-s \pmod{n}$ , since the subscripts  $4r+3+s$  may be larger than  $n (= 4p+1)$ , so we should set

$$4r+3+s - (4p+1) = 4p-2r-s.$$

Then  $s = 4p - 3r - 1$ , consequently,  $i = j = r + 1$ . We obtain

$$\begin{aligned} a_{r+1,r+1} &= 2(p+r+1)(4p+1) - (4p-3r-1) \\ &= (2n+3)r + (n^2+n+4)/2, \quad 0 \leq r < p. \end{aligned} \quad (7)$$

Now the downing diagonal sum is the adding of the sums of (4)–(7), which is

$$\begin{aligned} S &= a_{3p+1,3p+1} + \sum_{r=0}^{p-1} (\text{I}+\text{II}+\text{III}+\text{IV}) \\ &= \frac{4n^2 - 3n - 1}{4} + 4 \sum_{r=0}^{p-1} (2n+3)r + \frac{n-1}{4} \\ &\quad \left( \frac{3n+5}{4} + \frac{5n+11}{4} + \frac{n^2+1}{2} + \frac{n^2+n+4}{2} \right) \\ &\quad - n \sum_{0 \leq r < p, 3r+1 > p} 1 - n \sum_{0 \leq r < p, 3r > 2p} 1 \\ &= \frac{4n^2 - 3n - 1}{4} + (2n+3)(n-1) \cdot \frac{n-5}{8} \\ &\quad + \frac{n-1}{4} \left( n^2 + \frac{5n}{2} + \frac{13}{2} \right) - n\Sigma \\ &= \frac{n(n^2+1)}{2}. \quad \left( \Sigma = \frac{n-8}{4} \right) \end{aligned}$$

A **magic cube** of order  $n$  is an  $n \times n \times n$  cubical array constructed out of integers  $1, 2, \dots, n^3$  in such way that the sum  $s$  of the integers in the  $n$  cells of each of the following straight lines is the same:

- 1) lines parallel to an edge of the cube;
- 2) two diagonals of each plane cross section;
- 3) four space diagonals.

The number  $s$  is called the **magic sum** of the magic cube and has the value (since there are  $n^2$  vertical lines)

$$s = \frac{n^3(n^3 + 1)}{2} \cdot \frac{1}{n^2} = \frac{n(n^3 + 1)}{2}.$$

*There is no magic cube of order 3.*

Suppose there is a magic cube of order 3. Its magic sum should be  $\frac{3(3^3+1)}{2} = 42$ .

For a magic cube of order 3,  $1 \leq i, j, k \leq 3$ , consider any  $3 \times 3$  plane cross section

$$\begin{bmatrix} a & b & c \\ u & v & w \\ x & y & z \end{bmatrix}$$

Then

$$a + b + c = 42 \tag{8}$$

$$x + y + z = 42 \tag{9}$$

$$a + v + z = 42 \tag{10}$$

$$b + v + y = 42 \tag{11}$$

$$c + v + x = 42 \tag{12}$$

Do operation (3) + (4) + (5) – (1) – (2), we have  $3v = 42$  and  $v = 14$ . This means that 14 has to be the center for any plane cross section of the magic cube. However, there are more than one such plane centers, and 14 can only occupy one place. This is a contradiction.

It is much more difficult to show that there is no magic cube of order 4. A magic cube of order 8 is given in an article by Gardner, “Mathematical games,” *Scientific American*, January (1976), 118–123.

## 2.4 The Four-Color Problem

## 2.5 The Problem of 36 Officers

Given 36 officers of 6 ranks and from 6 regiments, no two have both the same rank and from the same regiments, can they be arranged in a  $6 \times 6$  formation so that in each row and column there is one officer of each rank and one officer from each regiment? This problem can be stated as follows:

Can the 36 ordered pairs  $(i, j)$  ( $i = 1, 2, \dots, 6; j = 1, 2, \dots, 6$ ) be arranged in a  $6 \times 6$  array so that in each row and each column, the numbers in the first position form a permutation of  $\{1, 2, \dots, 6\}$  and the numbers in the second position form a permutation of  $\{1, 2, \dots, 6\}$ ?

Such an array can be split into two  $6 \times 6$  arrays, one corresponding to the first positions of the ordered pairs (the rank array) and the other to the second positions (the regiment array). Thus the problem can be stated:

Do there exist two  $6 \times 6$  arrays whose entries are taken from the integers  $1, 2, \dots, 6$  such that (1) in each row and in each column of these arrays the integers  $1, 2, \dots, 6$  occur in some order, and (2) when the two arrays are juxtaposed all of the 36 ordered pairs  $(i, j)$  ( $1 \leq i, j \leq 6$ ) occur?

To make the problem concrete and easy, suppose instead that there are 9 officers of 3 ranks and from 3 different regiments. Then a solution for the problem in this case is

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} (1, 1) & (2, 2) & (3, 3) \\ (3, 2) & (1, 3) & (2, 1) \\ (2, 3) & (3, 1) & (1, 2) \end{bmatrix}$$

A **Latin square** of rank  $n$  is an array of integers such that each row and each column of the array the integers  $1, 2, \dots, n$  occur in some order. The rank and regiment arrays above are Latin squares of order 3. The following are Latin squares of order 2 and 4:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}.$$

Two Latin squares of order  $n$  are called **orthogonal** if, when they are juxtaposed, all ordered pairs  $(i, j)$  ( $1 \leq i \leq n$  and  $1 \leq j \leq n$ ) occur.

Euler investigated the existence of orthogonal Latin squares of order  $n$ . It is easy to see that there is no pair of Latin squares of order 2.

Euler showed that how to construct a pair of orthogonal Latin squares of order  $n$  when  $n$  is odd or divisible by 4. Notice that this does not include  $n = 6$ . On the basis of many trials he concluded, but did not prove, that there is no pair of orthogonal Latin squares of order 6, and he conjectured that there is no such pair existed for any of integers  $n = 4k + 2$  with  $k \geq 1$ .

By exhaustive enumeration Terry in 1901 proved that Euler's conjecture is true for  $n = 6$ . Around 1960 Bose, Parker, and Shrikhande succeeded in proving that Euler's conjecture was false for all  $n > 6$ , i.e., for  $n = 4k + 2$  with  $k \geq 2$ .

## 2.6 Shortest Path Problem

## 2.7 The Game of Nim

**Nim** is a game played by two players with heaps of coins. Suppose there are  $k$  heaps of coins which contain, respectively,  $n_1, \dots, n_k$  coins. The object of the game is to select the last coin. The rules of the game are as follows:

1. The players alternate turns (let us call the player who makes the first move I and then call the other player II)
2. Each player, when it is their turn, selects one of the heaps and removes one or more coins from the selected heap. (The player may take all of the coins from the selected heap, thereby leaving an empty heap, which is now "out of play.")

The game ends when all the heaps are empty. The last player to make a move, that is, the player who takes the last coin(s), is the winner.

When  $k = 1$ , obviously, player I wins the game.

When  $k = 2$ , if  $n_1 \neq n_2$ , say,  $n_1 > n_2$ , player I may remove  $n_1 - n_2$  coins from the first heap to make the two heaps having the same amount of coins; such a move is called **balancing**. No matter how player II moves, the player I may adopt the **winning strategy** to remove the same amount of coins that

player II moves. This guarantees that player I wins the game. If  $n_1 = n_2$ , the game is already balanced, no matter how player I moves at beginning, player II can take the winning strategy to win the game.

For instance,

$$(9, 7) \xrightarrow{I} (7, 7) \xrightarrow{II} (4, 7) \xrightarrow{I} (4, 4) \xrightarrow{II} (3, 4) \xrightarrow{I} (3, 3) \\ \xrightarrow{II} (3, 2) \xrightarrow{I} (2, 2) \xrightarrow{II} (0, 2) \xrightarrow{I} (0, 0).$$

When there are  $k$  heaps of coins, say  $(n_1, n_2, \dots, n_k)$ , we write the numbers  $n_1, n_2, \dots, n_k$  as base 2 numerals:

$$\begin{aligned} n_1 &= a_s a_{s-1} \cdots a_2 a_1 a_0, \\ n_2 &= b_s b_{s-1} \cdots b_2 b_1 b_0, \\ &\vdots \\ n_k &= c_s c_{s-1} \cdots c_2 c_1 c_0. \end{aligned}$$

Dividing each heap into  $s + 1$  sub-heaps (some of them may be zero), the the game becomes a game having total  $k(s + 1)$  sub-heaps as the following:

$$(a_s, \dots, a_1, a_0; b_s, \dots, b_1, b_0; \dots; c_s, \dots, c_1, c_0).$$

A Nim game is said to be **balanced** if all **digit sums**

$$a_s + b_s + \cdots + c_s, \quad \dots, \quad a_1 + b_1 + \cdots + c_1, \quad a_0 + b_0 + \cdots + c_0$$

are even; otherwise, it is said to be **unbalanced**.

When the game is unbalanced, it is always possible to move certain amount of coins in the largest heap so that the game becomes balanced. When the game is unbalanced, player I wins the game since player I can always balance the game. When the game is balanced, player II wins the game.

**Theorem 2.3.** *If the game of Nim is unbalanced, then the player I can always take certain number of coins from one of the heaps to balance the game. More specifically, if  $s$  is the largest digit whose sum  $a_s + b_s + \cdots + c_s$  is odd, then one of the summand is 1, say  $a_s = 1$ ; the player I can move*

$$d = \sum_{i=0}^s \varepsilon_i 2^i$$

coins from the first heap, where

$$\varepsilon_i = \begin{cases} 1 & \text{if the } i\text{th digit sum is odd and } a_i = 1, \\ -1 & \text{if the } i\text{th digit sum is odd and } a_i = 0, \\ 0 & \text{if the } i\text{th digit sum is even.} \end{cases}$$

*Proof.* In the case of unbalanced, say,  $s$  is the largest digit such that

$$a_s + b_s + \cdots + c_s$$

is odd and  $a_s = 1$ , remove  $d$  coins from the first heap to have  $n'_1 := n_1 - d$ , whose binary expression is

$$a'_s a'_{s-1} \dots a'_2 a'_1 a'_0, \quad \text{where } a'_i = a_i - \varepsilon_i.$$

If  $a_i + b_i + \cdots + c_i$  is even, then  $a'_i + b_i + \cdots + c_i = a_i + b_i + \cdots + c_i$  is even. If  $a_i + b_i + \cdots + c_i$  is odd, then

$$a'_i + b_i + \cdots + c_i = (a_i - \varepsilon_i) + b_i + \cdots + c_i = a_i + b_i + \cdots + c_i \mp 1$$

is even. Now the game becomes balanced.

In the case of balanced and  $(n_1, n_2, \dots, n_k) \neq (0, 0, \dots, 0)$ , say,  $n_1 \neq 0$ , removing any  $d$  coins from the 1st heap will result a balanced game. Let  $d$  be written as  $d = \sum_{i=0}^s \delta_i 2^i$ . Let  $r$  be the smallest digit such that  $\delta_r = 1$ . Then  $r$ th digit sum is odd, and the game becomes unbalanced.  $\square$

For example,

Size of heaps	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$
6	0	1	1	0
10	1	0	1	0
13	1	1	0	1
15	1	1	1	1

We can take 6 coins away from the heap 2 to balance the game as

Size of heaps	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$
6	0	1	1	0
4	0	1	0	0
13	1	1	0	1
15	1	1	1	1

The game can be also balanced by removing 10 coins from the heap 3 or removing 14 coins from the heap 4.

### 3 Gambler's Ruin

Two persons A and B gamble dollars on the toss of a fair coin; A has \$70 and B has \$30. In each play either A wins \$1 from B or loss \$1 to B; and the game is played without stop until one wins all the money of the other or goes forever. Find the odds of the following possibilities:

- (a) A wins all the money of B.
- (b) A loses all his money to B.
- (c) The game continues forever.

*Solution.* Either A or B can keep track of the game simply by counting their own money; their position (amount of dollars) can be one of the numbers  $0, 1, 2, \dots, 100$ . Let  $p_n$  be the probability that A reaches 100 at position  $n$ , where  $n = 0, 1, 2, \dots, 99, 100$ ;  $p_0 = 0$  and  $p_{100} = 1$ . After one toss, player A enters into either position  $n + 1$  or position  $n - 1$ , where  $1 \leq n \leq 99$ . The new probability that A reaches 100 is either  $p_{n+1}$  or  $p_{n-1}$ . Since the probability of A moving to position  $n + 1$  or  $n - 1$  from  $n$  is  $50\% = 1/2$ , i.e.,  $p_n = \frac{1}{2}p_{n+1} + \frac{1}{2}p_{n-1}$ , where  $1 \leq n \leq 99$ . Let us consider the sequence  $p_n$  defined by the recurrence relation

$$\begin{cases} p_n &= \frac{1}{2}p_{n+1} + \frac{1}{2}p_{n-1} \\ p_0 &= 0 \\ p_{100} &= 1 \end{cases} \Leftrightarrow \begin{cases} p_{n+1} &= 2p_n - p_{n-1} \\ p_0 &= 0 \\ p_{100} &= 1 \end{cases}$$

The characteristic equation is  $r^2 - 2r + 1 = 0$ ; it has only one root  $r = 1$ . The general solutions is

$$p_n = c_1 + c_2n.$$

Apply the boundary conditions  $p_0 = 0$  and  $p_{100} = 1$ ; we have  $c_1 = 0$  and  $c_2 = \frac{1}{100}$ . Thus

$$p_n = \frac{n}{100}, \quad 0 \leq n \leq 100.$$



Of course,  $p_n = \frac{n}{100}$  for  $n > 100$  is nonsense to the original problem. The probabilities for (a), (b), and (c) are 70%, 30%, and 0, respectively.

The recurrence relation  $p_n = \frac{1}{2}p_{n+1} + \frac{1}{2}p_{n-1}$  can be directly solved. In fact, the recurrence relation can be written as

$$p_{n+1} - p_n = p_n - p_{n-1}.$$

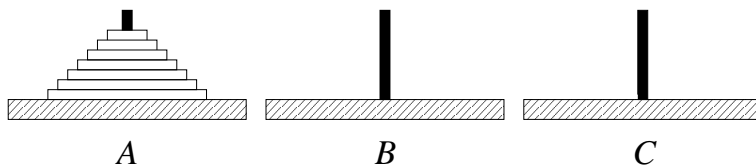
Then  $p_{n+1} - p_n = p_n - p_{n-1} = \cdots = p_1 - p_0$  is constant. Since  $p_0 = 0$ , we have  $p_n = p_{n-1} + p_1$ . Apply the recurrence relation again and again, we obtain

$$p_n = p_0 + np_1.$$

Apply the boundary conditions  $p_0 = 0$ ,  $p_{100} = 1$ ; we see that  $p_1 = \frac{1}{100}$ . Hence  $p_n = \frac{n}{100}$ .

## 4 Hanoi Tower

The game of **Hanoi Tower** is to play with a set of disks of graduated size with holes in their centers and a playing board having three spokes for holding the disks; see Figure 4. The object of the game is to transfer all the disks from spoke A to spoke C by moving one disk at a time without placing a larger disk on top of a smaller one. What is the minimal number of moves required when there are  $n$  disks?



*Solution.* Let  $a_n$  be the minimum number of moves to transfer  $n$  disks from one spoke to another. Then  $\{a_n : n \geq 1\}$  defines an **infinite sequence**. The first few terms of the sequence  $\{a_n\}$  can be listed as

$$1, 3, 7, 15, \dots$$

We are interested in finding a closed formula to compute  $a_n$  for arbitrary  $n$ .

In order to move  $n$  disks from spoke A to spoke C, one must move the first  $n - 1$  disks from spoke A to spoke B by  $a_{n-1}$  moves, then move the last (also the

largest) disk from spoke A to spoke C by one move, and then remove the  $n - 1$  disks again from spoke B to spoke C by  $a_{n-1}$  moves. Thus the total number of moves should be

$$a_n = a_{n-1} + 1 + a_{n-1} = 2a_{n-1} + 1.$$

This means that the sequence  $\{a_n \mid n \geq 1\}$  satisfies the recurrence relation

$$\begin{cases} a_n = 2a_{n-1} + 1, & n \geq 1 \\ a_1 = 1. \end{cases} \quad (13)$$

Given a recurrence relation for a sequence with initial conditions, solving the recurrence relation means to find a formula to express the general term  $a_n$  of the sequence.

For the sequence  $\{a_n \mid n \geq 0\}$  defined by the recurrence relation (13), if we apply the recurrence relation again and again, we have

$$\begin{aligned} a_1 &= 2a_0 + 1 \\ a_2 &= 2a_1 + 1 = 2(2a_0 + 1) + 1 = 2^2a_0 + 2 + 1 \\ a_3 &= 2a_2 + 1 = 2(2^2a_0 + 2 + 1) = 2^3a_0 + 2^2 + 2 + 1 \\ a_4 &= 2a_3 + 1 = 2(2^3a_0 + 2^2 + 2 + 1) = 2^4a_0 + 2^3 + 2^2 + 2 + 1 \\ &\vdots \\ a_n &= 2^n a_0 + 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 = 2^n a_0 + 2^n - 1. \end{aligned}$$

Let  $a_0 = 0$ . The general term is given by

$$a_n = 2^n - 1, \quad n \geq 1.$$

## 5 Marriage Problem