Week 2

1 Pigeonhole Principle: Simple Form

Theorem 1.1. *If $n+1$ objects are put into n boxes, then at least one box contains two or more objects.*

Proof. Trivial. $\square$

Example 1.1. Among any 13 people there are two having their birthdays in the same month.

Example 1.2. There are n married couples. How many of the $2n$ people must be selected in order to guarantee that a married couple is selected?

Other principles related to the pigeonhole principle:

- If n objects are put into n boxes and no box is empty, then each box contains exactly one object.
- If n objects are put into n boxes and no box gets more than one object, then each box has an object.

The abstract formulation of the three principles: Let X and Y be finite sets and let $f : X \to Y$ be a function.

- If X has more elements than Y, then f is not injective (one-to-one).
- If X and Y have the same number of elements and f is surjective (onto), then f is injective (one-to-one).
- If X and Y have the same number of elements and f is injective (one-to-one), then f is surjective (onto).

Example 1.3. In any group of n people there are at least two persons having the same number of friends. (It is assumed that if a person x is a friend of y then y is also a friend of x.)

Proof. The number of friends of a person x is an integer k with $0 \le k \le n-1$. If there is a person y whose number of friends is $n-1$, then everyone is a friend of y, that is, no one has 0 friend. This means that 0 and $n-1$ can not be simultaneously the numbers of friends of some people in the group. The pigeonhole principle tells us that there are at least two people having the same number of friends. □

Example 1.4. Given n integers $a_1, a_2, \dots, a_n$, not necessarily distinct, there exist some consecutive terms whose sum is a multiple of n, i.e., there exist integers k and l with $0 \le k < l \le n$ such that $a_{k+1} + a_{k+2} + \cdots + a_l$ is a multiple of n.

Proof. Consider the n integers

$$a_1, \quad a_1 + a_2, \quad a_1 + a_2 + a_3, \quad \dots, \quad a_1 + a_2 + \cdots + a_n.$$

Dividing these integers by n, we have

$$a_1 + a_2 + \cdots + a_i = q_i n + r_i, \quad 0 \le r_i \le n-1, \quad i = 1, 2, \dots, n.$$

If one of the remainders $r_1, r_2, \dots, r_n$ is zero, say, $r_k = 0$, then $a_1 + a_2 + \cdots + a_k$ is a multiple of n. If none of $r_1, r_2, \dots, r_n$ is zero, then two of them must be the same (since $1 \le r_i \le n-1$ for all i), say, $r_k = r_l$ with $k < l$. This means that the two integers $a_1 + a_2 + \cdots + a_k$ and $a_1 + a_2 + \cdots + a_l$ have the same remainder. Thus $a_{k+1} + a_{k+2} + \cdots + a_l$ is a multiple of n. □

Example 1.5. A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day but, in order not to tire himself, he decides not to play more than 12 games during any calendar week. Show that there exists a succession of consecutive days during which the chess master will have played exactly 21 games.

Proof. Let a_1 be the number of games played on the first day, a_2 the total number of games played on the first and second days, a_3 the total number games played on the first, second, and third days, and so on. Since at least one game is played each day, the sequence of numbers $a_1, a_2, \dots, a_{77}$ is strictly increasing, that is, $a_1 < a_2 < \cdots < a_{77}$ and $a_1 \ge 1$. Since at most 12 games are played during any one week, we have $a_{77} \le 12 \times 11 = 132$. Thus

$$1 \le a_1 < a_2 < \cdots < a_{77} \le 132.$$

Note that the sequence $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ is also strictly increasing, and

$$22 \le a_1 + 21 < a_2 + 21 < \cdots < a_{77} + 21 \le 132 + 21 = 153.$$

Now consider the 154 numbers

$$a_1,\ a_2,\ \ldots,\ a_{77},\ a_1 + 21,\ a_2 + 21,\ \ldots,\ a_{77} + 21;$$

each of them is between 1 and 153. It follows that two of them must be equal. Since $a_1, a_2, \ldots, a_{77}$ are distinct and $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ are also distinct, the two equal numbers must be an a_i and an $a_j + 21$. Since the number of games played up to the ith day is $a_i = a_j + 21$, we conclude that on the days $j + 1$, $j + 2$, $\ldots$, i the chess master played a total of 21 games. □

Example 1.6. Given 101 integers from $1, 2, \ldots, 200$, there are at least two integers such that one of them is divisible by the other.

Proof. By factoring out as many 2's as possible, we see that any integer can be written in the form $2^k \cdot a$, where $k \ge 0$ and a is odd. The number a can be one of the 100 odd numbers $1, 3, 5, \ldots, 199$. Thus among the 101 integers chosen, two of them must have the same a's when they are written in the form, say, $2^r \cdot a$ and $2^s \cdot a$. If $r \le s$, then the first one divides the second. If $r > s$, then the second one divides the first one. □

Example 1.7 (Chinese Remainder Theorem)**.** Let m and n be positive integers and relatively prime, i.e., $\gcd(m, n) = 1$. Then there exist solutions for the system

$$\begin{cases} x \equiv a \pmod{m} \\ x \equiv b \pmod{n} \end{cases}$$

Proof. We may assume that $0 \le a < m$ and $0 \le b < n$. Let us consider the n integers

$$a, \quad m + a, \quad 2m + a, \quad \ldots, \quad (n-1)m + a.$$

Each of these integers has remainder a when divided by m. Suppose that two of them had the same remainder r when divided by n. Let the two numbers be

$im + a$ and $jm + a$, where $0 \le i < j \le n - 1$. Then there are integers q_i, q_j such that

$$im + a = q_i n + r, \quad jm + a = q_j n + r.$$

Subtracting the first equation from the second, we obtain

$$(j - i)m = (q_j - q_i)n.$$

Since $\gcd(m, n) = 1$, we conclude that $n \mid (j - i)$. Note that $0 < j - i \le n - 1$. This is a contradiction. Thus the n integers $a, m + a, 2m + a, \ldots, (n-1)m + a$ have distinct remainders when divided by n. That is, each of the n numbers $0, 1, 2, \ldots, n-1$ occurs as a remainder. In particular, the number b does. Let p be the integer with $0 \le p \le n - 1$ such that the number $pm + a$ has remainder b when divided by n. Then $pm + a = qn + b$ for some integer q. Thus the integer

$$x = pm + a = qn + b$$

has the required property. □

2 Pigeonhole Principle: Strong Form

Theorem 2.1. *Let $q_1, q_2, \ldots, q_n$ be positive integers. If*

$$q_1 + q_2 + \cdots + q_n - n + 1$$

objects are put into n boxes, then either the 1st box contains at least q_1 objects, or the 2nd box contains at least q_2 objects, ..., or the nth box contains at least q_n objects.

Proof. Suppose it is not true, that is, the ith box contains at most $q_i - 1$ objects, $i = 1, 2, \ldots, n$. Then the total number of objects contained in the n boxes can be at most

$$(q_1 - 1) + (q_2 - 1) + \cdots + (q_n - 1) = q_1 + q_2 + \cdots + q_n - n,$$

which is one less than the number of objects distributed. This is a contradiction. □

The simple form of the pigeonhole principle is obtained from the strong form by taking $q_1 = q_2 = \cdots = q_n = 2$. Then

$$q_1 + q_2 + \cdots + q_n - n + 1 = 2n - n + 1 = n + 1.$$

In elementary mathematics the strong form of the pigeonhole principle is most often applied in the special case when $q_1 = q_2 = \cdots = q_n = r$. In this case the principle becomes:

- If $n(r-1)+1$ objects are put into n boxes, then at least one of the boxes contains r or more of the objects.
- If the average of n nonnegative integers $a_1, a_2, \ldots, a_n$ is greater than $r-1$, i.e.,
$$\frac{a_1 + a_2 + \cdots + a_n}{n} > r - 1,$$
then at leats one of the integers is greater than or equal to r.

Example 2.1. A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruit that should be put in the basket in order to guarantee that either there are at least 8 apples or at least 6 bananas or at least 9 oranges?

Answer: $8 + 6 + 9 - 3 + 1 = 21$.

Example 2.2. Given two disks, one is smaller than the other. Each disk is divided into $2N$ congruent sectors. In the larger disk, N sectors are chosen arbitrarily and painted red; the other N sectors are painted blue. In the smaller disk each sector is painted either red or blue with no stipulation on the number of red and blue sectors. The smaller disk is placed on the larger disk so that the centers and sectors coincide. Show that it is possible to align the two disks so that the number of sectors of the smaller disk whose color matches the corresponding sector of the larger disk is at least N.

Proof. We fix the larger disk first, then place the smaller disk on the top of the larger disk so that the centers and sectors coincide. There $2N$ ways to place the smaller disk in such a manner. For each such alignment, some sectors of the two disks may have the same color. Since each sector of the smaller disk will match the same color sector of the larger disk N times among all the $2N$ ways of alignment, and since there are $2N$ sectors in the smaller disk, the total number of matched color sectors among the $2N$ ways of alignment is

$$N \times 2N = 2N^2.$$

Note that there are only $2N$ ways. Then there is at least one way that the number of matched color sectors is $\frac{2N^2}{2N}$ $(= N)$ or more. $\square$

Example 2.3. Show that every sequence $a_1, a_2, \ldots, a_{n^2+1}$ of n^2+1 real numbers contains either an increasing (including equal) subsequence of length $n+1$ or a decreasing subsequence of length $n+1$.

Proof. Assume that there is no increasing subsequence of length $n+1$. We suffices to show that there must be a decreasing subsequence of length $n+1$.

Let ℓ_k be the length of the longest increasing subsequence which begins with a_k, $1 \le k \le n^2+1$. Since it is assumed that there is no increasing subsequence of length $n+1$, we have $1 \le \ell_k \le n$ for all k. By the strong form of the pigeonhole principle (if $n(r-1)+1$ objects are put into n boxes then at leat one of the boxes contains at least r objects; take $r = n+1$), $n+1$ of the n^2+1 integers $\ell_1, \ell_2, \ldots, \ell_{n^2+1}$ must be equal, say,

$$\ell_{k_1} = \ell_{k_2} = \cdots = \ell_{k_{n+1}},$$

where $1 \le k_1 < k_2 < \cdots < k_{n+1} \le n^2+1$. If there is one k_i $(1 \le i \le n)$ such that $a_{k_i} < a_{k_{i+1}}$, then any increasing subsequence of length $\ell_{k_{i+1}}$ beginning with $a_{k_{i+1}}$ will result a subsequence of length $\ell_{k_{i+1}}+1$ beginning with a_{k_i} by adding a_{k_i} in the front; so $\ell_{k_i} > \ell_{k_{i+1}}$, which is contradictory to $\ell_{k_i} = \ell_{k_{i+1}}$. Thus we must have

$$a_{k_1} \ge a_{k_2} \ge \cdots \ge a_{k_{n+1}},$$

which is a decreasing subsequence of length $n+1$. $\square$

3 Ramsey Theory

The following is the most popular and easily understood instance of the Ramsey Theory:

> Of `six` (or more) people, either there are `three` who are mutually acquainted, or there are `three` who are mutually unacquainted.

Let K_n denote the complete graph with n vertices $v_1, \ldots, v_n$, i.e., every pair of vertices are connected by an edge. Given a set V of n elements; we denote by K_V the complete graph whose vertex set is V. We adopt the following notation

$$K_6 \to K_3, K_3 \qquad \text{(read ``}K_6 \text{ implies } K_3, K_3\text{'')}.$$

This means that no matter how the edges of K_6 are colored with two colors, black and white, there is always a black triangle K_3, or there is always a white triangle K_3; in other words there is a monochromatic triangle K_3.

Proof. Let the edges of K_6 be colored either black or white arbitrarily. Consider the vertex v_1 which connects the other 5 vertices $v_2, v_3, \ldots, v_6$ by 5 edges. By the Pigeonhole Principle, among these 5 edges there are at least 3 edges colored black or colored white, say, the three edges incident with v_2, v_3, v_4 are black. If there is one edge colored black between the pairs of v_2, v_3, v_4, say, v_2v_3 is black, then there is already a black triangle K_3 with vertices v_1, v_2, v_3. Otherwise, all edges between the pairs of v_2, v_3, v_4 are colored white; they form a white triangle K_3 already. □

Let us consider the following slightly more general problem

$$K_p \to K_m, K_n,$$

which means the statement: given integers $m, n \geq 1$, if the edges of K_p are colored black or white arbitrarily, then there is either a black K_m (m vertices among them all edges are black) or there is a white K_n (n vertices among them all edges are white). Of course, this may not be true always, for instance, it is obviously not true when $p = 2$, and $m, n \geq 3$. However,

$$\text{If} \quad K_p \to K_m, K_n, \quad \text{then} \quad K_q \to K_m, K_n \quad \text{for all} \quad q \geq p.$$

We shall see that $K_p \to K_m, K_n$ is indeed true when p is large enough. The smallest such integer p is called the *Ramsey number of type* (m, n), denoted $R(m, n)$. We easily see by definition that

$$R(m, n) = R(n, m).$$

Example 3.1. $R(3,3) = 6$. This is the case $m = 3, n = 3$. We have seen that $R(3,3) \le 6$. We need to show that $K_5 \to K_3, K_3$ is not true. It is clear for the following coloring of K_5 (dotted lines represent white edges).

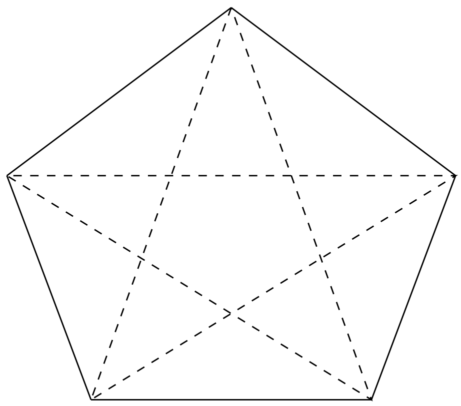

Figure 1: $K_5 \to K_3, K_3$ is not true.

Example 3.2. $R(2,n) = R(n,2) = n$.

Proof. It is clear that $R(1,2) = R(2,1) = 1$.

Assume $n \ge 2$. Consider K_n and have its edges colored black or white arbitrarily. If one of its edges is colored black, then a black K_2 is already found. Otherwise, all edges of K_n are colored white; then K_n itself is already a white K_n. So $R(2,n) \le n$.

Consider K_{n-1} and have its edges colored white. Then there is no black K_2 and there is no white K_n. So $R(2,n) > n-1$. Hence $R(2,n) = n$. □

Instead of using two colors, we may use k colors $c_1, c_2, \ldots, c_k$ with $k \ge 2$. Given integers $n_1, n_2, \ldots, n_k \ge 1$, we consider the problem

$$K_p \to K_{n_1}, K_{n_2}, \ldots, K_{n_k}.$$

This means that if the edges of K_p are colored with colors $c_1, c_2, \ldots, c_k$ arbitrarily, then there is either a K_{n_1} of color c_1 (existing n_1 vertices between the pairs of them all edges are colored c_1), or a K_{n_2} of color c_2 (existing n_2 vertices between the pairs of them all edges are colored c_2), ..., or a K_{n_k} of color c_k (existing n_k vertices between the pairs of them all edges are colored c_k). It is easy to see that

$$\text{If } K_p \to K_{n_1}, K_{n_2}, \ldots, K_{n_k}, \text{ then } K_q \to K_{n_1}, K_{n_2}, \ldots, K_{n_k} \text{ for } q \ge p.$$

Theorem 3.1 (**Ramsey Theorem - Simplified Version**). *Given integers $n_1, n_2, \ldots, n_k \geq 1$. There exists a smallest integer $R(n_1, n_2, \ldots, n_k)$, called the* **Ramsey number of type** $(n_1, n_2, \ldots, n_k)$, *such that if $p \geq R(n_1, n_2, \ldots, n_k)$ then $K_p \to K_{n_1}, K_{n_2}, \ldots, K_{n_k}$.*

It is clear that if σ is a permutation of $\{1, 2, \ldots, k\}$ then

$$R(n_1, n_2, \ldots, n_k) = R\big(n_{\sigma(1)}, n_{\sigma(2)}, \ldots, n_{\sigma(k)}\big).$$

Proposition 3.2. $R(2, n_1, \ldots, n_k)) = R(n_1, \ldots, n_k, 2) = R(n_1, \ldots, n_k)$, $k \geq 1$.

Proof. For $k = 1$, it is easy to see $R(n_1) = n_1$. Then $R(2, n_1) = R(n_1, 2) = n_1 = R(n_1)$ by Example 3.2.

Assume $k \geq 2$, we set $p := R(n_1, \ldots, n_k)$. Let the edges of K_p be colored arbitrarily with $k + 1$ colors $c_1, \ldots, c_{k+1}$. We first claim that $K_p \to K_{n_1}, \ldots, K_{n_k}, K_2$.

If there is one edge of K_p colored c_{k+1}, then a K_2 of color c_{k+1} is found. Otherwise, all edges of K_p are colored with colors $c_1, \ldots, c_k$. Since $p = R(n_1, \ldots, n_k)$, by definition of the Ramsey number there exists either a subgraph K_{n_1} whose edges are colored c_1, or a subgraph K_{n_2} whose edges are colored c_2, ..., or a subgraph K_{n_k} whose edges are colored c_k. We thus have $R(n_1, \ldots, n_k, 2) \leq p$.

Next we claim that $K_{p-1} \not\to K_{n_1}, \ldots, K_{n_k}, K_2$. Since $K_{p-1} \not\to K_{n_1}, \ldots, K_{n_k}$, the edges of K_{p-1} can be colored by $c_1, \ldots, c_k$ such that there is neither a subgraph K_{n_1} of color c_1, nor a subgraph K_{n_2} of color c_2, ..., nor a subgraph K_{n_k} of color c_k. Of course such a k-coloring of K_{p-1} can be viewed as a $(k + 1)$-coloring with colors $c_1, c_2, \ldots, c_{k+1}$ that no edges are colored c_{k+1}. It is clear that within such $(k+1)$-coloring there is neither a subgraph K_{n_1} of color c_1, nor a subgraph K_{n_2} of color c_2, ..., nor a subgraph K_{n_k} of color c_k, nor a subgraph K_2 of color c_{k+1}. Thus $R(n_1, \ldots, n_k, 2) \geq p$.

We have concluded that $R(n_1, \ldots, n_k, 2) = p$. □

W may view each edge of K_p with endpoints u, v as a subset $\{u, v\}$ of two elements. The collection of edges of K_p is the set of all 2-subsets of the vertex set V of K_p, where $|V| = p$. Then $K_p \to K_{n_1}, \ldots, K_{n_k}$ states that if all 2-subsets of V are colored with colors $c_1, \ldots, c_k$ arbitrarily, then there is either an n_1-subset $V_1 \subseteq V$ such that all 2-subsets of V_1 have the color c_1, or an n_2-subset $V_2 \subseteq V$ such that all 2-subsets of V_2 have the color c_2, ..., or an n_k-subset $V_k \subseteq V$ such that all 2-subsets of V_k have the color c_k.

We are not necessarily restrict ourselves to consider only 2-subsets of V. Fix an integer $t \geq 1$ and integers $q_1, \ldots, q_k \geq t$. We may consider the problem:

> *Let all t-subsets of V be colored with colors $c_1, \ldots, c_k$ arbitrarily. There is either a q_1-subset $V_1 \subseteq V$ such that all t-subsets of V_1 are colored c_1, or a q_2-subset $V_2 \subseteq V$ such that all t-subsets of V_2 are colored c_2, ..., or a q_k-subset $V_k \subseteq V$ such that all t-subsets of V_k are colored c_k.*

We use the notation

$$\begin{Bmatrix} p \\ t \end{Bmatrix} \to \begin{Bmatrix} q_1 \\ t \end{Bmatrix}, \begin{Bmatrix} q_2 \\ t \end{Bmatrix}, \ldots, \begin{Bmatrix} q_k \\ t \end{Bmatrix}$$

to denote the statement. It is easy to see that if $q \geq p$, then

$$\begin{Bmatrix} p \\ t \end{Bmatrix} \to \begin{Bmatrix} q_1 \\ t \end{Bmatrix}, \begin{Bmatrix} q_2 \\ t \end{Bmatrix}, \ldots, \begin{Bmatrix} q_k \\ t \end{Bmatrix} \Rightarrow \begin{Bmatrix} q \\ t \end{Bmatrix} \to \begin{Bmatrix} q_1 \\ t \end{Bmatrix}, \begin{Bmatrix} q_2 \\ t \end{Bmatrix}, \ldots, \begin{Bmatrix} q_k \\ t \end{Bmatrix}.$$

Let V be a set of p elements, called a **p-set**. A collection of 2-subsets of V may be viewed as a symmetric binary relation on V. We may consdier *ternary, quaternary, quinary* relations on V. Let $\boldsymbol{P}_t(V)$ denote the set of all t-subsets of V, called a **complete t-family of size** p.

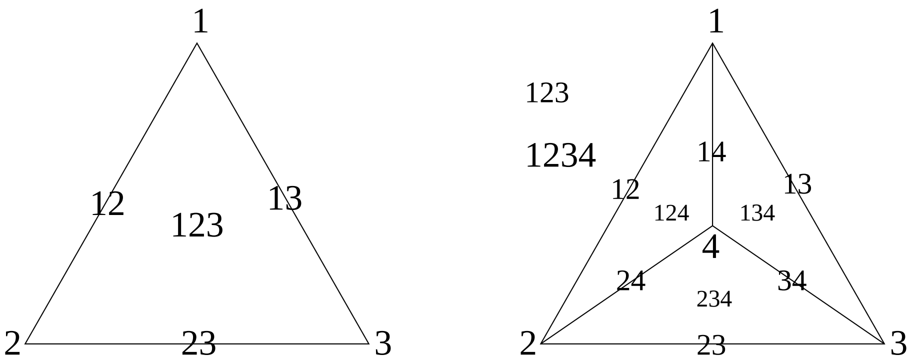

Figure 2: A triangle and a tetrahedron

An *n-simplex* is interpreted as the collection of all nonempty subsets of an $(n+1)$-set. For $n = 0, 1, 2, 3$, they are called *vertex, edge, triangle, and tetrahedron* respectively.

Definition 3.3. Let V be a set of n elements. Given a positive integer t and integers $q_1, \ldots, q_k \geq t$. A k-coloring of $\boldsymbol{P}_t(V)$ by colors $c_1, \ldots, c_k$ is said to satisfy the **Ramsey property of type** $(q_1, \ldots, q_k)$, provided that there exists either a q_1-subset $V_1 \subseteq V$ such that all members of $\boldsymbol{P}_t(V_1)$ are colored c_1, or a q_2-subset $V_2 \subseteq V$ such that all members of $\boldsymbol{P}_t(V_2)$ are colored c_2, ..., or a q_k-subset $V_k \subseteq V$ such that all members of $\boldsymbol{P}_t(V_k)$ are colored c_k.

A complete t-family of size n satisfies the **Ramsey property of type** $(q_1, \ldots, q_k)$ if its every k-coloring satisfies the **Ramsey property of type** $(q_1, \ldots, q_k)$. The smallest number n that its complete t-family satisfies the Ramsey property of type $(q_1, \ldots, q_k)$ is called the **Ramsey number of type** $(q_1, \ldots, q_k)$ **and size** t, denoted

$$R_t(q_1, \ldots, q_k).$$

Ramsey numbers satisfy the following trivial properties:

(a) If σ is a permutation of $\{1, 2, \ldots, k\}$, then

$$R_t(q_1, q_2, \ldots, q_k) = R_t\big(q_{\sigma(1)}, q_{\sigma(2)}, \ldots, q_{\sigma(k)}\big).$$

(b) If a complete t-family of size n satisfies the Ramsey property of type $(q_1, \ldots, q_k)$, then any complete t-family of size $m \geq n$ satisfies the Ramsey property of type $(q_1, \ldots, q_k)$.

(c) If a k-coloring of a complete t-family satisfies the Ramsey property of type $(q_1, \ldots, q_k)$, then the coloring can be considered as a $(k+l)$-coloring, and satisfies the Ramsey property of type $(q_1, \ldots, q_k, p_1, \ldots, p_l)$.

(d) If a k-coloring of a complete t-family does not satisfy the Ramsey property of type $(q_1, \ldots, q_k)$, then the coloring can be considered as a $(k+l)$-coloring, and does not satisfy the Ramsey property of type $(q_1, \ldots, q_k, p_1, \ldots, p_l)$.

Theorem 3.4 (**Ramsey Theorem – Complete Version**). *Given a positive integer t and integers $q_1, \ldots, q_k \geq t$. There exists a smallest integer $R_t(q_1, \ldots, q_k)$ such that if t-subsets of an n-set V are colored with colors $c_1, \ldots, c_k$ arbitrarily, then there is* **either**

(1) a q_1-subset $V_1 \subseteq V$ such that all t-subsets of V_1 are colored c_1, **or**

(2) a q_2-subset $V_2 \subseteq V$ such that all t-subsets of V_2 are colored c_2, **or**

$\ldots\ldots\ldots$, **or**

(k) *a q_k-subset $V_k \subseteq V$ such that all t-subsets of V_k are colored c_k.*

The number $R_t(q_1, \ldots, q_k)$ is called the **Ramsey number of size t and type** $(q_1, \ldots, q_k)$.

In the special case $t = 1$, the set of all 1-subsets of V may be considered as the set V by identifying each 1-subset $\{v\}$ to the element v itself. Then a coloring of all 1-subsets of V is just a coloring of the set V. Given a subset $A \subseteq V$; all 1-subsets of A having the same color means all elements of A are colored the same. Thus $K_p^1 \to K_{q_1}^1, \ldots, K_{q_k}^1$ means that if p elements are colored with k colors $c_1, \ldots, c_k$, then there is either at least q_1 elements colored c_1, or at least q_2 elements colored c_2, ..., or at least q_k elements colored c_k. Thus the Ramsey number of size 1 and of type $(q_1, \ldots, q_k)$ is

$$R_1(q_1, \ldots, q_k) = q_1 + \cdots + q_k - k + 1.$$

Lemma 3.5. *$R_t(q) = q$ and*

$$R_t(t, q_1, \ldots, q_k) = R_t(q_1, \ldots, q_k).$$

Proof. Let V be a q-set and members of $\boldsymbol{P}_t(V)$ be colored into one class $\boldsymbol{C}_1$. If $q \geq q_1 \geq t$, take any q_1-subset $V_1 \subseteq V$, we have $\boldsymbol{P}_t(V_1) \subseteq \boldsymbol{C}_1$. Thus $R_t(q_1) \leq q_1$. Let V be a $(q_1 - 1)$-set and members of $\boldsymbol{P}_t(V)$ be colored into one class $\boldsymbol{C}_1$. There is no q_1-subset of V such that $\boldsymbol{P}_t(V_1) \subseteq \boldsymbol{C}_1$. So $R_t(q_1) > q_1 - 1$, i.e., $R_t(q_1) \geq q_1$. Therefore $R_t(q_1) = q_1$.

Let $k \geq 1$ and set $p := R_t(q_1, \ldots, q_k)$. We claim $R_t(t, q_1, \ldots, q_k) \leq p$. Given a p-set V. It suffices to show that $\boldsymbol{P}_t(V)$ satisfies the Ramsey property of type $(t, q_1, \ldots, q_k)$. Let $\boldsymbol{P}_t(V)$ be colored arbitrarily by $k + 1$ colors $c_0, c_1, \ldots, c_k$.

Case 1: *There exists a member $V_0 \in \boldsymbol{P}_t(V)$ such that V_0 is colored c_0.* Since V_0 is a t-subset of V, we have $\boldsymbol{P}_t(V_0) = \{V_0\}$, then all members of $\boldsymbol{P}_t(V_0)$ are colored c_0. Thus the coloring satisfies the Ramsey property of type $(t, q_1, \ldots, q_k)$.

Case 2: *No members of $\boldsymbol{P}_t(V)$ are colored c_0, i.e., $\boldsymbol{P}_t(V)$ is colored by colors $c_1, \ldots, c_k$.* Now the $(k + 1)$-coloring becomes a k-coloring. Since $|V| = R_t(q_1, \ldots, q_k)$, the k-coloring satisfies the Ramsey property of type $(q_1, \ldots, q_k)$. The corresponding $(k + 1)$-coloring (without using the color c_0) satisfies the Ramsey property of type $(t, q_1, \ldots, q_k)$.

We have shown that an arbitrary $(k+1)$-coloring of $\boldsymbol{P}_t(V)$ satisfies the Ramsey property of type $(t, q_1, \ldots, q_k)$. By definition of Ramsey numbers, $R_t(t, q_1, \ldots, q_k) \leq p$.

Next we claim $R_t(t, q_1, \ldots, q_k) \geq p$.

It suffices to show that there exists a $(p-1)$-set V and a coloring of $\boldsymbol{P}_t(V)$ by colors $c_0, c_1, \ldots, c_k$, such that the $(k+1)$-coloring does not satisfy the Ramsey property of type $(t, q_1, \ldots, q_k)$. Since $p = R_t(q_1, \ldots, q_k)$, there exists a t-family of size $p-1$ such that one of its k-coloring does not satisfy the Ramsey property of type $(q_1, \ldots, q_k)$, i.e., the t-family $\boldsymbol{P}_t(V)$ with $|V| = p-1$ has a k-coloring (using colors $c_1, \ldots, c_k$) which does not satisfy the Ramsey property of type $(q_1, \ldots, q_k)$. Viewing this k-coloring as a $(k+1)$-coloring of $\boldsymbol{P}_t(V)$ by colors $c_0, c_1, \ldots, c_k$ (the color c_0 is not used), then the $(k+1)$-coloring of $\boldsymbol{P}_t(V)$ does not satisfy the Ramsey property of type $(t, q_1, \ldots, q_k)$. $\square$

Theorem 3.6 (**Ramsey Theorem – Special Version**). *Given positive integers $p, q \geq t$. There exists a smallest integer $R_t(p, q)$ such that, if $n \geq R_t(p, q)$, V is an n-set, and if each member of $\boldsymbol{P}_t(V)$ is colored either* **black** *or* **white** *arbitrarily, then there exists either a p-subset $X \subseteq V$ such that all members of $\boldsymbol{P}_t(X)$ are colored* **black***, or a q-subset $Y \subseteq V$ such that all members of $\boldsymbol{P}_t(Y)$ are colored* **white***.*

Proof. We proceed by induction on p, q and t. For $t = 1$ and arbitrary $p, q \geq 1$, we have $r_1(p, q) = p + q - 1$. Moreover, it has been shown in Lemma 3.5 that for all $t \geq 1$,

$$R_t(t, q) = q, \quad R_t(p, t) = p,$$

which are the induction bases.

Now for integers p, q, t such that $p, q \geq t \geq 2$, we show that $R_t(p, q)$ exists under the existence of $R_t(p-1, q)$, $R_t(p, q-1)$, and the existence of $R_{t-1}(a, b)$ for arbitrary integers $a, b \geq t-1$. This is achieved by establishing the following upper bound recurrence relation:

$$R_t(p, q) \leq R_{t-1}(p_1, q_1) + 1,$$

where $p_1 = R_t(p-1, q)$ and $q_1 = R_t(p, q-1)$.

Let V be an n-set with $n \geq R_{t-1}(p_1, q_1) + 1$. Fix an element $v \in V$ and consider the $(n-1)$-set $V' := V \smallsetminus \{v\}$. Note that $|V'| \geq R_{t-1}(p_1, q_1)$. Given an arbitrary 2-coloring $\{\boldsymbol{B}, \boldsymbol{W}\}$ of $\boldsymbol{P}_t(V)$. Let

$$\boldsymbol{B}(V') := \boldsymbol{B} \cap \boldsymbol{P}_t(V'), \quad \boldsymbol{W}(V') := \boldsymbol{W} \cap \boldsymbol{P}_t(V').$$

Then $\{\boldsymbol{B}(V'), \boldsymbol{W}(V')\}$ forms a 2-coloring of $\boldsymbol{P}_t(V')$. Let

$$\begin{aligned} \boldsymbol{B}' &:= \{A \in \boldsymbol{P}_{t-1}(V') \mid A \cup v \in \boldsymbol{B}\}, \\ \boldsymbol{W}' &:= \{A \in \boldsymbol{P}_{t-1}(V') \mid A \cup v \in \boldsymbol{W}\}. \end{aligned}$$

Then $\boldsymbol{B}'$ and $\boldsymbol{W}'$ are disjoint. For each $A \in \boldsymbol{P}_{t-1}(V')$, we have either $A \cup v \in \boldsymbol{B}$ or $A \cup v \in \boldsymbol{W}$; i.e., either $A \in \boldsymbol{B}'$ or $A \in \boldsymbol{W}'$. Thus $\{\boldsymbol{B}', \boldsymbol{W}'\}$ forms a 2-coloring of $\boldsymbol{P}_{t-1}(V')$. Since $|V'| \geq R_{t-1}(p_1, q_1)$, by induction on t there exists

(1) either a p_1-subset $X_1 \subseteq V'$ such that $\boldsymbol{P}_{t-1}(X_1) \subseteq \boldsymbol{B}'$,

(2) or a q_1-subset $Y_1 \subseteq V'$ such that $\boldsymbol{P}_{t-1}(Y_1) \subseteq \boldsymbol{W}'$.

We elaborate each of the two cases.

Case (1): $X_1 \subseteq V'$, $|X_1| = p_1 = R_t(p-1, q)$, and $\boldsymbol{P}_{t-1}(X_1) \subseteq \boldsymbol{B}'$. Since $\{\boldsymbol{B}, \boldsymbol{W}\}$ is a 2-coloring of $\boldsymbol{P}_t(V)$, its restriction to $\boldsymbol{P}_t(X_1)$ induces a 2-coloring $\{\boldsymbol{B}(X_1), \boldsymbol{W}(X_1)\}$ of $\boldsymbol{P}_t(X_1)$, where

$$\boldsymbol{B}(X_1) := \boldsymbol{B} \cap \boldsymbol{P}_t(X_1), \quad \boldsymbol{W}(X_1) := \boldsymbol{W} \cap \boldsymbol{P}_t(X_1).$$

By induction on p (when t is fixed), there exists

- either a $(p-1)$-subset $X_2 \subseteq X_1$ such that $\boldsymbol{P}_t(X_2) \subseteq \boldsymbol{B}(X_1)$ $(\subseteq \boldsymbol{B})$,
- or a q-subset $Y_2 \subseteq X_1$ such that $\boldsymbol{P}_t(Y_2) \subseteq \boldsymbol{W}(X_1)$ $(\subseteq \boldsymbol{W})$.

In the former case, consider the p-subset $X := X_2 \cup v \subseteq V$. For each t-subset $A \in \boldsymbol{P}_t(X)$, if $v \notin A$, then $A \subseteq X_2$, thus $A \in \boldsymbol{P}_t(X_2) \subseteq \boldsymbol{B}$; if $v \in A$, then $A \smallsetminus v$ is a $(t-1)$-subset of X_2 $(\subseteq X_1)$, consequently, $A \smallsetminus v \in \boldsymbol{P}_{t-1}(X_1) \subseteq \boldsymbol{B}'$, thus $A = (A \smallsetminus v) \cup v \in \boldsymbol{B}$ by definition of $\boldsymbol{B}'$. We have a p-subset $X \subseteq V$ such that $\boldsymbol{P}_t(X) \subseteq \boldsymbol{B}$. In the latter case, we already have a q-subset $Y := Y_2 \subseteq V$ such that $\boldsymbol{P}_t(Y) \subseteq \boldsymbol{W}$. We see that the coloring $\{\boldsymbol{B}, \boldsymbol{W}\}$ of $\boldsymbol{P}_t(V)$ satisfies the Ramsey property of type (p, q).

Case (2): $Y_1 \subseteq V'$, $|Y_1| = q_1 = R_t(p, q-1)$, and $\boldsymbol{P}_{t-1}(Y_1) \subseteq \boldsymbol{W}'$. Since $\{\boldsymbol{B}(V'), \boldsymbol{W}(V')\}$ is a 2-coloring of $\boldsymbol{P}_t(V')$, its restriction to $\boldsymbol{P}_t(Y_1)$ induces a 2-coloring $\{\boldsymbol{B}(Y_1), \boldsymbol{W}(Y_1)\}$ of $\boldsymbol{P}_t(Y_1)$, where

$$\boldsymbol{B}(Y_1) := \boldsymbol{B} \cap \boldsymbol{P}_t(Y_1), \quad \boldsymbol{W}(Y_1) := \boldsymbol{W} \cap \boldsymbol{P}_t(Y_1).$$

By induction on q (when t is fixed), there exists

- either a p-subset $X_2 \subseteq Y_1$ such that $\boldsymbol{P}_t(X_2) \subseteq \boldsymbol{B}(Y_1)$ $(\subseteq \boldsymbol{B})$,
- or a $(q-1)$-subset $Y_2 \subseteq Y_1$ such that $\boldsymbol{P}_t(Y_2) \subseteq \boldsymbol{W}(Y_1)$ $(\subseteq \boldsymbol{W})$.

In the former case, we already have a p-subset

$$X := X_2 \subseteq V \text{ such that } \boldsymbol{P}_t(X) \subseteq \boldsymbol{B}.$$

In the latter case, consider the q-subset $Y := Y_2 \cup v \subseteq V$. For each $A \in \boldsymbol{P}_t(Y)$, if $v \notin A$, then $A \in \boldsymbol{P}_t(Y_2) \subseteq \boldsymbol{W}$; if $v \in A$, then $A \smallsetminus v \subseteq Y_2 \subseteq Y_1$, i.e., $A \smallsetminus v \in \boldsymbol{P}_{t-1}(Y_1) \subseteq \boldsymbol{W}'$, thus $A = (A \smallsetminus v) \cup v \in \boldsymbol{W}$ by definition of $\boldsymbol{W}'$. So we have $\boldsymbol{P}_t(Y) \subseteq \boldsymbol{W}$. Now we see that the coloring $\{\boldsymbol{B}, \boldsymbol{W}\}$ of $\boldsymbol{P}_t(V)$ satisfies the Ramsey property of type (p, q).

We have established the upper bound recurrence relation:

$$R_t(p, q) \leq R_{t-1}\big(R_t(p-1, q), R_t(p, q-1)\big) + 1.$$

$\square$

Theorem 3.7 (**Ramsey Theorem - General Version**). *Given positive integers $q_1, q_2, \ldots, q_k \geq t$. There exists a smallest integer $R_t(q_1, q_2, \ldots, q_k)$ such that, if $n \geq R_t(q_1, q_2, \ldots, q_k)$, V is an n-set, and $\boldsymbol{P}_t(V)$ is arbitrarily colored into k color classes $\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_k$, then there is at least one i $(1 \leq i \leq k)$ and a q_i-subset $V_i \subseteq V$ such that $\boldsymbol{P}_t(V_i) \subseteq \boldsymbol{C}_i$.*

Proof. We proceed by induction on k. For $k = 1$, let members of $\boldsymbol{P}_t(V)$ be colored into one class $\boldsymbol{C}_1$. If $n \geq n_1$, take any n_1-subset $V_1 \subseteq V$, we have $\boldsymbol{P}_t(V_1) \subseteq \boldsymbol{C}_1$. Thus $R_t(n_1) \leq n_1$. If $n < n_1$, it is impossible to find an n_1-subset of V. So $R_t(n_1) \geq n_1$. Hence $R_t(n_1) = n_1$. For $k = 2$, it is Theorem 3.6.

For $k \geq 3$, fix integers $q_1, \ldots, q_k$. By induction there exist the Ramsey numbers $q'_{k-1} := R_t(q_{k-1}, q_k)$ and (consequently) $R_t(q_1, \ldots, q_{k-2}, q'_{k-1})$. Let V be an n-set with

$$n \geq R_t(q_1, \ldots, q_{k-2}, q'_{k-1}).$$

Let $\{\boldsymbol{C}_1, \ldots, \boldsymbol{C}_k\}$ be an arbitrary k-coloring of $\boldsymbol{P}_t(V)$. Consider the $(k-1)$-coloring $\{\boldsymbol{D}_1, \ldots, \boldsymbol{D}_{k-1}\}$ of $\boldsymbol{P}_t(V)$, where $\boldsymbol{D}_i = \boldsymbol{C}_i$ for $i = 1, \ldots, k-2$ and $\boldsymbol{D}_{k-1} = \boldsymbol{C}_{k-1} \cup \boldsymbol{C}_k$. By induction there is at least one q_i-subset $V_i \subseteq V$ with $1 \le i \le k-2$ such that $\boldsymbol{P}_t(V_i) \subseteq \boldsymbol{D}_i$ $(= \boldsymbol{C}_i)$, or a q'_{k-1}-subset $V'_{k-1} \subseteq V$ such that $\boldsymbol{P}_t(V'_{k-1}) \subseteq \boldsymbol{D}_{k-1} = \boldsymbol{C}_{k-1} \cup \boldsymbol{C}_k$. In the formal case, nothing is to be proved. In the latter case, let

$$\boldsymbol{D}'_{k-1} := \boldsymbol{P}_t(V'_{k-1}) \cap \boldsymbol{C}_{k-1}, \quad \boldsymbol{D}'_k := \boldsymbol{P}_t(V'_{k-1}) \cap \boldsymbol{C}_k.$$

Then $\{\boldsymbol{D}'_{k-1}, \boldsymbol{D}'_k\}$ is a 2-coloring of $\boldsymbol{P}_t(V'_{k-1})$. Since $|V'_{k-1}| = q'_{k-1} = R_t(q_{k-1}, q_k)$, there exists either a q_{k-1}-subset $V_{k-1} \subseteq V'_{k-1}$ such that $\boldsymbol{P}_t(V_{k-1}) \subseteq \boldsymbol{D}'_{k-1}$ $(\subseteq \boldsymbol{C}_{k-1})$ or a q_k-subset $V_k \subseteq V'_{k-1}$ such that $\boldsymbol{P}_t(V_k) \subseteq \boldsymbol{D}'_k$ $(\subseteq \boldsymbol{C}_k)$. Thus there exists either a q_{k-1}-subset $V_{k-1} \subseteq V$ such that $\boldsymbol{P}_t(V_{k-1}) \subseteq \boldsymbol{C}_{k-1}$ or a q_k-subset $V_k \subseteq V$ such that $\boldsymbol{P}_t(V_k) \subseteq \boldsymbol{C}_k$.

Summarizing the above argument we obtain the upper bound recurrence relation:

$$\begin{aligned} R_t(q_1, q_2, \ldots, q_k) &\le R_t\big(q_1, q_2, \ldots, q_{k-2}, q'_{k-1}\big) \\ &= R_t\big(q_1, q_2, \ldots, q_{k-2}, R_t(q_{k-1}, q_k)\big). \end{aligned} \tag{1}$$

□

Proposition 3.8 (Strong Form of Pigeonhole Principle)**.** *If $t = 1$, then the Ramsey number $r_1(q_1, q_2, \ldots, q_k)$ is the smallest integer n such that if the elements of an n-set are colored with k colors $c_1, c_2, \ldots, c_k$, then either there are q_1 elements of color c_1, or q_2 elements of color c_2, ..., or q_k elements of color c_k. Moreover,*

$$R_1(q_1, q_2, \ldots, q_k) = q_1 + q_2 + \cdots + q_k - k + 1.$$

The known Ramsey numbers:

$$R_2(3,3) = 6, \quad R_2(3,4) = R_2(4,3) = 9, \quad R_2(3,5) = R_2(5,3) = 14(or 13?),$$

$$R_2(3,6) = R_2(6,3) = 18, \quad R_2(3,7) = R_2(7,3) = 23,$$

$$R_2(3,8) = r_2(8,3) = 28, \quad R_2(3,9) = R_2(9,3) = 36,$$

$$R_2(4,4) = 18, \quad R_2(4,5) = R_2(5,4) = 25.$$

There are estimates: $40 \le R_2(3,10) = R_2(10,3) \le 43, \quad 43 \le R_2(5,5) \le 49.$

Example 3.3. Consider $V = \{1, 2, 3, 4, 5\}$. Then

$$\boldsymbol{P}_3(V) = \{123, 124, 125, 134, 135, 145, 234, 235, 245, 345\}$$

Many 2-colorings of $\boldsymbol{P}_3(V)$ does not satisfy the Ramsey property of type $(4, 4)$. For instance, the 2-coloring $\{\boldsymbol{C}_1, \boldsymbol{C}_2\}$, where

$$\boldsymbol{C}_1 = \{123, 124, 125, 134, 135\} \quad \text{and} \quad \boldsymbol{C}_2 = \{145, 234, 235, 245, 345\},$$

does not satisfy the Ramsey property of type $(4, 5)$, since none of

$$\boldsymbol{P}_3(1234), \quad \boldsymbol{P}_3(1235), \quad \boldsymbol{P}_3(1245), \quad \boldsymbol{P}_3(1345), \quad \boldsymbol{P}_3(2345)$$

is contained in $\boldsymbol{C}_1$, and $\boldsymbol{P}(12345)$ is not contained in $\boldsymbol{C}_2$. So $r_3(4, 5) > 5$.

The 2-coloring $\{\boldsymbol{C}_1, \boldsymbol{C}_2\}$ of the following does not satisfy the Ramsey property of type $(4, 4)$, where

$$\boldsymbol{C}_1 = \{123, 124, 125, 134\} \quad \text{and} \quad \boldsymbol{C}_2 = \{135, 145, 234, 235, 245, 345\}.$$

So $r_3(4, 4) > 5$. However, the 2-coloring $\{\boldsymbol{C}_1, \boldsymbol{C}_2\}$, where

$$\boldsymbol{C}_1 = \{123, 124, 134, 234, 125\} \quad \text{and} \quad \boldsymbol{C}_2 = \{135, 145, 235, 245, 345\},$$

does satisfy the Ramsey property of type $(4, 4)$, since $\boldsymbol{P}_3(1234)$ is contained in $\boldsymbol{C}_1$. The Ramsey number $r_3(4, 4)$ can be estimated as follows: Since $r_2(4, 4) = 18$, we have

$$r_3(4, 4) \leq r_2(r_3(3, 4), r_3(4, 3)) + 1 = r_2(4, 4) + 1 = 18 + 1 = 19.$$

So $6 \leq r_3(4, 4) \leq 19$.

4 Applications of the Ramsey Theorem

Theorem 4.1. *For positive integers $q_1, \ldots, q_k$ there exists a smallest positive integer $R(q_1, \ldots, q_k)$ such that, if $n \geq R(q_1, \ldots, q_k)$ and for any edge coloring of the complete graph K_n with k colors $c_1, \ldots, c_k$, there is at least one i $(1 \leq i \leq k)$ such that K_n contains a complete subgraph K_{q_i} of the color c_i.*

Proof. Each edge of K_n can be considered as a 2-subset of its vertices. □

Theorem 4.2 (Erdös-Szekeres)**.** *For any integer $k \geq 3$ there exists a smallest integer $N(k)$ such that, if $n \geq N(k)$ and for any n points on a plane having no three points through a line, then there is a convex k-gon whose vertices are among the given n points.*

Before proving the theorem we prove the following two lemmas.

Lemma 4.3. *Among any 5 points on a plane, no three points through a line, 4 of them must form a convex quadrangle.*

Proof. Join every pair of two points by a segment to have a configuration of 10 segments. The circumference of the configuration forms a convex polygon. If the convex polygon is a pentagon or a quadrangle, the problem is done. Otherwise the polygon must be a triangle, and the other two points must be located inside the triangle. Draw a straight line through the two points; two of the three vertices must be located in one side of the straight line. The two vertices on the same side and the two points inside the triangle form a quadrangle. □

Lemma 4.4. *Given $k \geq 4$ points on a plane, no 3 points through a line. If any 4 points are vertices of a convex quadrangle, then the k points are actually the vertices of a convex k-gon.*

Proof. Join every pair of two points by a segment to have a configuration of $k(k-1)/2$ segments. The circumference of the configuration forms a convex l-polygon. If $l = k$, the problem is solved. If $l < k$, there must be at least one point inside the l-polygon. Let $v_1, v_2, \ldots, v_l$ be the vertices of the convex l-polygon, and draw segments between v_1 and $v_3, v_4, \ldots, v_{l-1}$ respectively. The point inside the convex l-polygon must be located in one of the triangles $\triangle v_1v_2v_3, \triangle v_1v_3v_4, \ldots, \triangle v_1v_{l-1}v_l$. Obviously, the three vertices of the triangle with a given point inside together with the given point do not form a convex quadrangle. This is a contradiction. □

Proof of Theorem 4.2. We apply the Ramsey theorem to prove Theorem 4.2. For $k = 3$, it is obviously true. Now for $k \geq 4$, if $n \geq r_4(k, 5)$, we divide the 4-subsets of the n points into a class $\mathcal{C}$ of 4-subsets whose points are vertices of a convex quadrangle, and another class $\mathcal{D}$ of 4-subsets whose points are not vertices of any convex quadrangle. By the Ramsey theorem, there is either k points whose any 4-subset belongs to $\mathcal{C}$, or 5 points whose any 4-subset belongs to $\mathcal{D}$. In the formal case, the problem is solved by Lemma 4.4. In the latter case, it is impossible by Lemma 4.3. □

Theorem 4.5 (Schur)**.** *For any positive integer k there exists a smallest integer N_k such that, if $n \geq N_k$ and for any k-coloring of $[1, n]$, there is a monochromatic sequence $x_1, x_2, \ldots, x_l$ $(l \geq 2)$ such that $x_l = \sum_{i=1}^{l-1} x_i$.*

Proof. Let $n \geq R(l, \ldots, l)$ and let $\{A_1, \ldots, A_k\}$ be a k-coloring of $[1, n]$. Let $\{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$ be a k-coloring of $P_2([1, n])$ defined by

$$\{a, b\} \in \mathcal{C}_i \quad \text{if and only if} \quad |a - b| \in A_i, \quad \text{where} \quad 1 \leq i \leq k.$$

By the Ramsey theorem, there is one r $(1 \leq r \leq k)$ and an l-subset $A = \{a_1, a_2, \ldots, a_l\} \subseteq [1, n]$ such that $P_2(A) \subseteq \mathcal{C}_r$. We may assume $a_1 < a_2 < \cdots < a_l$. Then

$$\{a_i, a_j\} \in \mathcal{C}_r \quad \text{and} \quad a_j - a_i \in A_r \quad \text{for all } i < j.$$

Let $x_i = a_{i+1} - a_i$ for $1 \leq i \leq l - 1$ and $x_l = a_l - a_1$. Then $x_i \in A_r$ for all $1 \leq i \leq l$ and $x_l = \sum_{i=1}^{l-1} x_i$. □

5 Van der Waerden Theorem

The Van der Waerden theorem states that for any k-coloring of $\mathbb{Z}_+$, the set of positive integers, there always exists a monochromatic arithmetic progression of arbitrary length. An **arithmetic progression** (AP) (or **arithmetic sequence**) is a finite sequence of numbers such that the difference between the consecutive terms is constant. For instance, $3, 7, 11, 15, 19, 23, 27, 31, 35$ is an arithmetic progression of difference 4 and length 9. The following statements are equivalent.

Theorem 5.1. *(a) If $\mathbb{Z}_+ = C_1 \cup C_2 \cup \cdots \cup C_k$, then some C_i contains arbitrarily long arithmetic progression.*

(b) Given positive integers k and l. There exists a constant $N(k, l)$, known as **Van der Waerden number***, such that if $n \geq N(k, l)$ and*

$\{1, 2, \ldots, n\} \subseteq C_1 \cup C_2 \cup \cdots \cup C_k$, then some C_i contains an arithmetic progression of length l.

(c) Let $a_0, a_1, a_2, \ldots$ be an infinite sequence such that $0 < a_{i+1} - a_i < r$ for all i, where r is a fixed number. Then the sequence contains arbitrarily long arithmetic progressions.

(d) For integers $k, r \geq 1$, there exists a constant $M(k, r)$ such that if $m \geq M(k, l)$ and $a_1, \ldots, a_m$ is a sequence satisfying $0 < a_{i+1} - a_i \leq r$ for all i, then k of the numbers $a_1, \ldots, a_m$ are in arithmetic progression.

Let $[a, b]$ denote the set of integers x such that $a \leq x \leq b$. Two tuples $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ of $[1, l]^m$ are said to be l-*equivalent*, written

$$(x_1, \ldots, x_m) \sim (y_1, \ldots, y_m),$$

if all entries before the last l in each tuple are the same. For instance, for $l = 5$ and $m = 4$,

$$(3, 5, 2, 5) \sim (3, 5, 2, 5), \quad (2, 4, 5, 2) \sim (2, 4, 5, 4),$$

$$(4, 3, 1, 4) \sim (2, 3, 2, 1), \quad (3, 5, 5, 1) \not\sim (3, 5, 2, 4).$$

Obviously, l-equivalence is an equivalence relation on $[0, l]^m$. All tuples of $[0, l-1]^m$ are l-equivalent.

Definition 5.2. For integers $l, m \geq 1$, let $A(l, m)$ denote the statement: For any integer $k \geq 1$ there exists a smallest integer $N(l, m, k)$ such that, if $n \geq N(l, m, k)$ and $[1, n]$ is k-colored, then there are integers $a, d_1, d_2, \ldots, d_m \geq 1$ such that $a + l \sum_{i=1}^{m} d_i \leq n$ and for each l-equivalence class E of $[0, l]^m$,

$$\left\{ a + \sum_{i=1}^{m} x_i d_i : (x_1, \ldots, x_m) \in E \right\}$$

is monochromatic (having the same color).

When $m = 1$, there are only two l-equivalence classes for $[0, l]^m$, i.e.,

$$\{0, 1, 2, \ldots, l-1\} \quad \text{and} \quad \{l\}.$$

The statement $A(l, 1)$ means that for any integer $k \geq 1$ there exists a smallest integer $N(l, 1, k)$ such that, if $n \geq N(l, 1, k)$ and $[1, n]$ is k-colored, then there are integers $a, d \geq 1$ such that $a + ld \leq n$ and the sequence, $a, a + d, a + 2d, \ldots, a + (l-1)d$, is monochromatic.

Theorem 5.3 (Graham-Rothschild)**.** *The statement $A(l, m)$ is true for all integers $l, m \geq 1$.*

Proof. We proceed induction on l and m. For $l = m = 1$, the 1-equivalence classes of $[0, 1]$ are $\{0\}$ and $\{1\}$; the statement $A(1, 1)$ states that for any integer $k \geq 1$ there exists a smallest integer $N(1, 1; k)$ such that, if $n \geq N(1, 1; k)$ and $[1, n]$ is k-colored, then there are integers $a, d \geq 1$ such that $a+d \leq n$, and both $\{a\}$ and $\{a+d\}$ are monochromatic. This is obviously true and $N(1, 1; k) = 2$. We divide the induction argument into two statements:

(I) *If $A(l, m)$ is true for some $m \geq 1$ then $A(l, m+1)$ is true.*

(II) *If $A(l, m)$ is true for all $m \geq 1$ then $A(l+1, 1)$ is true.*

The induction goes as follows: The truth of $A(1, 1)$ implies the truth of $A(1, m)$ for all $m \geq 1$ by (I). Then by (II) the statement $A(2, 1)$ is true. Again by (I) the statement $A(2, m)$ is true for all $m \geq 1$. Continuing this procedure we obtain that $A(l, m)$ is true for all $l, m \geq 1$.

First Proof. Let the integer $k \geq 1$ be fixed. Since $A(l, m)$ is true, the integer $N(l, m, k)$ exists and set $p = N(l, m, k)$. Since $A(l, 1)$ is true, the integer $N(l, 1, k^p)$ exists and we set $q = N(l, 1, k^p)$, $N = pq$. Let $\phi : [1, N] \longrightarrow [1, k]$ be a k-coloring of $[1, N]$. Let $\psi : [1, q] \longrightarrow [1, k]^p$ be a k^p-coloring of $[1, q]$ defined by

$$\psi(i) = \Big(\phi((i-1)p+1),\ \phi((i-1)p+2),\ \ldots,\ \phi((i-1)p+p)\Big), \quad 1 \leq i \leq q. \tag{2}$$

Since $A(l, 1)$ is true, then for the k^p-coloring ψ of $[1, q]$ there are integers $a, d \geq 1$ such that

$$a + ld \leq q$$

and

$$\{a + xd : x = 0, 1, 2, \ldots, l-1\}$$

is monochromatic, i.e.,

$$\psi(a + xd) = \text{constant}, \quad x = 0, 1, 2, \ldots, l-1. \tag{3}$$

Note that $[(a-1)p+1, ap] \subseteq [1, pq]$ because $a \leq q$. Since $A(l, m)$ is true, then when ϕ is restricted to the p-set $[(a-1)p+1, ap]$ there are integers $b, d_1, d_2, \ldots, d_m \geq 1$ such that

$$(a-1)p+1 \leq b, \quad b + l\sum_{i=1}^{m} d_i \leq ap,$$

and for each l-equivalence class E of $[0, l]^m$,

$$\left\{ b + \sum_{i=1}^{m} x_i d_i : (x_1, \ldots, x_m) \in E \right\}$$

is monochromatic, i.e.,

$$\phi\left(b + \sum_{i=1}^{m} x_i d_i\right) = \text{constant}, \quad (x_1, \ldots, x_m) \in E. \tag{4}$$

Recall that our job is to prove that $A(l, m+1)$ is true. For the k-coloring ϕ of $[1, N]$, we have had the integers

$$b, d_1, d_2, \ldots, d_{m+1} \geq 1, \quad \text{where} \quad d_{m+1} = dp.$$

Since $b + l\sum_{i=1}^{m} d_i \leq ap$ and $a + ld \leq q$, we have

$$b + l\sum_{i=1}^{m+1} d_i \leq ap + ldp = (a+dl)p \leq pq = N.$$

Now for any two l-equivalent tuples $(x_1, \ldots, x_{m+1})$ and $(y_1, \ldots, y_{m+1})$ of $[0, l]^{m+1}$, consider the numbers

$$\alpha = b + \textstyle\sum_{i=1}^{m+1} x_i d_i, \;\; \beta = b + \sum_{i=1}^{m+1} y_i d_i,$$

$$\alpha_0 = b + \textstyle\sum_{i=1}^{m} x_i d_i, \;\; \beta_0 = b + \sum_{i=1}^{m} y_i d_i.$$

Notice that our job is to show that α and β have the same color, i.e., $\phi(\alpha) = \phi(\beta)$. We divide the job into three cases:

Case 1. $x_{m+1} = y_{m+1} = l$. Then $x_i = y_i$ for all $1 \leq i \leq m$. Thus $\alpha = \beta$, and obviously, $\phi(\alpha) = \phi(\beta)$.

Case 2. $x_{m+1} = l$ and $y_{m+1} \leq l-1$, or, $x_{m+1} \leq l-1$ and $y_{m+1} = l$. This implies that $(x_1, \ldots, x_{m+1})$ and $(y_1, \ldots, y_{m+1})$ are not l-equivalent. This is a contradiction.

Case 3. $x_{m+1}, y_{m+1} \in [0, l-1]$. Then $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ are l-equivalent. It follows from (3) that $\psi(a) = \psi(a + x_{m+1}d)$, and by definition (2) of ψ, the corresponding coordinates of $\psi(a)$ and $\psi(a + x_{m+1}d)$ are equal, i.e.,

$$\phi\left((a-1)p + i\right) = \phi\left((a + x_{m+1}d - 1)p + i\right), \quad i = 1, 2, \ldots, p.$$

Since $(a-1)p + 1 \leq b \leq \alpha_0 \leq b + l\sum_{i=1}^{m} d_i \leq ap = (a-1)p + p$, there exists $j \in [1, p]$ such that $\alpha_0 = (a-1)p + j$. We then have

$$\alpha = \alpha_0 + x_{m+1}dp = (a-1)p + j + x_{m+1}dp = (a + x_{m+1}d - 1)p + j.$$

Thus

$$\phi(\alpha) = \phi\left((a + x_{m+1}d - 1)p + j\right) = \phi\left((a-1)p + j\right) = \phi(\alpha_0).$$

Similarly, $\phi(\beta) = \phi(\beta_0)$. Since $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ are l-equivalent, it follows from (4) that $\phi(\alpha_0) = \phi(\beta_0)$. Therefore $\phi(\alpha) = \phi(\beta)$. This means that $A(l, m+1)$ is true and

$$N(l, m+1, k) \leq N(l, m, k) \cdot N\left(l, 1, k^{N(l,m,k)}\right).$$

Second Proof. Fix an integer $k \geq 1$. Since $A(l, m)$ is true for all $m \geq 1$, the statement $A(l, k)$ is true and $N(l, k, k)$ exists. Let $N = 2N(l, k, k)$ and let ϕ be a k-coloring of $[1, N]$. Notice that the restriction of ϕ on $[1, N(l, k, k)]$ is a k-coloring. Then there are integers $a, d_1, d_2, \ldots, d_k \geq 1$ such that

$$a + l\sum_{i=1}^{k} d_i \leq N(l, k, k),$$

and for l-equivalent tuples $(x_1, \ldots, x_k), (y_1, \ldots, y_k) \in [0, l]^k$,

$$\phi\left(a + \sum_{i=1}^{k} x_i d_i\right) = \phi\left(a + \sum_{i=1}^{k} y_i d_i\right).$$

Consider the $k+1$ tuples (none of them are l-equivalent)

$$(0,0,\ldots,0),\ (l,0,\ldots,0),\ (l,l,\ldots,0),\ \ldots,\ (l,l,\ldots,l)$$

of $[0,l]^k$ to have $k+1$ distinct integers

$$a,\ a+ld_1,\ a+l(d_1+d_2),\ \ldots,\ a+l(d_1+d_2+\cdots+d_k).$$

At least two of them, say $a+l(d_1+\cdots+d_\lambda)$ and $a+l(d_1+\cdots+d_\mu)$ with $\lambda<\mu$, must have the same color, i.e.,

$$\phi\left(a+l\sum_{i=1}^{\lambda}d_i\right)=\phi\left(a+l\sum_{i=1}^{\mu}d_i\right). \tag{5}$$

For any $x\in[0,l-1]$, the two tuples

$$(\underbrace{l,\ldots,l}_{\lambda},\underbrace{x,\ldots,x}_{\mu-\lambda},0,\ldots,0)\quad\text{and}\quad(\underbrace{l,\ldots,l}_{\lambda},\underbrace{0,\ldots,0}_{\mu-\lambda},0,\ldots,0)$$

of $[0,l]^k$ are l-equivalent. Thus the numbers $a+l\sum_{i=1}^{\lambda}d_i+x\sum_{i=\lambda+1}^{\mu}d_i$ for $x\in[0,l-1]$ have the same color by ϕ, i.e.,

$$\phi\left(a+l\sum_{i=1}^{\lambda}d_i+x\sum_{i=\lambda+1}^{\mu}d_i\right)=\text{constant},\quad x=0,1,2,\ldots,l-1.$$

Combining this with (5) we have

$$\phi\left(a+l\sum_{i=1}^{\lambda}d_i+x\sum_{i=\lambda+1}^{\mu}d_i\right)=\text{constant},\quad x=0,1,2,\ldots,l-1,l.$$

Recall that our job is to prove the truth of $A(l+1,1)$. Let $b=a+l\sum_{i=1}^{\lambda}d_i$ and $d=\sum_{i=\lambda+1}^{\mu}d_i$. Then we have had the integers $b,d\geq 1$ such that

$$b+(l+1)d=a+l\sum_{i=1}^{\lambda}d_i+(l+1)\sum_{i=\lambda+1}^{\mu}d_i=a+l\sum_{i=1}^{\mu}d_i+\sum_{i=\lambda+1}^{\mu}d_i\leq N(l,k,k)+N(l,k,$$

and for the k-coloring ϕ of $[1,N]$, the l-equivalence class $\{0,1,2,\ldots,l\}$ of $[0,l+1]^1$ have the same color, i.e.,

$$\phi(b+xd)=\text{constant},\quad x=0,1,2,\ldots,l.$$

This means that the statement $A(l+1,1)$ is true. $\square$

The truth of $A(l, m)$ for $m = 1$ is called the **Van der Waerden theorem**.

Corollary 5.4 (Van der Waerden Theorem). *For any positive integers k and l there exists a smallest integer $N(l, k)$ such that, if $n \geq N(l, k)$ and $[1, n]$ is k-colored, then there is a monochromatic arithmetic sequence of length l in $[1, n]$.*

Supplementary Exercises

1. For the game of Nim, let us restrict that each player can move one or two coins. Find the winning strategy for each player.

2. Let n be a positive integer. In the game of Nim let us restrict that each player can move only $i \in \{1, 2, \ldots, n\}$ coins each time from one heap. Find the winning strategy for each player.

3. Given $m(m-1)^2 + 1$ integral points on a plane, where m is odd. Show that there exists m points whose center is also an integral point.