BIJECTIVE COUNTING

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1. Binomial and Multinomial Coefficients

Definition 1.1. *An* r**-permutation** *of* n *objects is a linearly ordered selection of* r *objects from an* n*-set. The number of* r*-permutations of* n *objects is denoted by*

$$P(n, r).$$

An n*-permutation of* n *objects is just called a* **permutation** *of the* n *objects. The number of permutations of* n *objects is denoted by* $n!$*, read "*n *factorial".*

Definition 1.2. *An* r**-combination** *of* n *objects is a selection of* r *objects from a set of* n *objects without order. The number of* r*-combinations of* n *objects is denoted by*

$$\binom{n}{r},$$

read "n *choose* r*." These numbers are called* **binomial coefficients** .

Definition 1.3. *An* r**-combination with repetition of** n **objects** *is a selection of* r *objects from a set of* n *objects without order and objects can be selected repeatedly. The number of* r*-combinations of* n *objects with repetition allowed is denoted by*

$$\left\langle {n \atop r} \right\rangle,$$

read "n *choose* r *with repetition."*

For sake of brevity, we frequently call a set with n objects an n-set, and a subset with r objects of any set an r-subset. Elements of a set are always considered to be distinct. When considering indistinguishable objects we need the concept of multisets. By a **multiset** we mean a collection of objects such that some of them may be identically same, said to be **indistinguishable**. Given a set S; by a **multiset** M over S we mean a function $v : S \to \mathbb{N} = \{0, 1, 2, \ldots\}$, written $M = (S, v)$; the **cardinality** of $M = (S, v)$ is

$$|M| = \sum_{x \in S} v(x);$$

if $|M| = n$, we call M an n-multiset. For example, $M = \{a, a, b, b, b, c, c, e\}$ is an 8-multiset over $S = \{a, b, c, d, e\}$ with $v(a) = 2$, $v(b) = 3$, $v(c) = 2$, $v(d) = 0$, $v(e) = 1$. An n-multiset M over a k-set S is said to be of **type** $(r_1, \ldots, r_k)$ or an $(r_1, \ldots, r_k)$**-multiset**, if the ith object of S appears r_i times in M, $1 \le i \le k$. A **submultiset** of $M = (S, v)$ is a multiset $L = (S, u)$ such that $u(x) \le v(x)$ for all $x \in S$.

The number of permutations of an n-multiset of type $(r_1, \ldots, r_k)$ is denoted by

$$\binom{n}{r_1, \ldots, r_k},$$

called a **multinomial coefficient** of type $(n; r_1, \ldots, r_k)$. See (5) of Proposition 1.5.

Proposition 1.4. (1) *The number of r-permutations of n objects is given by*

$$P(n,r) = n(n-1)\cdots(n-r+1).$$

(2) *The number of r-combinations of n objects is given by*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

(3) *The number of permutations of an n-multiset of type $(r_1, \ldots, r_k)$ is the same as the number of ways to partition an n-set into k subsets of cardinalities $r_1, \ldots, r_k$, and is given by*

$$\binom{n}{i_1, \ldots, i_k} = \frac{n!}{r_1! \cdots r_k!}.$$

(4) *The number of n-combinations of r objects with repetition allowed equals the number of non-negative integer solutions of $x_1 + \cdots + x_r = n$, and is given by*

$$\left\langle {r \atop n} \right\rangle = \binom{n+r-1}{n}.$$

Proposition 1.5. (1) *The Pascal identity: $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$.*

(2) *An recurrence relation: $\binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}$.*

(3) *The Vandermonde convolution: $\binom{m+n}{r} = \sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i}$*

(4) *The binomial expansion: $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$.*

(5) *The multinomial expansion: $(x_1 + \cdots + x_k)^n = \sum_{i_1+\cdots+i_k=n,\, i_1,\ldots,i_k \geq 0} \binom{n}{i_1,\ldots,i_k} x_1^{i_1} \cdots x_k^{i_k}$.*

Proof. (1) Let $A_n = \{a_1, \ldots, a_n\}$ be an n-set and $A_{n-1} = \{a_1, \ldots, a_{n-1}\}$. The r-subsets of A_n are divided into two types: (i) r-subsets of A_{n-1}; and (ii) r-subsets of A_n, but not subsets of A_{n-1}. There are $\binom{n-1}{r}$ r-subsets of type (i). Each r-subset of type (ii) must contain the element a_n; and each such r-subset can be obtained by selecting an $(r-1)$-subsets of A_{n-1} first then adding the element a_n to it. Thus there are $\binom{n-1}{r-1}$ r-subsets of type (ii). Adding the number of r-subsets of two types, we have $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$.

(2) Let $A_i = \{a_1, \ldots, a_i\}$, $1 \leq i \leq n+1$. Then $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_{n+1}$. For each $(r+1)$-subset $S \subseteq A_{n+1}$, there exists a unique k $(r \leq k \leq n)$ such that $S \not\subseteq A_k$ and $S \subseteq A_{k+1}$. Thus $a_{k+1} \in S$ and $S' = S - \{a_{k+1}\}$ is an r-subset of A_k. Of course, each such r-subset $S' \subseteq A_k$ $(r \leq k \leq n)$ produces a unique $(r+1)$-subset $S = S' \cup \{a_{k+1}\}$ of A_{n+1}. Therefore $\binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}$.

(3) Let A be a set of m black balls and B a set of n white balls. Let $S = A \cup B$. Each r-subset of S is divided into a unique i-subset of A and a unique $(r-i)$-subset of B, and vice versa. The identity follows from the counting in two different ways. □

Proposition 1.6. (1) $\left\langle {n \atop m} \right\rangle = \left\langle {n \atop m-1} \right\rangle + \left\langle {n-1 \atop m} \right\rangle$.

(2) $\left\langle {n+1 \atop m} \right\rangle = \sum_{k=0}^{m} \left\langle {n \atop k} \right\rangle$.

Proof. (1) Let $A_n = \{a_1, \ldots, a_n\}$. Each m-multiset M of A_n either contains the element a_n or does not contain a_n. If M contains a_n, then $M \smallsetminus \{a_n\}$ is an $(m-1)$-multiset over A_n, and there are $\left\langle {n \atop m-1} \right\rangle$ such m-multisets. If M does not contain a_n, then M is an m-multiset of A_{n-1}, and there are $\left\langle {n-1 \atop m} \right\rangle$ such m-multisets.

(2) For each m-multiset M of $A_{n+1} = \{a_1, \ldots, a_{n+1}\}$, let k be the number of times that the element a_n appears in M. Clearly, $0 \leq k \leq m$. Deleting all multiple copies of a_n in M we obtain an $(m-k)$-multiset of A_n. Thus $\left\langle {n+1 \atop m} \right\rangle = \sum_{k=0}^{m} \left\langle {n \atop m-k} \right\rangle$. □

2. Counting of Functions

Given sets M and N, we have the following classes of functions from M to N.

$$\begin{aligned} \mathrm{Map}(M,N) &= \{f : M \to N\}, \\ \mathrm{Inj}(M,N) &= \{f : M \to N \mid f \text{ is injective}\}, \\ \mathrm{Sur}(M,N) &= \{f : M \to N \mid f \text{ is surjective}\}, \\ \mathrm{Bij}(M,N) &= \{f : M \to N \mid f \text{ is bijective}\}. \end{aligned}$$

Whenever M, N are linearly ordered sets, we say that a function $f : M \to N$ is **monotonic** provided that $x \leq y$ in M implies $f(x) \leq f(y)$ in N. We have the class of functions

$$\mathrm{Mon}(M,N) = \{f : M \to N \mid f \text{ is monotonic}\}.$$

Proposition 2.1. *Let M and N be finite sets with cardinalities $|M| = m$ and $|N| = n$. Then*

(1) $|\mathrm{Map}(M,N)| = n^m$;

(2) $|\mathrm{Inj}(M,N)| = n(n-1)\cdots(n-m+1)$;

(3) $|\mathrm{Sur}(M,N)| = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^m$;

(4) $|\mathrm{Bij}(M,N)| = \begin{cases} n! & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$

Proof. The cases (1), (2), (4) are obvious. The case (3) follows from the inclusion-exclusion principle. In fact, let $N = \{b_1, \ldots, b_n\}$. For $1 \leq k \leq n$, we have

$$\left|\mathrm{Map}\big(M, \{b_{i_1}, \ldots, b_{i_k}\}\big)\right| = k^m \quad \text{for} \quad i_1 < \cdots < i_k.$$

Then

$$\begin{aligned} |\mathrm{Sur}(M,N)| &= \left| \mathrm{Map}(M,N) \smallsetminus \bigcup_{i=1}^{n} \mathrm{Map}\big(M, N \smallsetminus \{b_i\}\big) \right| \\ &= n^m - \sum_{k=1}^{n} (-1)^{k+1} \sum_{i_1 < \cdots < i_k} \left|\mathrm{Map}\big(M, N \smallsetminus \{b_{i_1}, \ldots, b_{i_k}\}\big)\right| \\ &= n^m + \sum_{k=1}^{n} (-1)^k \binom{n}{k} (n-k)^m = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^m. \end{aligned}$$

□

Note: The power set $\mathcal{P}(N)$ of N is a poset under the partial order of inclusion. Then

$$|\mathrm{Map}(M,T)| = \sum_{S \subseteq T} |\mathrm{Sur}(M,S)|, \quad T \subseteq N.$$

By the Möbius inversion, we have

$$|\mathrm{Sur}(M,T)| = \sum_{S\subseteq T} (-1)^{|T\smallsetminus S|} |\mathrm{Map}(M,S)|, \quad T\subseteq N.$$

Definition 2.2. *The* **falling factorial** *of length n is*

$$[x]_{(n)} = x(x-1)\cdots(x-n+1), \quad n\geq 1$$

and $[x]_{(0)} = 1$. *The* **rising factorial** *of length n is expression*

$$[x]^{(n)} = x(x+1)\cdots(x+n-1), \quad n\geq 1$$

and $[x]^{(0)} = 1$.

Proposition 2.3. (Reciprocity Law) *For integers $n\geq 1$,*

$$[-x]_{(n)} = (-1)^n [x]^{(n)}, \tag{2.1}$$

$$[-x]^{(n)} = (-1)^n [x]_{(n)}. \tag{2.2}$$

Proposition 2.4. *Let N and X be linearly ordered finite sets with cardinalities $|N| = n$ and $|X| = x$. Then*

$$|\mathrm{Mon}(N,X)| = \frac{[x]^{(n)}}{n!}. \tag{2.3}$$

Proof. Let $N = \{1,2,\ldots,n\}$, $X = \{1,2,\ldots,x\}$, $Y = \{1,2,\ldots,x+n-1\}$, and be linearly ordered by the natural order of numbers. Consider the map $\Phi : \mathrm{Mon}(N,X) \to \binom{Y}{n}$, defined for $f\in \mathrm{Mon}(N,X)$ by

$$\Phi(f) = \big\{f(1), f(2)+1, \ldots, f(n)+n-1\big\},$$

where $\binom{Y}{n}$ is the set of all n-subsets of Y. It is easy to see that Φ is a bijection. The inverse of Φ is given by

$$\Phi^{-1}\big(\{y_1,\ldots,y_n\}\big)(i) = y_i - i + 1, \quad 1\leq i\leq n;$$

where $\{y_1,\ldots,y_n\}$ is an n-subset of Y with $y_1 < \cdots < y_n$. Thus

$$\mathrm{Mon}(N,X)| = \binom{x+n-1}{n} = \frac{(x+n-1)(x+n-2)\cdots(x+1)x}{n!},$$

which is the form $[x]^{(n)}/n!$. □

Let M, N be either sets whose objects are distinguishable or multisets whose objects are indistinguishable, having cardinalities $|M| = m$, $|N| = n$. We use 'D' and 'I' to indicate distinguishability and indistinguishability respectively. A function from N to M can be considered as distributing objects of N into boxes indexed by the members of M. A function from N to M can be also considered as selecting $|N|$ objects from M, with repetition allowed, and put them into boxes indexed by members of N so that each box contains exactly one object.

If N is indistinguishable and M is distinguishable, then there are $\left\langle {m \atop n} \right\rangle$ ways to select n objects from M with repetition allowed, and there is only one way to put them into boxes indexed by the members of N; so $|\mathrm{Map}(N,M)| = \left\langle {m \atop n} \right\rangle$.

If N is distinguishable and M is indistinguishable, then each function from N to M is a distribution of N into identical boxes, which induces a partition of N, and the number of parts ranges from 1 to m.

If both N, M are indistinguishable, then a function from N to M is a partition of n identical objects into some nonempty parts, which is a partition of the integer n, and the number of parts ranges from 1 to m.

Let $S_{n,k}$ denote the number of partitions of an n-set into k parts. Let $P_k(n)$ denote the number of partitions of the integer n with k parts. We have the following table.

N	M	Map	Inj ($n \le m$)	Sur ($n \ge m$)	Bij ($n = m$)
D	D	m^n	$[m]_{(n)}$	$m!S_{n,m}$	$n!$
I	D	$\left\langle {m \atop n} \right\rangle$	$\binom{m}{n}$	$\left\langle {m \atop n-m} \right\rangle$	1
D	I	$\sum_{k=1}^{m} S_{n,k}$	$S_{n,n} = 1$	$S_{n,m}$	1
I	I	$\sum_{k=1}^{m} P_k(n)$	$P_n(n) = 1$	$P_m(n)$	1

3. Counting of Permutations

A **permutation** of an n-set $[n] = \{1, 2, \ldots, n\}$ is a bijection $\sigma : N \to N$, written

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

For simplicity, we frequently write σ as a word $s_1 s_2 \ldots s_n$, where $s_i = \sigma(i)$, $1 \le i \le n$. For each $i \in [n]$, the sequence $i, \sigma(i), \sigma^2(i), \sigma^3(i), \ldots$ must return to i for some terms. Let $\ell_i = \ell_i(\sigma)$ be the smallest integer such that $\sigma^{\ell_i}(i) = i$. We call the sequence

$$\big(i \;\; \sigma(i) \;\; \sigma^2(i) \;\; \cdots \;\; \sigma^{\ell_i - 1}(i)\big)$$

a **cycle** of the permutation σ and ℓ_i (the number of elements in the cycle) the **cycle length**. Since $\sigma^{\ell_i}(\sigma^j(i)) = \sigma^{j+\ell_i}(i) = \sigma^j(i))$, one can write the above cycle by starting any element $\sigma^j(i)$ with $0 \le j \le \ell_i - 1$. We require to write the cycle so that the leading element is largest, and to write the whole permutation σ in increasing order of the leading elements of its cycles; such a writing is called the **standard cycle notation** of σ, denoted $\operatorname{cyc}(\sigma)$. For instance, the standard cycle notation of the permutation 857162394 of $\{1, 2 \ldots, 9\}$ is

$$\operatorname{cyc}(857162394) = (625)(73)(9418).$$

If we delete the parenthesis in $\operatorname{cyc}(\sigma)$, we obtain a permutation $\hat{\sigma} = \widehat{\operatorname{cyc}}(\sigma)$. For instance,

$$\widehat{\operatorname{cyc}}(857162394) = 625739418,$$

whose standard cycle notation is (2)(53)(74)(9816). We denote by $\mathfrak{S}_n$ the **symmetric group** of all permutations of $\{1, 2, \ldots, n\}$.

Proposition 3.1. *The map* $\widehat{\mathrm{cyc}} : \mathfrak{S}_n \to \mathfrak{S}_n$ *is a bijection.*

Proof. It is clear that the map $\widehat{\mathrm{cyc}}$ is well-defined. We claim that $\widehat{\mathrm{cyc}}$ is surjective. For each permutation $t_1 t_2 \ldots t_n$, we construct a permutation σ such that $\widehat{\mathrm{cyc}}(\sigma) = t_1 t_2 \ldots t_n$. In fact, the standard representation of σ can be obtained by inserting parentheses into $t_1 t_2 \ldots t_n$ as follows: First write a left parenthesis to the left of t_1 and a right parenthesis to the right of t_n. If $t_i < t_j$ for all $i < j$, where $j \neq 1$, write a right and a left parentheses)(between t_{j-1} and t_j to have $(t_1 \ldots t_{j-1})(t_j \ldots t_n)$. Continue this procedure for $(t_j \ldots t_n)$. Alternatively, one can define the map σ as follows:

$$\sigma(t_j) = \begin{cases} t_{j+1} & \text{if there exists an } i \leq j \text{ s.t. } t_i > t_{j+1}, \\ t & \text{if } t_i \leq t_{j+1} \text{ for all } i \leq j, \end{cases}$$

where t is the unique element of the singleton $\{t_1, \ldots, t_j\} \smallsetminus \{\sigma(t_1), \ldots, \sigma(t_{j-1})\}$. Now it forces that the surjective map $\widehat{\mathrm{cyc}}$ is bijective, for $\mathfrak{S}_n$ is finite. □

A permutation σ of $[n]$ is said to be of **cycle-type** $1^{\lambda_1} 2^{\lambda_2} \ldots n^{\lambda_n}$, if it has

$$\begin{aligned} &\lambda_1(\sigma) \text{ cycles of length } 1, \\ &\lambda_2(\sigma) \text{ cycles of length } 2, \\ &\qquad \cdots\cdots\cdots \\ &\lambda_n(\sigma) \text{ cycles of length } n; \end{aligned}$$

the cycle-type of σ is denoted by

$$\mathrm{type}(\sigma) = 1^{\lambda_1(\sigma)} 2^{\lambda_2(\sigma)} \ldots n^{\lambda_n(\sigma)}.$$

Clearly,

$$\sum_{i=1}^{n} i\, \lambda_i(\sigma) = n.$$

Proposition 3.2. *The number of permutations of an n-set of type* $1^{\lambda_1} 2^{\lambda_2} \ldots n^{\lambda_n}$*, where* $\sum_{i=1}^{n} i\, \lambda_i = n$*, is given by*

$$\frac{n!}{1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n} (\lambda_1!)(\lambda_2!) \cdots (\lambda_n!)}. \tag{3.1}$$

Proof. Let $\mathfrak{S}(1^{\lambda_1} 2^{\lambda_2} \ldots n^{\lambda_n})$ denote the set of permutations of type $1^{\lambda_1} 2^{\lambda_2} \ldots n^{\lambda_n}$. Let P_i denote a linearly ordered λ_i pairs of parentheses, each pair of parentheses contains i linearly ordered positions. Then there are n linearly ordered positions in the arrangement $P = P_1 P_2 \ldots P_n$. For each permutation $\sigma = s_1 s_2 \ldots s_n$, let $\Phi(\sigma)$ denote the placement of the n elements $s_1, s_2, \ldots, s_n$ placed into the n positions of P in the same order. Then $\Phi(\sigma)$ defines a permutation of type $1^{\lambda_1} 2^{\lambda_2} \ldots n^{\lambda_n}$, which may not be in standard cycle representations. Thus $\Phi : \mathfrak{S}_n \to \mathfrak{S}(1^{\lambda_1} 2^{\lambda_2} \ldots n^{\lambda_n})$ defines a surjective map.

Notice that each filled pair of parentheses with i positions has i representations; and there are λ_i such pairs of parentheses. Then there are $i^{\lambda_i}(\lambda_i!)$ ways to rearrange the elements in P_i to have the same λ_i cycles of length i. Since the rearrangements in $P_1, \ldots, P_n$, respectively, are independent, it follows that the fiber of Φ over each member of $\mathfrak{S}(1^{\lambda_1} 2^{\lambda_2} \ldots n^{\lambda_n})$ has the cardinality

$$1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n} (\lambda_1!)(\lambda_2!) \cdots (\lambda_n!).$$

The formula (3.1) follows immediately. □

A **partition** of a set S is a collection of disjoint nonempty subsets whose union is S. For an n-set S, a partition of S is said to be of **type** $1^{\lambda_1}2^{\lambda_2}\ldots n^{\lambda_n}$ if the number of i-subsets of the partition is λ_i. Clearly, we have $\sum_{i=1}^{n} i\,\lambda_i = n$.

Proposition 3.3. *The number of partitions of an n-set of type $1^{\lambda_1}2^{\lambda_2}\ldots n^{\lambda_n}$ is*

$$\frac{n!}{(1!)^{\lambda_1}(2!)^{\lambda_2}\cdots(n!)^{\lambda_n}(\lambda_1!)(\lambda_2!)\cdots(\lambda_n!)}. \tag{3.2}$$

Proof. Let $\mathrm{Par}(1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n})$ be the set of all partitions of an n-set N of type $1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}$. Let B_i denote a linearly ordered λ_i boxes, each box contains i linearly ordered positions. There are total n positions in the arrangement $B = B_1B_2\cdots B_n$. For each permutation $\sigma = s_1s_2\ldots s_n$, let $\Psi(\sigma)$ denote the placement of the n elements $s_1, s_2, \ldots, s_n$ placed into the n positions of B in the same order. Then $\Psi(\sigma)$ defines a partition of type $1^{\lambda_1}2^{\lambda_2}\ldots n^{\lambda_n}$. Thus $\Psi : \mathfrak{S}_n \to \mathrm{Par}(1^{\lambda_1}2^{\lambda_2}\ldots n^{\lambda_n})$ defines a surjective map. Notice that the i elements in each box with i positions can be arranged in $i!$ ways; and there are λ_i such boxes. Then there are $(i!)^{\lambda_i}(\lambda_i!)$ ways to rearrange the elements in B_i to have the same λ_i i-subsets. Since the rearrangements in $B_1, \ldots, B_n$, respectively, are independent, it follows that the fiber of Ψ over any element of $\mathrm{Par}(1^{\lambda_1}2^{\lambda_2}\ldots n^{\lambda_n})$ has the cardinality

$$(1!)^{\lambda_1}(2!)^{\lambda_2}\cdots(n!)^{\lambda_n}(\lambda_1!)(\lambda_2!)\cdots(\lambda_n!).$$

The formula (3.2) follows immediately. □

Definition 3.4. *The number of permutations of an n-set with exactly k cycles is denoted by $c_{n,k}$, where $n \geq k \geq 0$ and $c_{0,0} = 1$.*

Proposition 3.5. *The numbers $c_{n,k}$ satisfy the recurrence relation*

$$\begin{cases} c_{0,0} = c_{n,n} = 1 & \text{for} \quad n \geq 0 \\ c_{n,0} = 0 & \text{for} \quad n \geq 1 \\ c_{n+1,k} = c_{n,k-1} + nc_{n,k} & \text{for} \quad n \geq k \geq 1 \end{cases} \tag{3.3}$$

Proof. The initial conditions are obvious.

We consider the set of $c_{n+1,k}$ permutations of an $(n+1)$-set N with k cycles. Fix an element $w \in N$ and divide permutations of N into two kinds:

(i) Permutations where (w) is a cycle of length 1. There are $c_{n,k-1}$ such permutations.

(ii) Permutations where w is contained in a cycle of length at least 2. Such permutations can be obtained from permutations of $N \smallsetminus \{w\}$ with k cycles by inserting the element w into one of the k cycles, and there are exactly n independent ways of making the insertion. So there are $nc_{n,k}$ such permutations. □

Theorem 3.6. $\sum_{k=0}^{n} c_{n,k}x^k = x(x+1)(x+2)\cdots(x+n-1)$.

Proof. Let x be a positive integer and let $C(\sigma)$ denote the set of cycles of a permutation σ of $[n]$ in standard cycle notation. The left-hand side counts all pairs (σ, f), where σ is a permutation of $[n]$ and f is a function from $C(\sigma)$ to $[x]$. The right-hand side counts the integer sequences $(a_1, a_2, \ldots, a_n)$, where $1 \leq a_i \leq x + n - i$. We define a map $(a_1, \ldots, a_n) \mapsto (\sigma, f)$ as follows:

(1) Write down the number n and regard it as a cycle $C = (n)$. Let $\sigma = C$ and define $f(C) = a_n$.

(2) Whenever $i+1, i+2, \ldots, n$ have been inserted into the cycles of σ, consider to insert i into σ. There are two situations:
   (a) If $a_i \in [1, x]$, start a new cycle $C' = (i)$ with the element i to the left of existing cycles of σ, and define $f(C') = a_i$.
   (b) If $a_i = x+k \in [x+1, x+n-i]$ with $1 \le k \le n-i$, insert i into a cycle of σ so that it appears to the right of exactly k previously inserted elements. (The $n-i$ numbers $a_{i+1}, \ldots, a_n$ were inserted previously.)

It follows that a_i is the leading element in a cycle C_i of σ iff $a_i \in [1, x]$, $f(C_i) = a_i$, and if $a_i \in [x+1, x+n-i]$, then i is placed in σ such that there are exactly k elements larger than and to the right of i.

For example, for $n = 9$, $x = 5$, $(a_1, \ldots, a_9) = (6, 9, 10, 1, 6, 8, 4, 6, 3)$, the permutation σ and the function f can be constructed as the following:

(9)	$a_9 = 3 \in [1,5]$	$f(9) = a_9 = 3$
(98)	$a_8 = 6 = 5+1$ $\in [6, 5+1] = [6,6]$	$f(98) = 3$
$(7)(98)$	$a_7 = 4 \in [1,5]$	$f(7) = a_7 = 4$ $f(98) = 3$
$(7)(986)$	$a_6 = 8 = 5+3$ $\in [6, 5+3] = [6,8]$	$f(7) = 4$ $f(986) = 3$
$(75)(986)$	$a_5 = 6 = 5+1$ $\in [6,9]$	$f(75) = 4$ $f(986) = 3$
$(4)(75)(986)$	$a_4 = 1 \in [1,5]$	$f(4) = a_4 = 1$ $f(75) = 4$ $f(986) = 3$
$(4)(75)(9836)$	$a_3 = 10 = 5+5$ $\in [6, 5+6] = [6,11]$	$f(4) = 1$ $f(75) = 4$ $f(9836) = 3$
$(4)(75)(92836)$	$a_2 = 9 = 5+4$ $\in [6, 5+7] = [6,12]$	$f(4) = 1$ $f(75) = 4$ $f(92836) = 3$
$(41)(75)(92836)$	$a_1 = 6 = 5+1$ $\in [6, 5+8] = [6,13]$	$f(41) = 1$ $f(75) = 4$ $f(92836) = 3$

It is clear that the map is injective. In fact, for $(a_1, \ldots, a_n) \neq (a'_1, \ldots, a'_n)$, there exists an index j such that $a_j \neq a'_j$ and $a_i = a'_i$ for all $i < j$. If both $a_j, a'_j \in [1, x]$, then $f \neq f'$, since the values of f, f' at the cycle of σ, σ' with the leading term j are a_j, a'_j respectively; otherwise, $\sigma \neq \sigma'$, since the numbers of terms on the left side of and larger than j in σ, σ' respectively are distinct.

For surjectivity, for a pair (σ, f) of permutation σ and function $f : C(\sigma) \to [1, x]$, let $(\sigma, f) \mapsto (a_1, \ldots, a_n)$ be defined by

$$a_i = \begin{cases} f(C) & \text{if } i \text{ is the leading term of a cycle } C \text{ of } \sigma, \\ x+k & \text{otherwise, where } k \text{ is the number of terms} \\ & \text{on the left-sdie of and larger than } i. \end{cases}$$

□

Exercise 1. Find the inverse map $(\sigma, f) \mapsto (a_1, \dots, a_n)$ explicitly, letting that σ be written in the standard cycle notation and the values of f be given on cycles.

Let $\sigma = s_1 s_2 \dots s_n$ be a permutation of $[n]$. An **inversion** of σ is a pair (s_i, s_j) such that $i < j$ but $s_i > s_j$. For each $k \in [n]$, let a_k denote the number of terms that precede k in $s_1 s_2 \dots s_n$ and are greater than k, i.e.,

$$a_k := \#\big\{s_i \mid s_i > s_j = k,\, i < j\big\} = \#\big\{\sigma(i) \mid \sigma(i) > \sigma(j) = k,\, i < j\big\}.$$

It measures how much k is out of order by counting number of integers larger than k but located before k. The tuple $(\sigma) := (a_1, \dots, a_n)$ is called the **inversion sequence** (or **inversion table**) of σ, and the sum

$$\operatorname{inv}(\sigma) := a_1 + \cdots + a_n$$

is called the **inversion number** of σ, measuring the total disorder of σ. Clearly, $0 \le a_i \le n - i$.

Proposition 3.7. *Let $n \ge k \ge 1$. Then $c_{n,k}$ counts the number of integer sequences $(a_1, \dots, a_n)$ such that $0 \le a_i \le n - i$ and exactly k values of a_i equal to 0.*

Proof. Let $x = 1$ in Theorem 3.6. The function f in the pair (σ, f) is a constant function. Then $1 \le a_i \le x + n - i$ becomes $1 \le a_i \le n - i + 1$, which can be equivalently reduced to $0 \le a_i \le n - i$ by shifting the values by 1 unit. Note that a_i produces a cycle if and only if $a_i \in [1, x] = [1, 1]$, i.e., $a_i = 1$, equivalently, $a_i = 0$ after shifting by 1. Hence, permutations with k cycles correspond to inversion sequences $(a_1, \dots, a_n)$ having exactly k values of a_i equal to 0. □

Corollary 3.8. *The map $\mathfrak{S}_n \to \prod_{i=1}^{n} [0, n-i] \cap \mathbb{Z}$, sending each permutation σ to its inversion sequence, is a bijection.*

Proposition 3.9. *The inversion generating polynomial has the factorization*

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{inv}(\sigma)} = \prod_{i=1}^{n-1} \left(1 + q + \cdots + q^i\right).$$

Proof. Note that $\operatorname{inv}(\sigma) = a_1 + \cdots + a_n$ for each permutation σ with inversion table $(a_1, \dots, a_n)$. We have

$$\begin{aligned} \sum_{\sigma \in S_n} q^{\operatorname{inv}(\sigma)} &= \sum_{a_1=0}^{n-1} \sum_{a_2=0}^{n-2} \cdots \sum_{a_n=0}^{0} q^{a_1 + a_2 + \cdots + a_n} \\ &= \left(\sum_{a_1=0}^{n-1} q^{a_1}\right)\left(\sum_{a_2=0}^{n-2} q^{a_2}\right) \cdots \left(\sum_{a_n=0}^{0} q^{a_n}\right). \end{aligned}$$

□

Given a permutation $\sigma = s_1 s_2 \dots s_n$ of $[n]$. The **descent set** of σ is the set

$$\operatorname{Des}(\sigma) := \big\{i \in [n] \mid s_i > s_{i+1}\big\}; \tag{3.4}$$

its cardinality $\operatorname{des}(\sigma) := |\operatorname{Des}(\sigma)|$ is called the **descent** of σ. Some authors include n into the set $\operatorname{Des}(\sigma)$ by saying that σ goes down from s_n to zero at position n. We do not include n in $\operatorname{Des}(\sigma)$. Likewise, the **ascent set** of σ is the set

$$\operatorname{Asc}(\sigma) := \big\{i \in [n] \mid s_i < s_{i+1}\big\}; \tag{3.5}$$

its cardinality $\mathrm{asc}(\sigma) := |\mathrm{Asc}(\sigma)|$ is called the **ascent** of σ. Clearly, we have

$$0 \leq \mathrm{des}(\sigma) \leq n-1, \quad 0 \leq \mathrm{asc}(\sigma) \leq n-1.$$

We introduce two integer-valued functions α, β on the power set $2^{[n]}$ of $[n]$ as follows: For each subset $S \subseteq [n]$,

$$\alpha(S) = \#\big\{\sigma \in \mathfrak{S}_n : \mathrm{Des}(\sigma) \subseteq S\big\},$$
$$\beta(S) = \#\big\{\sigma \in \mathfrak{S}_n : \mathrm{Des}(\sigma) = S\big\}.$$

Clearly, we have

$$\alpha(T) = \sum_{S \subseteq T} \beta(S), \quad T \subseteq [n].$$

It is equivalent to (by the Möbius inversion)

$$\beta(T) = \sum_{S \subseteq T} (-1)^{|T \setminus S|} \alpha(S), \quad T \subseteq [n].$$

Proposition 3.10. *Let $a_1, \ldots, a_k$ be nonnegative integers such that $a_1 + \cdots + a_k = n$ and $S = \{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_k\}$. Then*

$$\alpha(S) = \binom{n}{a_1, a_2, \ldots, a_k}.$$

Proof. For simplicity we may assume $a_i \geq 1$. We count all permutations $\sigma = s_1 s_2 \ldots s_n$ such that $\mathrm{Des}(\sigma) \subseteq S$, i.e.,

$$s_1 < s_2 < \cdots < s_{a_1} > s_{a_1+1},$$
$$s_{a_1+1} < s_{a_1+a_2+2} < \cdots < s_{a_1+a_2} > s_{a_1+a_2+1},$$
$$\cdots\cdots\cdots$$
$$s_{a_1+\cdots+a_{k-1}+1} < s_{a_1+\cdots+a_{k-1}+2} < \cdots < s_{a_1+\cdots+a_k} = s_n.$$

We choose $s_1 < \cdots < s_{a_1}$ in $\binom{n}{a_1}$ ways; then choose $s_{a_1+1} < \cdots < s_{a_1+a_2}$ in $\binom{n-a_1}{a_2}$ ways; and so on. We thus have

$$\begin{aligned} \alpha(S) &= \binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-a_1-\cdots-a_{k-1}}{a_k} \\ &= \binom{n}{a_1, a_2, \ldots, a_k}. \end{aligned}$$

It is easily modified to the case of some $a_i = 0$. □

Definition 3.11. *The* **Eulerian polynomial** *is the generating polynomial*

$$A_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{des}(\sigma)} = \sum_{k=0}^{n} A_{n,k} x^k, \tag{3.6}$$

whose coefficients $A_{n,k}$ are **Eulerian numbers***, counting the number of n-permutations with k descents. We assume $A_{0,0} = 1$.*

Proposition 3.12. *The Eulerian numbers satisfy the symmetric property:*

$$A_{n,k} = A_{n,n-k-1}$$

and the recurrence relation:

$$\begin{cases} A_{0,0} = 1, & \\ A_{n,n} = 0,\ A_{n,0} = A_{n,n-1} = 1 & \text{for} \quad n \geq 1 \\ A_{n,k} = (k+1)A_{n-1,k} + (n-k)A_{n-1,k-1} & \text{for} \quad n > k \geq 1 \end{cases}$$

Proof. The map $\mathfrak{S}_n \to \mathfrak{S}_n$, $s_1 \dots s_n \mapsto t_1 \dots t_n$ with $t_i = n - s_i$, is a bijection, sending an n-permutation with exactly k descents to an n-permutation with exactly k ascents. The number of n-permutations with k descents is the same as the number of n-permutations with exactly k ascents.

Given a permutation $\sigma = s_1 \dots s_n$ and $i \in [n-1]$, we have either a descent $s_i > s_{i+1}$ or an ascent $s_i > s_{i+1}$. So σ has exactly k descent iff it has exactly $(n-1) - k$ ascents. It follows that $A_{n,k} = A_{n,n-k-1}$.

Permutations of $[n]$ with k descending positions can be obtained as follows: (i) each permutation σ of $[n-1]$ with k descending positions produces exactly k permutations of $[n]$ with k descending positions by inserting n behind each of the k descending positions of σ, plus one more by placing n rightmost; (ii) each permutation σ of $[n-1]$ with $k-1$ descending positions produces $(n-1)-(k-1)$ permutations of $[n]$ with k descending positions by inserting n anywhere (total $n-1$ positions, left sides of members of $[n-1]$) but not behind each of the $k-1$ descending positions of σ. It is clear that permutations of $[n]$ obtained in (i) and (ii) are distinct; and each permutation of $[n]$ with k descending positions can be obtained in this way. □

Proposition 3.13 (Worpitzky Identity)**.** *The* **Euler numbers** $A_{n,k}$ *satisfy the relation:*

$$x^n = \sum_{k=0}^{n-1} A_{n,k} \binom{x+k}{n} = \sum_{k=0}^{n-1} \frac{A_{n,k}}{n!} [x+k]_{(n)}. \tag{3.7}$$

Proof. Let I_n denote the right-hand side of (3.7). We show the identity by induction on n. For $n = 0, 1$, it is easily verified to be true. Now for $n+1$, we have

$$\begin{aligned} I_{n+1} &= A_{n,0}\binom{x}{n+1} + \sum_{k=1}^{n}\Big((k+1)A_{n,k} + (n+1-k)A_{n,k-1}\Big)\binom{x+k}{n+1} \\ &= \sum_{k=0}^{n} A_{n,k}\binom{x+k}{n}\cdot\frac{(k+1)(x+k-n)}{n+1} \\ &\qquad + \sum_{k=1}^{n} A_{n,k-1}\binom{x+k-1}{n}\cdot\frac{(n+1-k)(x+k)}{n+1} \\ &= \sum_{k=0}^{n-1} A_{n,k}\binom{x+k}{n}\cdot\frac{(k+1)(x+k-n)+(n-k)(x+k+1)}{n+1} \\ &= x\sum_{k=0}^{n-1} A_{n,k}\binom{x+k}{n} = x^{n+1}. \end{aligned}$$

□

Exercise 2. For $n \geq 0$,

$$\sum_{k=1}^{\infty} k^n x^k = \frac{1}{(1-x)^{n+1}} \sum_{j=0}^{n-1} A_{n,j} x^{j+1}.$$

For $n = 0$, we have

$$\text{LHS} = \sum_{k\geq 1} x^k = \frac{x}{1-x}, \quad \text{RHS} = \frac{A_{0,0}x}{1-x} = \frac{x}{1-x}.$$

For $n = 1$, we have

$$\text{LHS} = \sum_{k\geq 1} kx^k = x\frac{d}{dx}\sum_{k\geq 1} x^k = x\frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{x}{(1-x)^2},$$

$$\text{RHS} = \frac{A_{1,0}x}{(1-x)^2} = \frac{x}{(1-x)^2}.$$

For $n = 2$, LHS $= \sum_{k\geq 1} k^2 x^k$,

$$\begin{aligned}
\text{RHS} &= \frac{A_{2,0}x + A_{2,1}x^2}{(1-x)^3} = \frac{x+x^2}{(1-x)^3} \\
&= (x+x^2)\sum_{k\geq 0}\binom{-3}{k}(-x)^k \\
&= (x+x^2)\sum_{k\geq 0}\binom{k+2}{k}x^k \\
&= \sum_{k\geq 1}\binom{k+1}{k-1}x^k + \sum_{k\geq 2}\binom{k}{k-2}x^k \\
&= x + \sum_{k\geq 2} x^k\left[\binom{k+1}{k-1} + \binom{k}{k-2}\right] \\
&= x + \sum_{k\geq 2} k^2 x^k = \text{LHS}.
\end{aligned}$$

For $n = 3$, LHS $= \sum_{k\geq 1} k^3 x^k$,

$$\begin{aligned}
\text{RHS} &= \frac{A_{3,0}x + A_{3,1}x^2 + A_{3,2}x^3}{(1-x)^4} \\
&= \frac{x+4x^2+x^3}{(1-x)^4} = (x+4x^2+x^3)\sum_{k\geq 0}\binom{k+3}{k}x^k \\
&= x + 4x^2 + \sum_{k\geq 3} x^k\left[\binom{k+2}{k-1} + 4\binom{k+1}{k-2} + \binom{k}{k-3}\right] \\
&= x + 4x^2 + \sum_{k\geq 3} k^3 x^k.
\end{aligned}$$

For arbitrary n, recall $k^n = \sum_{j=0}^{n-1} A_{n,j}\binom{k+j}{n} = \sum_{j=0}^{n-1} A_{n,j}\cdot\frac{[k+j]_{(n)}}{n!}$. Then

$$\sum_{k=1}^{\infty} k^n x^k = \sum_{k=1}^{\infty}\sum_{j=0}^{n-1} A_{n,j}\binom{k+j}{n}x^k = \sum_{j=0}^{n-1} A_{n,j}\cdot\frac{1}{n!}\sum_{k=1}^{\infty}[k+j]_{(n)}x^k.$$

Note that $j < n$ and

$$\begin{aligned} S : &= \frac{1}{n!}\sum_{k=1}^{\infty}[k+j]_{(n)}x^k \\ &= \frac{x^{n-j}}{n!}\sum_{k=1}^{\infty}\frac{d^n}{dx^n}(x^{k+j}) \\ &= \frac{x^{n-j}}{n!}\cdot\frac{d^n}{dx^n}\sum_{k=1}^{\infty}x^{k+j} \\ &= \frac{x^{n-j}}{n!}\cdot\frac{d^n}{dx^n}\left(\frac{x^{j+1}}{1-x}\right). \end{aligned}$$

Applying the Leibliz rule, we have

$$\begin{aligned} S &= \frac{x^{n-j}}{n!}\sum_{i=0}^{n}\binom{n}{i}\frac{d^i}{dx^i}(x^{j+1})\frac{d^{n-i}}{dx^{n-i}}(1-x)^{-1} \\ &= \frac{x^{n-j}}{n!}\sum_{i=0}^{n}\binom{n}{i}[j+1]_{(i)}x^{j-i+1}[-1]_{(n-i)}(1-x)^{i-n-1}(-1)^{n-i} \\ &= \frac{x^{n-j}}{n!}\sum_{i=0}^{n}\frac{n!(j+1)!(n-i)!}{i!(n-i)!(j-i+1)!}\cdot\frac{x^{j-i+1}}{(1-x)^{n-i+1}} \\ &= \sum_{i=0}^{n}\frac{(j+1)!}{i!(j-i+1)!}\left(\frac{x}{1-x}\right)^{n-i+1}. \end{aligned}$$

Now S becomes

$$\begin{aligned} S &= \left(\frac{x}{1-x}\right)^{n+1}\sum_{i=0}^{n}\binom{j+1}{i}\left(\frac{1-x}{x}\right)^{i} \\ &= \left(\frac{x}{1-x}\right)^{n+1}\left(\frac{1-x}{x}+1\right)^{j+1} \\ &= \left(\frac{x}{1-x}\right)^{n+1}\left(\frac{1}{x}\right)^{j+1} = \frac{x^{n-j}}{(1-x)^{n+1}}. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{k=1}^{\infty}k^n x^k &= \frac{1}{(1-x)^{n+1}}\sum_{j=0}^{n-1}A_{n,j}x^{n-j} \\ &= \frac{1}{(1-x)^{n+1}}\sum_{j=0}^{n-1}A_{n,n-j-1}x^{n-j} \\ &= \frac{1}{(1-x)^{n+1}}\sum_{i=0}^{n-1}A_{n,i}x^{i+1} \end{aligned}$$

Given a permutation $\sigma = s_1 s_2 \ldots s_n \in \mathfrak{S}_n$. An **exceedance** of σ is a number i such that $\sigma(i) > i$. The set of all exceedances of σ is

$$\mathrm{Exc}(\sigma) = \big\{i \in [n] : s_i > i\big\} = \big\{i \in [n] : \sigma(i) > i\big\}. \tag{3.8}$$

The number of exceedances of σ is the cardinality $\mathrm{exc}(\sigma) := |\mathrm{Exc}(\sigma)|$. A week exceedance of σ is a number $i \in [n]$ such that $\sigma(i) \geq i$. The set of weak exceedances is

$$\text{w-Exc}(\sigma) = \big\{ i \in [n] : s_i \geq i \big\} = \big\{ i \in [n] : \sigma(i) \geq i \big\}. \tag{3.9}$$

Proposition 3.14. *The Eulerian number $A_{n,k}$ counts the number of n-permutations with k exceedances, i.e.,*

$$A_{n,k} = \#\{\sigma \in \mathfrak{S}_n : \mathrm{exc}(\sigma) = k\} = \#\{\sigma \in \mathfrak{S}_n : \text{w-exc}(\sigma) = k+1\}.$$

Proof. The bijection $\sigma \mapsto \hat{\mathrm{cyc}}(\sigma)$ gives another description of the Eulerian numbers.

Given a permuation $\sigma = s_1 s_2 \ldots s_n \in \mathfrak{S}_n$ having the standard cycle notation

$$\mathrm{cyc}(\sigma) = (t_1 t_2 \ldots t_{\ell_1})(t_{\ell_1+1} t_{\ell_1+2} \ldots t_{\ell_2}) \ldots (t_{\ell_{m-1}+1} t_{\ell_{m-1}+2} \ldots t_n),$$

where $t_1 = t_{\ell_0+1}$, $t_n = t_{\ell_m}$, and $t_{\ell_0+1}, t_{\ell_1+1}, \ldots, t_{\ell_{m-1}+1}$ are the m largest elements in the corresponding k cycles of σ and are arranged in increasing order. Then $\hat{\sigma} := \hat{\mathrm{cyc}}(\sigma) = t_1 t_2 \ldots t_n$ is a permutation. Note that $\sigma(t_j) = t_{j+1}$ for $\ell_{i-1}+1 \leq j < \ell_i$, and $\sigma(t_{\ell_i}) = t_{\ell_{i-1}+1} \geq t_{\ell_i}$ (the equality holds for cycles of length 1, i.e., $\ell_i = \ell_{i-1}+1$).

Assume that $t_j < t_{j+1}$, where $j < n$, which is is automatically true when $j = \ell_i$ for some i. There exists an i such that either $\ell_{i-1}+1 \leq j < \ell_i$ or $j = \ell_i$. Then either $\sigma(t_j) = t_{j+1} > t_j$ or $\sigma(t_{\ell_i}) = t_{\ell_{i-1}+1} > t_{\ell_i}$ if $\ell_i > \ell_{i-1}+1$, or $\sigma(t_{\ell_i}) = t_{\ell_i}$ if $\ell_i = \ell_{i-1}+1$. So $\sigma(t_j) \geq t_j$ for all $j < n$ (it is also true for $j = n$). Conversely, assume that $\sigma(t_j) \geq t_j$. There exists an i such that either $\ell_{i-1}+1 \leq j < \ell_i$ or $j = \ell_i$. Then either $\sigma(t_j) = t_{j+1} \neq t_j$, i.e., $t_j < t_{j+1}$, or $t_{\ell_i} < t_{\ell_i+1}$ if $\ell_i \neq n$. It then follows that $\sigma(t_j) \geq t_j$ iff $j = n$ or $t_j < t_{j+1}$ for $j < n$.

Recall that $\mathrm{Asc}(\hat{\sigma}) = \{j \in [n-1] : t_j < t_{j+1}\} = [n-1] \smallsetminus \mathrm{Des}(\hat{\sigma})$. Then

$$[n] \smallsetminus \mathrm{Des}(\hat{\sigma}) = \mathrm{Asc}(\hat{\sigma}) \cup \{n\} = \{j \in [n] : \sigma(t_j) \geq t_j\}.$$

Thus

$$\begin{aligned} n - \mathrm{des}(\hat{\sigma}) &= |\{j \in [n] : \sigma(t_j) \geq t_j\}| \\ &= |\{t_j \in [n] : \sigma(t_j) \geq t_j\}| \\ &= \text{w-exc}(\sigma). \end{aligned}$$

Since $\sigma \mapsto \hat{\mathrm{cyc}}(\sigma)$ is a bijection, we see that

$$A_{n,k} = |\{\hat{\sigma} : \mathrm{des}(\hat{\sigma}) = k\}| = |\{\sigma : \text{w-exc}(\sigma) = n-k\}|.$$

Moreover, for each permutation $\pi = u_1 u_2 \ldots u_n$, let $\tilde{\pi} = v_1 v_2 \ldots v_n$, where $v_i = n+1-u_{n+1-i}$. Note that π has $n-k$ weak exceedances iff π has k indices i such that $u_i < i$. Since $u_i < i$ iff $v_{n+1-i} > n+1-i$, we see that π has $n-k$ weak exceedances iff $\tilde{\pi}$ has k exceedances. Thus

$$A_{n,k} = |\{\sigma : \text{w-exc}(\sigma) = n-k\}| = |\{\tilde{\sigma} : \mathrm{exc}(\tilde{\sigma}) = k\}|.$$

Applying the formula above to $A_{n,n-k-1}$, we have

$$A_{n,k} = A_{n,n-k-1} = |\{\sigma \in \mathfrak{S}_n : \text{w-exc}(\sigma) = k+1\}|.$$

□

4. q-Analogs

Let $\mathbb{F}_q$ be a finite field of q elements. Let $V = \mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$. Given nonnegative integers $a_1, a_2, \ldots, a_m$ such that

$$a_1 + a_2 + \cdots + a_m = n.$$

We denote by $\mathrm{Fl}(a_1, \ldots, a_m)$ the set of flags

$$\{0\} \subseteq V_1 \subseteq \cdots \subseteq V_m \subseteq V$$

of length m such that $\dim(V_i/V_{i-1}) = a_i$, $1 \le i \le m$, called the **flag space of V of type** $(a_1, a_2, \ldots, a_m)$. The set of all flags of length m is denoted by Fl_m, called the **flag space of V of length** m. For $m = 1$ and $a_1 = k$, the set $\mathrm{Fl}(k)$ can be identified as the collection of all k-subspaces of $\mathbb{F}_q^n$, called the **Grassmannian** of k-subspaces of V, denoted $\mathrm{Gr}(V, k)$. The cardinality of $\mathrm{Gr}(V, k)$ is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1})}{(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})},$$

which is actually a polynomial, called the **Gaussian polynomial** of the q-analog of binomial coefficients. In general, we introduce the notations

$$[n]_q := 1 + q + \cdots + q^{n-1},$$
$$[n]_q! := [n]_q[n-1]_q \cdots [2]_q[1]_q.$$

For nonnegative integers $a_1, a_2, \ldots, a_m$ such that $a_1 + a_2 + \cdots + a_m = n$, we define the **$q$-analog of multinomial coefficient** (or just **q-multinomial coefficient**)

$$\begin{bmatrix} n \\ a_1, a_2, \ldots, a_m \end{bmatrix}_q := \frac{[n]_q!}{[a_1]_q![a_2]_q!\cdots[a_m]_q!}. \tag{4.1}$$

Let $\mathcal{M}$ denote the vector space of all $n \times n$ matrices over $\mathbb{F}_q$. We denote by $\mathcal{M}^n$ be the set of $n \times n$ matrices of rank n. For each $M \in \mathcal{M}$, we divide M into a block matrix of the form

$$M = \begin{pmatrix} M_1 \\ \vdots \\ M_m \end{pmatrix},$$

where M_i is an $a_i \times n$ matrix, $1 \le i \le m$. For sake of convenience, we write $M = (M_1, M_2, \ldots, M_m)$ in the row form. There is a canonical projection

$$\pi : \mathcal{M} \to \mathrm{Fl}(a_1, \ldots, a_m),$$

defined by

$$\pi(M) = \{0\} \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_m \subseteq V,$$

where V_i is the row space of the submatrix $(M_1, M_2, \ldots, M_i)$, $1 \le i \le m$. The restriction

$$\pi : \mathcal{M}^n \to \mathrm{Fl}(a_1, \ldots, a_m)$$

is surjective. Note that

$$\begin{aligned} \#(\mathcal{M}^n) &= (q^n - 1)(q^n - q)\cdots(q^n - q^{n-1}) \\ &= q^{n(n-1)/2}(q^n - 1)(q^{n-1} - 1)\cdots(q - 1) \\ &= q^{n(n-1)/2}(q-1)^n[n]_q[n-1]_q\cdots[2]_q[1]_q \\ &= q^{n(n-1)/2}(q-1)^n[n]_q!. \end{aligned}$$

For each $F \in \mathrm{Fl}(a_1,\ldots,a_m)$ in $\mathcal{M}^n$, the fiber $\pi^{-1}(F)$ in $\mathcal{M}^n$ has the cardinality

$$\begin{aligned}\#(\pi^{-1}(F)) = {} & \underbrace{(q^{a_1}-1)\cdots(q^{a_1}-q^{a_1-1})}_{a_1}\times \\ & \underbrace{(q^{a_1+a_2}-q^{a_1})\cdots(q^{a_1+a_2}-q^{a_1+a_2-1})}_{a_2}\times \\ & \cdots\times\underbrace{(q^n-q^{a_1+\cdots+a_{m-1}})\cdots(q^n-q^{n-1})}_{a_m}. \end{aligned} \tag{4.2}$$

It follows that

$$\begin{aligned}\#(\pi^{-1}(F)) = {} & q^e\,\underbrace{(q^{a_1}-1)(q^{a_1-1}-1)\cdots(q-1)}_{a_1}\times \\ & \underbrace{(q^{a_2}-1)(q^{a_2-1}-1)\cdots(q-1)}_{a_2}\times \\ & \cdots\times\underbrace{(q^{a_m}-1)(q^{a_m-1}-1)\cdots(q-1)}_{a_m}, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} e = {} & [1+2+\cdots+(a_1-1)] + [a_1+(a_1+1)+2+\cdots+(a_1+a_2-1)] \\ & +\cdots+[(a_1+\cdots+a_{m-1})+(a_1+\cdots+a_{m-1}+1)+\cdots+(n-1)] \\ = {} & n(n-1)/2. \end{aligned}$$

We then have

$$\begin{aligned}\#(\pi^{-1}(F)) &= q^e\prod_{i=1}^m (q-1)^{a_i}[a_i]_q[a_i-1]_q\cdots[2]_q[1]_q \\ &= q^e(q-1)^{\sum_{i=1}^m a_i}\prod_{i=1}^m [a_i]_q! \\ &= q^e(q-1)^n[a_1]_q![a_2]_q!\cdots[a_m]_q!; \end{aligned}$$

Since

$$\#(\mathcal{M}^n) = \#(\mathrm{Fl}(a_1,\ldots,a_m))\cdot\#(\pi^{-1}(F)),$$

we obtain

$$\#(\mathrm{Fl}(a_1,\ldots,a_m)) = \frac{[n]_q!}{[a_1]_q![a_2]_q!\cdots[a_m]_q!} = \begin{bmatrix} n \\ a_1,\ldots,a_m \end{bmatrix}_q.$$

We shall see that $\left[{n \atop a_1,\ldots,a_m}\right]_q$ is a polynomial of q.

Let $B(a_1,\ldots,a_m)$ denote a subgroup of the **general linear group** $\mathrm{GL}(n,\mathbb{F}_q)$ of $n\times n$ invertible matrices over $\mathbb{F}_q$, consisting of the block lower triangular matrices of the form

$$A = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix},$$

where A_{kl} are $a_k \times a_l$ matrices and A_{kk} are invertible. Then $B(a_1, \ldots, a_m)$ is a subgroup of $\mathrm{GL}(n, \mathbb{F}_q^n)$, acting on $\mathcal{M}^n$ on the left by multiplication, i.e.,

$$AM = A \begin{pmatrix} M_1 \\ \vdots \\ M_m \end{pmatrix} = \begin{pmatrix} M_1' \\ \vdots \\ M_m' \end{pmatrix} = M',$$

where $M_k' = A_{k1}M_1 + \cdots + A_{kk}M_k$, $1 \le k \le m$. The projection $\pi : \mathcal{M}^n \to \mathrm{Fl}(a_1, \ldots, a_m)$ induces a quotient bijection

$$\pi : \mathcal{M}^n / B(a_1, \ldots, a_m) \to \mathrm{Fl}(a_1, \ldots, a_m).$$

Let $\mathrm{Row}(M_1, \ldots, M_k), \mathrm{Row}(M_1', \ldots, M_k')$ denote the row spaces of the submatrices

$$\begin{pmatrix} M_1 \\ \vdots \\ M_k \end{pmatrix}, \quad \begin{pmatrix} M_1' \\ \vdots \\ M_k' \end{pmatrix}$$

respectively. It is clear that $\mathrm{Row}(M_1', \ldots, M_k') \subseteq \mathrm{Row}(M_1, \ldots, M_k)$. Since A is invertible and $A^{-1} \in B(a_1, \ldots, a_m)$, we see that $\mathrm{Row}(M_1', \ldots, M_k') = \mathrm{Row}(M_1, \ldots, M_k)$. So the quotient map is well-defined.

The surjectivity is trivial. For injectivity, assume that $\pi(M) = \pi(M')$ for two matrices M and M', i.e.,

$$\mathrm{Row}(M_1, \ldots, M_k) = \mathrm{Row}(M_1', \ldots, M_k'), \quad 1 \le k \le m. \tag{4.4}$$

We need to show that there exists a matrix $A \in B(a_1, \ldots, a_m)$ such that $AM = M'$.

Since $\mathrm{Row}(M_1) = \mathrm{Row}(M_1')$, there exits an invertible matrix A_{11} such that $A_{11}M_1 = M_1'$. Next, since $\mathrm{Row}(M_1, M_2) = \mathrm{Row}(M_1', M_2')$, then each row of M_2' is a linear combination of rows of M_1 and M_2. This means that there exit matrices A_{21} and A_{22} such that $M_2' = A_{21}M_1 + A_{22}M_2$. We then have

$$\begin{pmatrix} M_1' \\ M_2' \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix},$$

where A_{22} must be invertible. Continue this procedure, one obtains matrices A_{kl} $(1 \le l \le k)$ such that $M_k' = \sum_{l=1}^{k} A_{kl}M_l$ and A_{kk} are invertible, $1 \le k \le m$. Set $A = [A_{kl}]$, where $A_{kl} = 0$ for $k < l$. Then $A \in B(a_1, \ldots, a_m)$ and $AM = M'$. This means that $\pi : \mathcal{M}^n / B(a_1, \ldots, a_m) \to \mathrm{Fl}(a_1, \ldots, a_m)$ is a bijection.

A **reduced block echelon matrix** of type $(a_1, \ldots, a_m)$ is a block $n \times n$ matrix

$$E = \begin{pmatrix} E_1 \\ \vdots \\ E_m \end{pmatrix},$$

where each E_k is a reduced row echelon matrix justified from right and bottom, pivot positions are in different rows and different columns, and all entries below a pivot position are zero.

Each non-singular $n \times n$ block matrix M of type $(a_1, \ldots, a_m)$ can be converted into a reduced block echelon matrix of the same type by multiplying a matrix $A \in B(a_1, \ldots, a_m)$ to the left of M. In other words, each orbit of $\mathcal{M}^n / B(a_1, \ldots, a_m)$ has a representative of reduced block echelon matrix, which can be obtained as follows.

Step 1: Find the rightmost nonzero column of M_1, called the **pivot column** of Block 1; the bottom position of the pivot column is called a **pivot position**. If the entry of the pivot position is zero, interchange the bottom row of M_1 and one row of M_1 whose entry in the pivot column is nonzero; now the pivot entry is nonzero. Reduce the nonzero pivot entry to 1 and the entries above it in M_1 to zero by row operations. Next, cover the the bottom row of Block 1 to obtain a matrix M_1'; repeat the procedure until all rows of Block 1 are covered. We then obtain a reduced row echelon matrix E_1 of M_1. There exists an invertible matrix A_{11} such that $E_1 = A_{11}M_1$.

Step 2: Reduce all entries of M_k $(2 \le k \le m)$ below the pivot positions of $E_1, \ldots, E_{k-1}$ to zero by multiplying matrices $A_{k1}, \ldots, A_{k(k-1)}$ to $E_1, \ldots, E_{k-1}$ respectively. We then obtain a block matrix

$$M' = \begin{pmatrix} E_1 \\ \vdots \\ E_{k-1} \\ M_k' \\ \vdots \\ M_m' \end{pmatrix}.$$

Cover the blocks $E_1, \ldots, E_{k-1}$ of M' and apply Step 1 to the block M_k'.

Step 3: Repeat until every block becomes reduced row echelon matrix. Finally, a reduced block echelon matrix $E = (E_1, \ldots, E_m)$ is obtained.

There are n pivot positions in a block row echelon form E, located in distinct rows and distinct columns. Entries beyond each pivot position on the right and below are zero. Entries of a pivot column in the block of the pivot position are zero, except the pivot entry, which is 1.

Let the pivot position of the ith row be located in (i, s_i). Then the reduced block echelon matrix E can be indexed by a permutation

$$\sigma = s_1 s_2 \ldots s_n, \quad \sigma(i) = s_i.$$

Let $b_k = a_1 + \cdots + a_k$, $1 \le k \le m$. Recall that $\mathrm{Des}(\sigma)$ is the set of indices where σ decreases. Note that σ increases strictly at integers inside intervals (b_{k-1}, b_k) for each block M_k of M. The descents of σ can only occur at the indices b_k of each block. So we have

$$\mathrm{Des}(\sigma) \subseteq \{b_1, b_2, \ldots, b_m\}, \quad b_k = a_1 + \cdots + a_k. \tag{4.5}$$

Conversely, each permutation σ satisfying (4.5) determines a block echelon form.

Next we show the injectivity of π on the reduced block echelon matrices of type $(a_1, \ldots, a_m)$. Given two reduced block echelon matrices E, E'. If $\pi(E) = \pi(E')$, i.e.,

$$\mathrm{Row}(E_1, \ldots, E_k) = \mathrm{Row}(E_1', \ldots, E_k'), \quad 1 \le k \le m, \tag{4.6}$$

we claim that $E = E'$. Suppose $E \ne E'$. We may assume that $E_1 = E_1'$, $\ldots$, $E_{k-1} = E_{k-1}'$, and $E_k \ne E_k'$.

Let F, F' be matrices obtained from E, E' respectively by row operations to reduce all entries above each pivot position in the first b_k rows. And let $\boldsymbol{v}_i, \boldsymbol{v}_i'$ denote the ith rows of F, F' respectively.

Suppose that E_k, E_k' have distinct pivot positions. Let l be the largest row index of E_k, E_k' such that $s_l \ne s_l'$. Assume $s_l < s_l'$. Then $s_i < s_l'$ for $i \in (b_{k-1}, l]$; and

$s_i = s'_i$ for $i \in [1, b_{k-1}] \cup (l, b_k]$. In particular, $s_i \neq s'_l$ for all $i = 1, \ldots, b_k$. Since (4.6), we have $\boldsymbol{v}'_l = \sum_{i=1}^{b_k} c_i \boldsymbol{v}_i$, i.e.,

$$v'_{lj} = \sum_{i=1}^{b_k} c_i v_{ij}, \quad j = 1, \ldots, n. \tag{4.7}$$

Note that $v'_{lj} = 0$ for all $j > s'_l$ by the echelon property of F'. If $s_{i_0} > s'_l$ for a row index $i_0 \in [1, b_k]$, then $s'_{i_0} = s_{i_0} > s'_l$, consequently, $v'_{ls_{i_0}} = 0$ by the echelon property of F'. Note that $v_{is_{i_0}} = \delta_{ii_0}$ for $i \in [1, , b_k]$ by the echelon property of F. Set $j = s_{i_0}$ in (4.7), we see that

$$0 = v'_{ls_{i_0}} = \sum_{i=1}^{b_k} c_i v_{is_{i_0}} = c_{i_0}.$$

It follows that (4.7) becomes

$$v'_{lj} = \sum_{1 \leq i \leq b_k,\, s_i < s'_l} c_i v_{ij}, \quad 1 \leq j \leq n. \tag{4.8}$$

Note that $v'_{ls'_l} = 1$ by the echelon property of F', and $v_{is'_l} = 0$ for all i such that $s_i < s'_l$ by the echelon property of F. Set $j = s'_l$ in (4.8); we obtain

$$1 = v'_{ls'_l} = \sum_{1 \leq i \leq b_k,\, s_i < s'_l} c_i v_{is'_l} = 0,$$

which is a contradiction. We must have $s_l \geq s'_l$. Likewise, $s'_l \geq s_l$. Hence $s_l = s'_l$, contradicting to the previous assumption. This shows that E, E' have the same pivot positions in the first b_k rows.

Now let l be the largest row index of E_k, E'_k such that their lth rows are distinct. Recall (4.6) again; there exists a $b_k \times b_k$ matrix $A_k \in B(a_1, \ldots, a_k)$ such that $\begin{pmatrix} E'_1 \\ \vdots \\ E'_k \end{pmatrix} = A_k \begin{pmatrix} E_1 \\ \vdots \\ E_k \end{pmatrix}$. The two linear systems

$$\begin{pmatrix} E_1 \\ \vdots \\ E_k \end{pmatrix} \boldsymbol{x} = \boldsymbol{0}, \quad \begin{pmatrix} E'_1 \\ \vdots \\ E'_k \end{pmatrix} \boldsymbol{x} = \boldsymbol{0}$$

have the same solution space. So do the two linear systems

$$\begin{pmatrix} F_1 \\ \vdots \\ F_k \end{pmatrix} \boldsymbol{x} = \boldsymbol{0}, \quad \begin{pmatrix} F'_1 \\ \vdots \\ F'_k \end{pmatrix} \boldsymbol{x} = \boldsymbol{0} \tag{4.9}$$

We construct a particular solution $\boldsymbol{x}$ of the first system in (4.9), which is not a solution of the second system in (4.9), resulting a contradiction.

Since $v_l \neq v'_l$ and $v_{ls_l} = v'_{ls_l} = 1$, let $j_0 \in [1, s_l)$ be the first column index such that $v_{lj_0} \neq v'_{lj_0}$. Notice that $j_0 \neq s_i$ for all $i \in [1, b_k]$ by the echelon property of F, F'. A typical solution $\boldsymbol{x}$ of the first system in (4.9) is given by

$$x_j = \begin{cases} 1 & \text{if } j = j_0 \\ -v_{ij_0} & \text{if } j = s_i,\, 1 \leq i \leq b_k, \\ 0 & \text{otherwise} \end{cases} \qquad j = 1, \ldots, n.$$

It is clear that such an $\boldsymbol{x}$ is not a solution of the second system in (4.9), for the lth equation of the second system is not satisfied. This is a contradiction.

Now each flag $\{0\} \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_m = V$ of type $(a_1, \ldots, a_m)$ is identified as one and only one of a reduced block echelon matrix of type $(a_1, \ldots, a_m)$ with certain values for each $*$. The space $\mathrm{Fl}(a_1, \ldots, a_m)$ is then decomposed into a disjoint union of affine subspaces corresponding to reduced block echelon forms.

Each reduced block echelon form E can be indexed by a permutation $\sigma = s_1 s_2 \ldots s_n$ of $\{1, 2, \ldots, n\}$, where (i, s_i) is the pivot position of the ith row in E. For each star position (i, j) of E, we have $j < s_i$, and there exists a unique $k > i$ such (k, j) is a pivot position; so $s_k := j < s_i$, i.e., (s_i, s_k) is an inversion of σ. Conversely, if (s_i, s_k) is an inversion, i.e., $i < k$ and $s_i > s_k$, then the row i and the row k of E cannot be in the same block, thus (i, s_k) must be a star position of E. So the number of inversions of the permutation σ equals the number of star positions of the reduced block echelon form E.

For example, given type $(a_1, a_2, a_3, a_4) = (3, 2, 2, 2)$, its reduced block echelon form is

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix} = \left(\begin{array}{ccccccccc} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & * & 1 & 0 \\ \hline * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline * & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & * & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$$

where each $*$ position can be filled with arbitrary values of $\mathbb{F}_q$. Notice that there is no star position in the last block. The affine subspace with the reduced block echelon form above is indexed by the permutation $\sigma = 368245917$, whose inversion table is $(a_1, \ldots, a_9) = (2, 4, 5, 1, 1, 1, 2, 0, 0)$. The number of star positions in its reduced block echelon form is the number of inversions of σ, i.e.,

$$\mathrm{inv}(\sigma) = \mathrm{inv}(368245917) = 16.$$

We have proved the following theorem.

Theorem 4.1. *Given nonnegative integers $a_1, \ldots, a_m$ such that $a_1 + \cdots + a_m = n$. Let $\mathfrak{S}_n(a_1, \ldots, a_m)$ denote the set of permutations σ of $[n]$ whose descent set satisfies*

$$\mathrm{Des}(\sigma) \subseteq \{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_m\}.$$

Then

$$\sum_{\sigma \in \mathfrak{S}_n(a_1, \ldots, a_m)} q^{\mathrm{inv}(\sigma)} = \begin{bmatrix} n \\ a_1, \ldots, a_m \end{bmatrix}_q . \tag{4.10}$$

Let $\mathbb{Z}_+$ be the set of positive integers. Let $\mathbb{F}_q^\infty$ denote the vector space of all functions from $\mathbb{Z}_+$ to $\mathbb{F}_q$ with finite support. We write each vector $\boldsymbol{v} \in \mathbb{F}_q^\infty$ as an infinite tuple

$$\boldsymbol{v} = (v_1, v_2, \ldots, v_n, 0, 0, \ldots).$$

Given nonnegative integers $a_1, \ldots, a_m$. Denote by $\mathrm{Fl}_\infty(a_1, \ldots, a_m)$ the set of flags

$$\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_m \subsetneq \mathbb{F}_q^\infty$$

of length m, such that $\dim(V_i/V_{i-1}) = a_i$, $1 \le i \le m$.

Let $\mathcal{M}_{n,\infty}$ denote the vector space of $n \times \infty$ matrices over $\mathbb{F}_q$, having only finitely many nonzero entries. Let $\mathcal{M}^n_{n,\infty}$ denote the subset of $\mathcal{M}_{n,\infty}$, consisting of matrices of rank n. Each member of $\mathcal{M}_{n,\infty}$ can be written as a block matrix

$$M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_m \end{pmatrix}.$$

where M_i is an $a_i \times \infty$ submatrix. There is canonical projection

$$\pi : \mathcal{M}^n_{n,\infty} \to \mathrm{Fl}_\infty(a_1, \ldots, a_m), \quad M \mapsto V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m,$$

where $V_0 = \{0\}$, $V_i = \mathrm{Row}(M_1, \ldots, M_i)$, $1 \le i \le m$. The parabolic group $B(a_1, \ldots, a_m)$ acts on $\mathcal{M}^n_{n,\infty}$ on the left by multiplication. We shall see that the orbit space $\mathcal{M}^n_{n,\infty}/B(a_1, \ldots, a_m)$ is isomorphic to the flag space $\mathrm{Fl}_\infty(a_1, \ldots, a_m)$.

We denote by $\mathfrak{S}_\infty$ the group of all bijections $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\sigma(k) = k$ for large enough $k \in \mathbb{Z}_+$, i.e., there exists an integer N such that $\sigma(k) = k$ for all $k > N$. For each $\sigma \in \mathfrak{S}_\infty$, the **inversion set** of σ is the collection

$$\mathrm{Inv}(\sigma) = \big\{(s_i, s_j) : i < j \text{ and } s_i > s_j\big\}$$

and $\mathrm{inv}(\sigma) = |\mathrm{Inv}(\sigma)|$. Given nonnegative integers $a_1, a_2, \ldots, a_m$; we denote by $\mathfrak{S}_\infty(a_1, \ldots, a_m)$ the set of permutations $\sigma \in \mathfrak{S}_\infty$ such that

$$\mathrm{Des}(\sigma) \subseteq \{b_1, b_2, \ldots, b_m\},$$

where $b_i = a_1 + \cdots + a_i$, $\quad 1 \le i \le m$.

For example, for $(a_1, a_2, a_3) = (3, 2, 2)$ we have $(b_1, b_2, b_3) = (3, 5, 7)$. For the permutation $\sigma = s_1 s_2 \ldots s_n \ldots$ with $s_1 s_2 \ldots s_9 = 368472915$ and $s_i = i$ for $i \ge 10$, we have

$$\mathrm{Inv}(\sigma) = \#\{(i,j) \in \mathbb{Z}_+^2 : i < j, s_i > s_j\},$$

$\mathrm{inv}(\sigma) = 19$, and $\mathrm{Des}(\sigma) = \{3, 5, 7\}$.

$$\left(\begin{array}{ccccccccc|cc}
* & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
* & * & 0 & * & * & 1 & 0 & 0 & 0 & 0 & \cdots \\
* & * & 0 & * & * & 0 & * & 1 & 0 & 0 & \cdots \\
\hline
* & * & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
* & * & 0 & 0 & * & 0 & 1 & 0 & 0 & 0 & \cdots \\
\hline
* & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
* & 0 & 0 & 0 & * & 0 & 0 & 0 & 1 & 0 & \cdots \\
\hline\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & I_\infty
\end{array}\right) \tag{4.11}$$

Definition 4.2. *For nonnegative integers $a_1, \ldots, a_m$, the q-analog of multinomial coefficient of infinite type $(\infty; a_1, \ldots, a_m)$ is*

$$\begin{bmatrix} \infty \\ a_1, \ldots, a_m \end{bmatrix}_q := \prod_{i=1}^{m} \frac{1}{(1-q)(1-q^2)\cdots(1-q^{a_i})}.$$

Theorem 4.3. *For non-negative integers $a_1, \ldots, a_m$, let $\mathfrak{S}_\infty(a_1, \ldots, a_m)$ denote the set of bijections $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\sigma(k) = k$ for k large enough. Then*

$$\sum_{\sigma \in \mathfrak{S}_\infty(a_1,\ldots,a_m)} q^{\mathrm{inv}(\sigma)} = \begin{bmatrix} \infty \\ a_1, \ldots, a_m \end{bmatrix}_q . \tag{4.12}$$

Proof. Fix a permutation $\sigma = s_1 s_2 \cdots \in \mathfrak{S}_\infty(a_1, \ldots, a_m)$. Let $(1, s_1), \ldots, (a_1, s_{a_1})$ denote the pivot positions of the echelon form E_1 for the first block. The number of stars in E_1 on the left of the s_1th column is $k_1 a_1$, where $k_1 \geq 0$. The number of stars in E_1 between the columns s_1 and s_2 is $k_2(a_1 - 1)$, where $k_2 \geq 0$. And the number of stars in E_1 between the columns s_{a_1-1} and s_{a_1} is $k_{a_1} \geq 0$. So the total number of stars in E_1 is $k_1 a_1 + k_2(a_2 - 1) + \cdots + k_{a_1-1} \cdot 2 + k_{a_1} \cdot 1$. Likewise, the total number of stars in the echelon form E_i of the ith block is

$$k_{i1} a_i + k_{i2}(a_i - 1) + \cdots + k_{i(a_i-1)} \cdot 2 + k_{ia_i} \cdot 1.$$

The left-hand side of (4.12) becomes

$$\begin{aligned} \text{LHS} &= \sum_{[k_{ij_i} \geq 0]_{1 \leq i \leq m, 1 \leq j_i \leq a_i}} \prod_{i=1}^{m} \prod_{j_i=1}^{a_i} q^{k_{ij_i}(a_i - j_i + 1)} \\ &= \prod_{i=1}^{m} \prod_{j_i=1}^{a_i} \sum_{k_{ij_i}=0}^{\infty} q^{k_{ij_i}(a_i - j_i + 1)} \\ &= \prod_{i=1}^{m} \prod_{j_i=1}^{a_i} \frac{1}{1 - q^{a_i - j_i + 1}}. \end{aligned}$$

□

5. Stirling Numbers

5.1. Stirling numbers of the first kind.

Definition 5.1. *The* **Stirling numbers of the first kind** *are the numbers $s_{n,k}$ determined by the expansion*

$$[x]_{(n)} = \sum_{k=0}^{n} s_{n,k}\, x^k, \quad n \geq k \geq 0$$

for $n \geq 1$ and with $s_{0,0} \equiv 1$.

Proposition 5.2. *The Stirling numbers of the first kind $s_{n,k}$ satisfy the recurrence relation:*

$$\begin{cases} s_{n,n} = 1 & \text{for } \; n \geq 0 \\ s_{n,0} = 0 & \text{for } \; n \geq 1 \\ s_{n+1,k} = s_{n,k-1} - n s_{n,k} & \text{for } \; n \geq k \geq 1 \end{cases} \tag{5.1}$$

Proof. Expanding the falling factorial $[x]_{(n)=}x(x-1)\cdots(x-n+1)$ for $n \geq 1$, it is clear that $s_{n,n}=1$ and $s_{n,0}=0$. Since

$$\begin{aligned}\sum_{k=0}^{n+1} s_{n+1,k}x^k &= [x]_{(n+1)} = [x]_{(n)}(x-n)\\ &= \sum_{k=0}^{n} s_{n,k}x^{k+1} - n\sum_{k=0}^{n} s_{n,k}x^k\\ &= \sum_{k=1}^{n+1} s_{n,k-1}x^k - \sum_{k=0}^{n} ns_{n,k}x^k,\end{aligned}$$

we see that $s_{n+1,k}=s_{n,k-1}-ns_{n,k}$ for $n \geq k \geq 1$. □

Exercise 3.

$$s_{n+1,k} = \sum_{i=0}^{n}(-1)^i[n]_{(i)}s_{n,k-1}, \quad n \geq k \geq 1.$$

Corollary 5.3. *The numbers* $a_{n,k} := (-1)^{n-k}c_{n,k}$ *satisfy the same recurrence relation* (5.1) *for the Stirling numbers of the first kind* $s_{n,k}$*. Thus*

$$s_{n,k} = (-1)^{n-k}c_{n,k}$$

and the absolute value $|s_{n,k}|$ *counts the number of permutations of an* n*-set with exactly* k *cycles.*

Proof. Obviously, $a_{0,0}=1$, and $a_{n,0}=0, a_{n,n}=1$ for all $n \geq 1$. For $n \geq k \geq 1$, we have

$$\begin{aligned}a_{n+1,k} &= (-1)^{n+1-k}c_{n+1,k}\\ &= (-1)^{n-k+1}c_{n,k-1} - n(-1)^{n-k}c_{n,k}\\ &= a_{n,k-1} - na_{n,k}.\end{aligned}$$

□

Proposition 5.4.

$$[x]^{(n)} = \sum_{k=0}^{n} c_{n,k}x^k = \sum_{k=0}^{n} |s_{n,k}|x^k.$$

Proof. By the Reciprocity Law for the rising factorial function and falling factorial function, we have

$$\begin{aligned}[x]^{(n)} &= (-1)^n[-x]_n = (-1)^n\sum_{k=0}^{n} s_{n,k}(-x)^k\\ &= \sum_{k=0}^{n}(-1)^{n-k}s_{n,k}x^k = \sum_{k=0}^{n} c_{n,k}x^k.\end{aligned}$$

□

5.2. **Stirling numbers of the second kind.**

Definition 5.5. *The* **Stirling number of the second kind** $S_{n,k}$ *is the number of ways to partition an* n*-set into* k *nonempty subsets. We take convention* $S_{0,0} = 1$ *and* $S_{n,0} = 1$ *for all* $n \geq 1$.

Proposition 5.6. *The Stirling numbers of the second kind* $S_{n,k}$ *are given by*

$$S_{n,k} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n, \quad n \geq k \geq 0.$$

Proof. The formula follows from the identity

$$|\mathrm{Sur}(N, K)| = k! S_{n,k},$$

where N and K are finite sets with $|N| = n$ and $|K| = k$. □

Proposition 5.7. *The numbers* $S_{n,k}$ *satisfy the recurrence relation:*

$$\begin{cases} S_{0,0} = S_{n,n} = 1 & \text{for} \quad n \geq 0 \\ S_{n,0} = 0 & \text{for} \quad n \geq 1 \\ S_{n,1} = 1 & \text{for} \quad n \geq 1 \\ S_{n+1,k} = S_{n,k-1} + kS_{n,k} & \text{for} \quad n \geq k \geq 1 \end{cases} \tag{5.2}$$

Proof. The initial conditions $S_{n,1} = S_{n,n} = 1$ for $n \geq 1$ are obvious. As for the recurrence relation, consider the set of all partitions of an $(n+1)$-set N into k non-empty subsets; there $S_{n+1,k}$ such partitions. Let w be an element of N. We divide these partitions into two kinds:

(a) Partitions that the singleton set $\{w\}$ is a block. There $S_{n,k-1}$ such partitions.

(b) Partitions that w is contained in a block of at least two elements. Such partitions can be obtained from the partitions of the set $N - \{w\}$ into k blocks by joining w into any of the k blocks. There are $kS_{n,k}$ such partitions. □

Proposition 5.8. *The sequence* $\{S_{n,k} \,|\, 0 \leq k \leq n\}$ *is unimodal for all* $n \geq 0$. *In fact, set* $M(n) = \max\{k \,|\, S_{n,k} = \max\}$. *The sequence* $\{S_{n,k}\}$ *has the one of the following two types:*

(1) $S_{n,0} < s_{n,1} < \cdots < S_{n,M(n)} > S_{n,M(n)+1} > \cdots > S_{n,n}$,
(2) $S_{n,0} < s_{n,1} < \cdots < S_{n,M(n)-1} = S_{n,M(n)} > \cdots > S_{n,n}$.

Proposition 5.9.

$$x^n = \sum_{k=0}^{n} S_{n,k} [x]_k.$$

Proof. For finite sets N and X with $|N| = n$ and $|X| = x$, we have

$$\mathrm{Map}(N, X) = \bigsqcup_{S \subset X} \mathrm{Sur}(N, S).$$

Then

$$\begin{aligned} x^n &= |\mathrm{Map}(N,X)| \\ &= \sum_{S\subset X} |\mathrm{Sur}(N,S)| \\ &= \sum_{k=0}^{n} \binom{x}{k} k! S_{n,k} \\ &= \sum_{k=0}^{n} S_{n,k}[x]_{(k)}. \end{aligned}$$

□

Theorem 5.10. *The Stirling inversion formula:*

$$[x]_{(n)} = \sum_{k=0}^{n} s_{n,k} x^k, \tag{5.3}$$

$$x^n = \sum_{k=0}^{n} S_{n,k}[x]_{(k)}. \tag{5.4}$$

Theorem 5.11.

$$\sum_{k=0}^{n} s_{n,k} S_{k,m} = \sum_{k=0}^{n} S_{n,k} s_{k,m} = \delta_{n,m}$$

Proposition 5.12.

$$S_{n+1,k} = \sum_{i=1}^{n} \binom{n}{i} S_{i,k-1}.$$

Exercise 4.

$$\sum_{n=k}^{\infty} \frac{S_{n,k}}{n!} t^n = \frac{(e^t-1)^k}{k!}.$$

5.3. **Lah Numbers.**

Definition 5.13. *The* **Lah numbers** $L_{n,k}$ *are defined by the identity*

$$[-x]_{(n)} = \sum_{k=0}^{n} L_{n,k}[x]_{(k)}, \quad n \geq k \geq 0$$

with convention $L_{0,0} = 1$.

Theorem 5.14. *The Lah inversion formula:*

$$[-x]_{(n)} = \sum_{k=0}^{n} L_{n,k}[x]_{(k)}, \tag{5.5}$$

$$[x]_{(n)} = \sum_{k=0}^{n} L_{n,k}[-x]_{(k)}. \tag{5.6}$$

Proposition 5.15. *The numbers* $L_{n,k}$ *satisfy the recurrence relation:*

$$\begin{cases} L_{n,n} = (-1)^n & \text{for } n \geq 0 \\ L_{n,0} = 0 & \text{for } n \geq 1 \\ L_{n+1,k} = -L_{n,k-1} - (n+k)L_{n,k} & \text{for } n \geq k \geq 1 \end{cases} \tag{5.7}$$

Proof. Since $[-x]_{(n)} = (-x)(-x-1)(-x-2)\cdots(-x-n+1)$, it follows that $L_{n,n} = (-1)^n$ and $L_{n,0} = 0$ (because there is no constant term) for all $n \geq 1$. The recursion formula follows from

$$\begin{aligned}
\sum_{k=0}^{n+1} L_{n+1,k}[x]_{(k)} &= [-x]_{(n+1)} = (-x-n)[-x]_{(n)} \\
&= (-x-n)\sum_{k=0}^{n} L_{n,k}[x]_{(k)} \\
&= \sum_{k=0}^{n} L_{n,k}(-(x-k)-(n+k))[x]_{(k)} \\
&= -\sum_{k=0}^{n} L_{n,k}[x]_{k+1} - (n+k)\sum_{k=0}^{n} L_{n,k}[x]_{(k)} \\
&= -\sum_{k=1}^{n+1} L_{n,k-1}[x]_{(k)} - (n+k)\sum_{k=0}^{n} L_{n,k}[x]_{(k)}.
\end{aligned}$$

□

Theorem 5.16. *The number of ways of placing n distinguishable objects into k indistinguishable boxes with no box left empty and objects in each box are linearly ordered, is given by*

$$d_{n,k} = \frac{n!}{k!}\binom{n-1}{k-1}, \quad n \geq k \geq 1. \tag{5.8}$$

Proof. Let the k indistinguishable boxes be divided into the distinguishable boxes $B_1, B_2, \ldots, B_k$ (linearly ordered) by inserting the bars "|" in between. Now we place the n objects of an n-set N into the distinguishable boxes so that no one is empty. Each such placement can be obtained from the permutations

$$a_1{}_\wedge a_2{}_\wedge a_3{}_\wedge a_4{}_\wedge \cdots {}_\wedge a_{n-1}{}_\wedge a_n$$

of N by inserting $k-1$ bars "|" in the $n-1$ positions indicated by "∧". There are $n!$ permutations and $\binom{n-1}{k-1}$ ways of insertion. So there are $n!\binom{n-1}{k-1}$ ways of placing n distinct objects into k distinct boxes so that no one is empty. Since the boxes in question are indistinguishable, the answer in question is given by $\frac{n!}{k!}\binom{n-1}{k-1}$. □

Proposition 5.17. *The sequence $d_{n,k}$ defined by* (5.8) *satisfy the recurrence relation:*

$$\begin{cases} d_{n,n} = 1 & \text{for} \quad n \geq 0 \\ d_{n,0} = 0 & \text{for} \quad n \geq 1 \\ d_{n+1,k} = d_{n,k-1} + (n+k)d_{n,k} & \text{for} \quad n \geq k \geq 1 \end{cases} \tag{5.9}$$

Proof. First, $d_{n+1,1} = (n+1)! = 0 + (n+1)\cdot n! = d_{n,0} + d_{n,1}$. For $n \geq k \geq 2$,

$$\begin{aligned}
d_{n,k-1} + (n+k)d_{n,k} &= \frac{n!(n-1)!}{(k-1)!(k-2)!(n-k+1)!} \\
&\quad + (n+k)\cdot\frac{n!(n-1)!}{k!(k-1)!(n-k)!} \\
&= \frac{(n+1)!n!}{k!(k-1)!(n-k+1)!} = d_{n+1,k}.
\end{aligned}$$

□

Theorem 5.18. *The Lah numbers can be expressed by*

$$L_{n,k} = (-1)^n \frac{n!}{k!} \binom{n-1}{k-1}, \quad n \geq k \geq 0;$$

and the absolute value $|L_{n,k}|$ counts the number of ways of placing n distinguishable objects into k indistinguishable boxes such that no boxes are empty and objects in each box are linearly ordered.

Proof. It follows from Proposition 5.17 that the sequence $b_{n,k} = (-1)^n d_{n,k}$ satisfies the same recurrence relation (5.7) of Lah numbers $L_{n,k}$. Hence $L_{n,k} = b_{n,k}$. □

Proposition 5.19. *The number of surjective monotone functions from a totally ordered n-set to a totally ordered r-set = the number of ordered r-partitions of a positive integer n, and is equal to*

$$\binom{n-1}{r-1}.$$

Proof. For $n = 1$ it is obviously true. For $n > 1$, the map

$$\phi : n_1 + n_2 + \cdots + n_r \mapsto (n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_{r-1})$$

from the set of r-partitions of n to the set of strict monotone words of length $r - 1$ in $\{1, 2, \ldots, n-1\}_<$ is a bijection because it has the inverse

$$\psi : (s_1, s_2, \ldots, s_{r-1}) \mapsto s_1 + (s_2 - s_1) + \cdots + (s_{r-1} - s_{r-2}) + (n - s_{r-1}).$$

Then the two sets have the same cardinality; and the second set has cardinality $\binom{n-1}{r-1}$. □

Proposition 5.20.

$$[x]^{(n)} = \sum_{k=0}^{n} |L_{n,k}|\, [x]_{(k)}. \tag{5.10}$$

Proof. Let N and X be totally ordered sets such that $|N| = n$ and $|X| = x$. Then

$$\mathrm{Mon}(N, X) = \bigsqcup_{S \subset X} \text{Surj-Mon}(N, S).$$

Since $|\text{Sur-Mon}(N, S)| = \binom{n-1}{k-1}$ for $|S| = k$ by Proposition 5.19, we have

$$\frac{[x]^{(n)}}{n!} = \sum_{S \subset X} |\text{Sur-Mon}\,(N, S)| = \sum_{k=0}^{n} \binom{x}{k} \binom{n-1}{k-1} = \sum_{k=0}^{n} \frac{1}{k!} \binom{n-1}{k-1} [x]_{(k)}.$$

Therefore

$$[x]^{(n)} = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n-1}{k-1} [x]_{(k)}.$$

□

Exercise 5. Prove the following identities.

(1) $\sum_{k=m}^{n} L_{n,k} L_{k,m} = \delta_{n,m}$.
(2) $L_{n,m} = \sum_{k=m}^{n} (-1)^k s_{n,k} S_{k,m}$.

5.4. **Bell Numbers.**

Definition 5.21. *The number of partitions of an n-set is called the* **Bell number** *and is denoted by B_n with $B_0 = 1$. In other words,*

$$B_n = \sum_{k=1}^{n} S_{n,k}.$$

Proposition 5.22. (Dobinski's Formula)

$$B_n = \frac{1}{e}\sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

Proposition 5.23. (Recursion for the Bell Numbers)

$$\begin{aligned} B_0 &= 1, \\ B_{n+1} &= \sum_{k=0}^{n} \binom{n}{k} B_k. \end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{B_n}{n!}\, t^n = e^{e^t - 1}.$$

6. Bernoulli Numbers and Eulerian Numbers

Definition 6.1. *The* **Bernoulli numbers** *B_n are defined by*

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}\, x^n.$$

Proposition 6.2. *$B_0 = 1$, $B_{2n+1} = 0$ for $n \geq 1$, and*

$$B_n = -\frac{1}{n}\sum_{k=0}^{n-2} \binom{n}{k} B_k.$$

Definition 6.3. *The* **Euler numbers** *$A_{n,k}$ are defined by*

$$x^n = \sum_{k=0}^{n} \binom{x+k-1}{n} A_{n,k}$$

with $A_{0,0} = 1$.

Proposition 6.4.

$$A_{n,k} = \sum_{i=1}^{k} (-1)^i \binom{n+1}{i} (k-i)^n.$$

7. Catalan, Fibonacci, and Lucas Numbers

Definition 7.1. *The* **Catalan numbers** *C_n are the positive integers*

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0.$$

Proposition 7.2. (1) *The number of diagonal triangulations of a labelled n-gon is given by*

$$C_{n-2}.$$

(2) *The number of associations to compute the noncommutative product* $a_1 a_2 \cdots a_n$ *is given by*

$$C_{n-1}.$$

(3) *The number of increasing lattice path from* $(0,0)$ *to* (n,n) *such that all intermediate points* (a,b) *satisfying* $a \leq b$*, is given by*

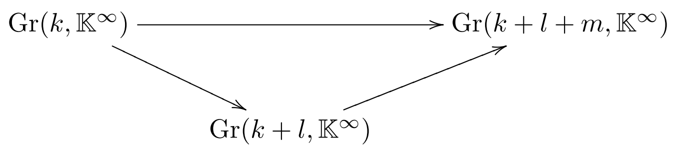
$$2C_n.$$

8. Grassmannian of ∞-dimensional subspaces

Let $\mathbb{K}$ be a field. Let $\mathbb{K}^\infty = \{(x_1, x_2, \ldots) : x_i \in \mathbb{K}, x_i = 0 \text{ for large enogh } i\}$. For each $k \geq 0$, let $\mathrm{Gr}(k, \mathbb{K}^\infty)$ be the Grassmannian of k-subspaces of $\mathbb{K}^\infty$. There are natural embeddings

$$\mathrm{Gr}(k, \mathbb{K}^\infty) \hookrightarrow \mathrm{Gr}(k+l, \mathbb{K}^\infty), \quad V \mapsto \mathbb{K}^l \times V$$

such that the following diagram is commutative:



So the collection $\mathrm{Gr} := \{\mathrm{Gr}(k, \mathbb{K}^\infty) \mid k \in \mathbb{Z}_{\geq 0}\}$ is a directed system. We define the *Grassmannian* $\mathrm{Gr}(\infty, \mathbb{K}^\infty)$ as the algebraic limit of the directed system Gr. If $\mathrm{Gr}(k, \mathbb{K}^\infty)$ is identified with the image under the embedding, then $\mathrm{Gr}(k, \mathbb{K}^\infty)$ is a subset of $\mathrm{Gr}(k+1, \mathbb{K}^\infty)$ and

$$\mathrm{Gr}(\infty, \mathbb{K}^\infty) = \bigcup_{k=0}^{\infty} \mathrm{Gr}(k, \mathbb{K}^\infty).$$

Each element of $\mathrm{Gr}(\infty, \mathbb{K}^\infty)$ can be viewed as a full flag of infinite length. Let $\mathrm{GL}_\infty(\mathbb{K})$ denote the group of all invertible $\infty \times \infty$ matrices of the form

$$\begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix},$$

where M is an invertible square matrix over $\mathbb{K}$ and I is an infinite identity matrix.

Theorem 8.1. *The space* $\mathrm{Gr}(\infty, \mathbb{K}^\infty)$ *can be viewed as the Grassmannian of* ∞*-dimensional subspaces of* $\mathbb{K}^\infty$*, and has the following cellular decomposition*

$$\mathrm{Gr}(\infty, \mathbb{K}^\infty) = \bigsqcup_{\sigma} X_\sigma,$$

where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ *is extended over all sequences such that* $2 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_k$*, and when* $k \geq 0$*,* $\sigma = (1)$*. Moreover,*

$$\sum_{n=0}^{\infty} p(n) q^n = \#\left(\mathrm{Gr}(\infty, \mathbb{K}_q^\infty)\right) = \sum_{\sigma \in \mathfrak{T}_\infty} q^{\mathrm{inv}(\sigma)} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

Proof. Let $\mathcal{M}(\infty)$ be the vector space of $\infty \times \infty$ matrices M over $\mathbb{K}$ such that the (i,j)-entry of M is zero when i or j is large enough. Let $\sim$ be the equivalence relation on $\mathcal{M}(\infty)$, generated by (1) $M \sim AM$, where $A \in \mathrm{GL}_\infty(\mathbb{K})$, and (2) $M \sim \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$. Then $\mathrm{Gr}(\infty, \mathbb{K}^\infty)$ can be viewed as the quotient space of $M(\infty)$ under the above equivalence relation.

Viewing in this way, for every element M of $\mathcal{M}(\infty)$, there exists a unique permutation σ of $\{1, 2, \ldots\}$ such that M is equivalent to the matrix of the schubert cell X_σ, where $\sigma = (1)$ or $\sigma = (\sigma_1, \ldots, \sigma_k)$, $\sigma_1 \geq 2$. For instance, the matrices of the Schubert cell

$$X_\tau = \left(\begin{array}{c|ccccccc} I_r & & & & O & & & \\ \hline & * & 1 & 0 & 0 & 0 & 0 & 0 \\ O & * & 0 & * & 1 & 0 & 0 & 0 \\ & * & 0 & * & 0 & * & * & 1 \end{array}\right),$$

where $\tau = (1) \cdots (r)(r+1, r+4, r+2)(r+3, r+5, r+6, r+7)$, are equivalent to the matrices of the Schubert cell

$$X_\sigma = \begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 & 0 & 0 \\ * & 0 & * & 0 & * & * & 1 \end{pmatrix},$$

where $\sigma = (1, 4, 2)(3, 5, 6, 7)$, respectively.

For each $\sigma \in \mathfrak{S}_\infty$, let $\sigma = (a_1, \ldots, a_i)(b_1, \ldots, b_j) \cdots (c_1, \ldots, c_k)$, where the leading entries $a_1, b_1, \ldots, c_1$ are the smallest in the corresponding cycles. For $0 \leq r < a_1$, we define

$$\sigma + r = (a_1 + r, \ldots, a_i + r)(b_1 + r, \ldots, b_j + r) \cdots (c_1 + r, \ldots, c_k + r).$$

Two permutations σ and τ are called *equivalent* if $\tau = \sigma + r$ for some $r \geq 0$. Let

$$\mathfrak{T}_\infty := \mathfrak{S}_\infty / \sim .$$

Let $\sim$ be an equivalence relation on $\mathfrak{S}_\infty$, defined by

$$\sigma \sim$$

$$\begin{array}{c} \begin{array}{ccccccccccc} & & \sigma_1 & & & \sigma_2 & & \sigma_3 & & & \end{array} \\ \left(\begin{array}{ccccccccc|cc} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ * & * & 0 & * & * & 1 & 0 & 0 & 0 & 0 & \cdots \\ * & * & 0 & * & * & 0 & * & 1 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) \end{array}.$$

$\mathrm{Gr}(\infty, \mathbb{K}^\infty)$ is decomposed into disjoint union

$$\mathrm{Gr}(\infty, \mathbb{K}^\infty) = \bigcup_\sigma X_\sigma$$

where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ is extended over all sequences such that $2 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_k$, and when $k \geq 0$, $\sigma = (1)$.