

# Counting Lattice Points of Rational Polyhedra

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## Abstract

The generating function  $F(P) = \sum_{\alpha \in P \cap \mathbf{Z}^N} x^\alpha$  for a rational polytope  $P$  carries all essential information of  $P$ . In this paper we show that for any positive integer  $n$ , the generating function  $F(P, n)$  of  $nP = \{nx : x \in P\}$  can be written as

$$F(P, n) = \sum_{\alpha \in A} P_\alpha(n) x^{n\alpha},$$

where  $A$  is the set of all vertices of  $P$  and each  $P_\alpha(n)$  is a certain periodic function of  $n$ . The Ehrhart reciprocity law follows automatically from the above formula. We also present a formula for the coefficients of Ehrhart polynomials in terms of elementary symmetric functions.

## 1 Introduction

Counting the lattice points in the integral dilates of a subset of Euclidean space  $\mathbf{R}^N$  is a well known and extensively studied problem. Specifically, let  $\Lambda = \mathbf{Z}^N$  be the standard  $N$ -dimensional lattice in  $\mathbf{R}^N$ . For a (metrically) bounded set  $P \subset \mathbf{R}^N$  and an integer  $n \geq 1$ , consider the number of lattice points in the dilated set  $nP = \{nx \mid x \in P\}$ :

$$L(P, n) = \text{the cardinality of } nP \cap \Lambda.$$

It is clear that for each  $n$  the mapping  $P \mapsto L(P, n)$  is a finitely additive measure on the class of bounded sets: for any bounded sets  $P$  and  $Q$ ,

$$L(P \cup Q, n) = L(P, n) + L(Q, n) - L(P \cap Q, n).$$

Ehrhart [6] first systematically studied the function  $n \mapsto L(P, n)$  in the case where  $P$  is a convex lattice polytope, i.e., the convex hull of a finite family of points of  $\Lambda$ . Ehrhart proved that in this case  $L(P, n)$  is a polynomial of  $n$  of degree  $\dim(P)$ . This result extends to any convex rational polytope  $P$ , i.e., to the convex hull of a finite family of points in  $\mathbf{R}^N$  with rational coordinates. Let  $d_P$  be the minimal positive integer such that all vertices of  $d_P P$  lie in  $\Lambda$ . Then  $L(P, n)$  is a quasi-polynomial of  $n$  of degree  $\dim(P)$  and period  $d_P$ , see

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for instance [8], Chapter 4. Here by a quasi-polynomial of degree  $m$  and period  $d$ , we mean a function  $n \mapsto \sum_{i=0}^m c_i(n)n^i$  where each  $c_i(n)$  is a periodic function of  $n$  with period  $d$ .

Brion [1] introduced a different approach to counting lattice points in a polytope. To describe his approach, consider the integral group ring  $\mathbf{Z}[\Lambda]$  of  $\Lambda$ . The additive generator of  $\mathbf{Z}[\Lambda]$  represented by a vector  $\alpha \in \Lambda$  will be denoted by  $x^\alpha$ . By definition,  $x^{\alpha+\beta} = x^\alpha x^\beta$  for any  $\alpha, \beta \in \Lambda$ . Note that the ring  $\mathbf{Z}[\Lambda]$  has no zero-divisors; we denote its field of quotients by  $\mathbf{Z}(\Lambda)$ .

For a bounded set  $P \subset \mathbf{R}^N$ , set

$$F(P) = \sum_{\alpha \in P \cap \Lambda} x^\alpha \in \mathbf{Z}[\Lambda].$$

Brion [1] proved that if  $P$  is a convex lattice polytope, then the function  $n \mapsto F(P, n) = F(nP)$  is *exponential* in  $n$ . More precisely, for any integer  $n \geq 1$ ,

$$F(P, n) = \sum_{\alpha \in A} P_\alpha x^{n\alpha} \tag{1}$$

where  $A$  is the set of vertices of  $P$  and  $P_\alpha \in \mathbf{Z}(\Lambda)$  for all  $\alpha \in A$ . The original proof of Formula (1) given in [1] uses the theory of toric varieties and a localization theorem in the equivariant  $K$ -theory. For an elementary proof, see [2].

In this paper we extend Formula (1) to rational polytopes. We need the following notation. For an integer  $d \geq 1$ , denote by  $d^{-1}\Lambda$  the set  $\{x \in \mathbf{R}^N \mid dx \in \Lambda\}$ . It is clear that  $d^{-1}\Lambda$  is a lattice in  $\mathbf{R}^N$  so that we can consider the group ring  $\mathbf{Z}[d^{-1}\Lambda]$  and its field of quotients  $\mathbf{Z}(d^{-1}\Lambda)$ .

**Theorem 1.1** *Let  $P$  be a convex rational polytope in  $\mathbf{R}^N$ . Let  $A$  be the set of vertices of  $P$  and  $d = d_P$  be the minimal positive integer such that  $A \subset d^{-1}\Lambda$ . Then there exist periodic functions  $\{P_\alpha : \mathbf{N} \rightarrow \mathbf{Z}(d^{-1}\Lambda)\}_{\alpha \in A}$  with period  $d$  such that for any integer  $n \geq 1$ ,*

$$F(P, n) = \sum_{\alpha \in A} P_\alpha(n) x^{n\alpha}. \tag{2}$$

It is easy to verify that the functions  $\{P_\alpha\}_{\alpha \in A}$  in this theorem are necessarily unique. In the case  $d = 1$ , we recover Brion's formula (1).

We can give a more precise description of the denominators of  $P_\alpha$  in Theorem 1.1. To this end, denote by  $I$  the kernel of the natural augmentation  $\mathbf{Z}[d^{-1}\Lambda] \rightarrow \mathbf{Z}$  determined by summation of coefficients. It is clear that  $I$  is a two-sided ideal in  $\mathbf{Z}[d^{-1}\Lambda]$ . For an integer  $m \geq 0$ , denote by  $I^m$  the  $m$ -th power of  $I$ .

**Theorem 1.2** *Under the conditions of Theorem 1.1, for all  $\alpha \in A$  and  $n \geq 1$ ,*

$$P_\alpha(n) \prod_{\substack{\alpha, \alpha' \in A \\ \alpha \neq \alpha'}} (x^{d\alpha} - x^{d\alpha'}) \in I^{M(M-1) - \dim(P)} \tag{3}$$

where  $M = \text{card}(A)$ .

The plan of the paper is as follows. As a warm up, we give in Section 2 an explicit formula for the Ehrhart polynomial of a rational simplex. Our exposition essentially follows the original ideas of Ehrhart, cf. also [8], Chapter 4 and [2]. In Section 3 we prove Theorems 1.1 and 1.2. In Section 4 we discuss the reciprocity law for rational polyhedra. In Section 5 we have collected a number of miscellaneous remarks and comments.

## 2 A closed formula for rational simplices

Let  $P$  be an  $m$ -dimensional closed simplex in  $\mathbf{R}^N$  with vertices  $\alpha_1, \alpha_2, \dots, \alpha_{m+1}$  whose coordinates are rational numbers. We can put the vertices of  $P$  in  $\mathbf{R}^{N+1}$  in the hyper-plane  $x_{N+1} = 1$  by lifting upward one unit. This gives  $m + 1$  linearly independent vectors

$$\beta_1 = (\alpha_1, 1), \beta_2 = (\alpha_2, 1), \dots, \beta_{m+1} = (\alpha_{m+1}, 1).$$

Denote by  $\tilde{P}$  the closed simplex in  $\mathbf{R}^{N+1}$  with the vertices  $\beta_1, \dots, \beta_{m+1}$ . For any integer  $n \geq 1$ , the number of lattice points in  $nP$  is the same as the number of lattice points in  $n\tilde{P}$ . Each lattice point in  $n\tilde{P}$  can be uniquely expanded as a linear combination  $b_1\beta_1 + \dots + b_{m+1}\beta_{m+1}$  where  $b_1, \dots, b_{m+1}$  are non-negative real numbers whose sum is equal to  $n$ . We have  $b_i = u_i + k_i d$ , where  $0 \leq u_i < d$ , and  $k_i = [b_i/d]$  is a non-negative integer. It is clear that the vector

$$d(k_1\beta_1 + \dots + k_{m+1}\beta_{m+1})$$

is a lattice point. Then  $u = u_1\beta_1 + \dots + u_{m+1}\beta_{m+1}$  must be a lattice point. Denote the set of such points by  $D(P)$ . Thus,

$$D(P) = \{u \in \mathbf{Z}^{N+1} \mid u = u_1\beta_1 + \dots + u_{m+1}\beta_{m+1} \text{ with } 0 \leq u_i < d\}.$$

Obviously,  $D(P)$  is a finite set containing the origin. It can be viewed geometrically as the set of lattice points in the ‘‘half-open’’ parallelotope generated by  $d\beta_1, \dots, d\beta_{m+1}$ .

For  $u \in \mathbf{Z}^{N+1}$ , denote by  $|u|$  the last coordinate of  $u$ . Clearly, if  $u = u_1\beta_1 + \dots + u_{m+1}\beta_{m+1} \in D(P)$  then

$$|u| = u_1 + \dots + u_{m+1} < (m + 1)d.$$

We summarize the discussion above in the following lemma.

**Lemma 2.1** *For any  $u \in D(P)$  and any non-negative integers  $k_1, \dots, k_{m+1}$  such that  $k_1 + \dots + k_{m+1} = (n - |u|)/d$ , the vector  $u + d(k_1\beta_1 + \dots + k_{m+1}\beta_{m+1})$  is a lattice point of  $n\tilde{P}$ . Each lattice point of  $n\tilde{P}$  can be uniquely written in this way.*

Now we give an explicit formula for the number of lattice points in the integral dilates of a closed rational simplex and a similar formula for an open rational simplex.

**Theorem 2.2** *Let  $P$  be an  $m$ -dimensional closed simplex in  $\mathbf{R}^N$  whose vertices have rational coordinates. Then for any integer  $n \geq 1$ , the number of lattice points of  $nP$  is*

$$L(P, n) = \sum_{\substack{u \in D(P) \\ d \mid (n - |u|)}} \binom{m + (n - |u|)/d}{m} \quad (4)$$

and the number of lattice points of the relative interior  $nP^0$  of  $nP$  is

$$L(P^0, n) = \sum_{\substack{u \in D(P) \\ d \mid (n + |u|)}} \binom{-1 + (n + |u|)/d}{m}. \quad (5)$$

*Proof* It follows from Lemma 2.1 that

$$L(P, n) = L(\tilde{P}, n) = \sum_{\substack{u \in D(P), |u| \leq n \\ d \mid (n - |u|)}} \sum_{\substack{k_1, \dots, k_{m+1} \geq 0 \\ k_1 + \dots + k_{m+1} = (n - |u|)/d}} 1. \quad (6)$$

The number of non-negative integer solutions  $(k_1, \dots, k_{m+1})$  for the linear equation  $k_1 + \dots + k_{m+1} = (n - |u|)/d$  is equal to the binomial coefficient

$$\binom{m + (n - |u|)/d}{m} = \binom{m + \frac{n - |u|}{d}}{m} \binom{m + \frac{n - |u|}{d} - 1}{m - 1} \dots \binom{\frac{n - |u|}{d} + 1}{1} / m!$$

appearing on the right hand side of Formula (4). This accounts for the binomial coefficients in Formula (4) corresponding to  $u \in D(P)$  with  $|u| \leq n$ . If  $n < |u|$ , then the binomial coefficient appearing in (4) is equal to 0. This results from the inequalities  $m > m + (n - |u|)/d \geq 0$  where the second inequality follows from the fact that  $|u| < (m + 1)d$  so that the integer  $(n - |u|)/d$  is bounded from below by  $-m$ . Therefore (4) follows from (6).

Let us prove Formula (5). Consider the finite set

$$D^0 = \{u \in \mathbf{Z}^{N+1} \mid u = u_1\beta_1 + \dots + u_{m+1}\beta_{m+1} \text{ with } 0 < u_i \leq d\}.$$

Denote the relative interior of  $\tilde{P}$  by  $\tilde{P}^0$ . Similar to Lemma 2.1, we note that for any  $u \in D^0$  and any non-negative integers  $k_1, \dots, k_{m+1}$  such that  $d(k_1 + \dots + k_{m+1}) = n - |u|$ , the vector  $u + d(k_1\beta_1 + \dots + k_{m+1}\beta_{m+1})$  is a lattice point of  $n\tilde{P}^0$ . Each lattice point of  $n\tilde{P}^0$  can be uniquely written in this way. Thus

$$\begin{aligned} L(P^0, n) &= L(nP^0) \\ &= L(n\tilde{P}^0) \\ &= \sum_{\substack{u \in D^0, |u| \leq n \\ d \mid (n - |u|)}} \sum_{\substack{k_1, \dots, k_{m+1} \geq 0 \\ k_1 + \dots + k_{m+1} = (n - |u|)/d}} 1 \\ &= \sum_{\substack{u \in D^0 \\ d \mid (n - |u|)}} \binom{m + (n - |u|)/d}{m}. \end{aligned} \quad (8)$$

Here we again use the fact that in the case  $n < |u|$ , the latter binomial coefficient vanishes. Notice that each vector  $u \in D^0$  can be uniquely written as  $d(\beta_1 + \dots + \beta_{m+1}) - v$  with  $v \in D(P)$ , and vice versa. Therefore (5) follows from (7) by changing the coordinates.  $\square$

The right hand sides of Formulas (4) and (5) are quasi-polynomials in  $n$  of degree  $m$  with period  $d$ . Since the number  $L(P, n)$  is additive with respect to disjoint union, we obtain the result mentioned in the introduction: for any convex rational polytope  $P$ , the number  $L(P, n)$  is a quasi-polynomial of  $n$  of degree  $\dim(P)$  with period  $d_P$ . In particular, if  $P$  is a convex lattice polytope, then  $L(P, n)$  is a polynomial of  $n$  of degree  $\dim(P)$ .

To end this section, we recall Ehrhart's reciprocity law: if  $P$  is a convex  $m$ -dimensional rational polytope in  $\mathbf{R}^N$  then  $L(P^0, -n) = (-1)^m L(P, n)$  for any integer  $n$ . In the case where  $P$  is a simplex, this formula directly follows from Theorem 2.2 and the reciprocity law for binomial coefficients,

$$\binom{m - a}{m} = (-1)^m \binom{a - 1}{m}.$$

For a discussion of the general case, see Section 4.

### 3 Proof of Theorems 1.1 and 1.2

We begin with two identities between  $m + 1$  commuting variables  $x_1, \dots, x_{m+1}$  where  $m \geq 0$ .

**Lemma 3.1** For any  $r = 0, 1, \dots, m - 1$ ,

$$\sum_{k=1}^{m+1} \left( x_k^r \prod_{\substack{i=1 \\ i \neq k}}^{m+1} \frac{1}{x_k - x_i} \right) = 0.$$

*Proof* For any integer  $r$ , consider the following  $(m + 1) \times (m + 1)$ -determinant

$$v_r(x_1, \dots, x_{m+1}) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{m+1} \\ \vdots & \vdots & & \vdots \\ x_1^{m-1} & x_2^{m-1} & \cdots & x_{m+1}^{m-1} \\ x_1^r & x_2^r & \cdots & x_{m+1}^r \end{vmatrix}.$$

In particular, for  $r = m$  we obtain the classical Vandermonde determinant

$$v(x_1, \dots, x_{m+1}) = v_m(x_1, \dots, x_{m+1}) = \pm \prod_{1 \leq i < j \leq m+1} (x_i - x_j).$$

(The sign on the right hand side equals  $(-1)^{m(m+1)/2}$  but we shall not use this.) Expanding the determinant  $v_r(x_1, \dots, x_{m+1})$  with respect to the last row we obtain

$$\begin{aligned} v_r(x_1, \dots, x_{m+1}) &= \sum_{k=1}^{m+1} (-1)^{m+1+k} x_k^r v(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{m+1}) \\ &= \pm \sum_{k=1}^{m+1} (-1)^k x_k^r \prod_{\substack{1 \leq i < j \leq m+1, \\ i \neq k, j \neq k}} (x_i - x_j) = \pm \left( \prod_{1 \leq i < j \leq m+1} (x_i - x_j) \right) \left( \sum_{k=1}^{m+1} x_k^r \prod_{\substack{i=1 \\ i \neq k}}^{m+1} \frac{1}{x_k - x_i} \right). \end{aligned}$$

It is obvious that for  $r = 0, 1, \dots, m - 1$ , the determinant  $v_r(x_1, \dots, x_{m+1})$  vanishes. This and the previous formula imply the claim of the lemma.  $\square$

**Lemma 3.2** For any integer  $r \geq 0$ ,

$$\sum_{\substack{k_1, \dots, k_{m+1} \geq 0 \\ k_1 + \dots + k_{m+1} = r}} x_1^{k_1} \dots x_{m+1}^{k_{m+1}} = \sum_{k=1}^{m+1} \left( x_k^{r+m} \prod_{\substack{i=1 \\ i \neq k}}^{m+1} \frac{1}{x_k - x_i} \right). \quad (9)$$

*Proof* The proof goes by induction on  $m$ . The case  $m = 0$  is obvious. Assume that for  $m$  variables Formula (9) holds and prove it for  $m + 1$  variables. By the inductive assumption,

$$\sum_{\substack{k_1, \dots, k_{m+1} \geq 0 \\ k_1 + \dots + k_{m+1} = r}} x_1^{k_1} \dots x_{m+1}^{k_{m+1}} = \sum_{k=0}^r x_{m+1}^k \sum_{\substack{k_1, \dots, k_m \geq 0 \\ k_1 + \dots + k_m = r-k}} x_1^{k_1} \dots x_m^{k_m}$$

$$\begin{aligned}
&= \sum_{k=0}^r x_{m+1}^k \sum_{l=1}^m \left( x_l^{r-k+m-1} \prod_{\substack{i=1 \\ i \neq l}}^m \frac{1}{x_l - x_i} \right) \\
&= \sum_{l=1}^m \left( x_l^{r+m-1} \cdot \prod_{\substack{i=1 \\ i \neq l}}^m \frac{1}{x_l - x_i} \cdot \sum_{k=0}^r (x_{m+1}/x_l)^k \right) \\
&= \sum_{l=1}^m \left( x_l^{r+m-1} \cdot \prod_{\substack{i=1 \\ i \neq l}}^{m+1} \frac{1}{x_l - x_i} \cdot \left( x_l - \frac{x_{m+1}^{r+1}}{x_l^r} \right) \right) \\
&= \sum_{l=1}^m \left( x_l^{r+m} \prod_{\substack{i=1 \\ i \neq l}}^{m+1} \frac{1}{x_l - x_i} \right) - x_{m+1}^{r+1} \sum_{l=1}^m \left( x_l^{m-1} \prod_{\substack{i=1 \\ i \neq l}}^{m+1} \frac{1}{x_l - x_i} \right).
\end{aligned}$$

By Lemma 3.1,

$$\sum_{l=1}^m \left( x_l^{m-1} \prod_{\substack{i=1 \\ i \neq l}}^{m+1} \frac{1}{x_l - x_i} \right) = -x_{m+1}^{m-1} \prod_{\substack{i=1 \\ i \neq m+1}}^{m+1} \frac{1}{x_l - x_i}.$$

Substituting this expression in the previous formula we obtain the claim of the lemma.  $\square$

### 3.1 Proof of Theorem 1.1

Let us first consider the case where  $P$  is an  $m$ -dimensional rational simplex with vertices  $\alpha_1, \dots, \alpha_{m+1}$ . Recall the set  $D(P)$  introduced in Section 2. For a vector  $u \in D(P) \subset \mathbf{Z}^{N+1} = \mathbf{Z}^N \oplus \mathbf{Z}$  denote by  $\hat{u}$  its projection into  $\mathbf{Z}^N$ . Lemma 2.1 implies that

$$\begin{aligned}
F(nP) &= \sum_{\substack{u \in D(P), k_1, \dots, k_{m+1} \geq 0 \\ |u| \leq n, d \mid (n-|u|), k_1 + \dots + k_{m+1} = (n-|u|)/d}} x^{\hat{u} + d(k_1 \alpha_1 + \dots + k_{m+1} \alpha_{m+1})} \\
&= \sum_{\substack{u \in D(P), |u| \leq n \\ d \mid (n-|u|)}} x^{\hat{u}} \sum_{\substack{k_1, \dots, k_{m+1} \geq 0 \\ k_1 + \dots + k_{m+1} = (n-|u|)/d}} (x^{d\alpha_1})^{k_1} \dots (x^{d\alpha_{m+1}})^{k_{m+1}}. \tag{10}
\end{aligned}$$

By Lemma 3.2 applied to  $x_1 = x^{d\alpha_1}, \dots, x_{m+1} = x^{d\alpha_{m+1}}$ ,

$$F(nP) = \sum_{\substack{u \in D(P) \\ d \mid (n-|u|)}} \left( x^{\hat{u}} \sum_{k=1}^{m+1} x^{(n-|u|+dm)\alpha_k} \prod_{\substack{i=1 \\ i \neq k}}^{m+1} \frac{1}{x^{d\alpha_k} - x^{d\alpha_i}} \right). \tag{11}$$

Note that the condition  $|u| \leq n$  does not appear on the right hand side because if this condition is not met, then (as in the proof of Formula 4) we have  $(n-|u|)/d+m \in \{0, 1, \dots, m-1\}$  and the corresponding summand vanishes by Lemma 3.1. Thus,

$$F(nP) = \sum_{k=1}^{m+1} P_{\alpha_k}(n) x^{n\alpha_k} \tag{12}$$

where

$$P_{\alpha_k}(n) = \left( \prod_{\substack{i=1 \\ i \neq k}}^{m+1} \frac{1}{x^{d\alpha_k} - x^{d\alpha_i}} \right) \sum_{\substack{u \in D(P) \\ d \mid (n-|u|)}} x^{\hat{u} + (dm-|u|)\alpha_k}. \tag{13}$$

It is obvious that  $P_{\alpha_k}(n)$  is a periodic function of  $n$  with period  $d$ . This proves the claim of the theorem for closed rational simplices. Note that each such simplex is the disjoint union of its open faces. The additivity of the function  $n \mapsto F(nP)$  and an easy induction imply that the claim of Theorem 1.1 holds also for open rational simplices.

Any convex rational polytope  $P$  can be presented as a disjoint union of open rational simplices whose vertices belong to the set of vertices of  $P$ . Now, the claim of the theorem follows from the additivity of  $F(nP)$ .

### 3.2 Proof of Theorem 1.2

We decompose  $P$  as a disjoint union of open rational simplices  $\{\Delta_k\}_k$  whose vertices belong to the set  $A$ . By additivity  $F(nP) = \sum_k F(n\Delta_k)$ . By Formulas (12, 13) and the obvious inclusion-exclusion argument, each  $F(n\Delta_k)$  is a finite sum of expressions  $p(n)x^{n\alpha}$  where  $p(n) \in \mathbf{Z}(d^{-1}\Lambda)$  is a rational function whose denominator is a product of  $\dim(\Delta_k) \leq \dim(P)$  factors of type  $x^{d\alpha} - x^{d\alpha'}$  where  $\alpha, \alpha' \in A$ . This directly implies the claim of the theorem.

## 4 Reciprocity

By a rational polyhedron we shall mean a finite simplicial complex in  $\mathbf{R}^N$  which admits a triangulation whose vertices have rational coordinates. It is clear from the discussion at the end of Section 3.1 that Theorem 1.1 directly extends to rational polyhedra. Note the following corollary of Theorem 1.1.

**Corollary 4.1** *Let  $P$  be a rational polyhedron in  $\mathbf{R}^N$  and let  $A$  be its set of vertices. Let  $d = d_P$  be the minimal positive integer such that  $A \subset d^{-1}\Lambda$ . If the underlying topological space of  $P$  is a topological manifold (possibly with boundary), then there are unique periodic functions  $\{P_\alpha^0 : \mathbf{N} \rightarrow \mathbf{Z}(d^{-1}\Lambda)\}_{\alpha \in A}$  of period  $d$  such that for any integer  $n \geq 1$ ,*

$$F(P^0, n) = \sum_{\alpha \in A} P_\alpha^0(n) x^{n\alpha} \quad (14)$$

where  $P^0 = P \setminus \partial P$  is the interior of  $P$ .

This corollary follows from the generalization of Theorem 1.1 to rational polyhedra mentioned above and the formula  $F(P^0, n) = F(P, n) - F(\partial P, n)$ .

As in the Ehrhart theory, we can use Formulas (2, 14) to extend both  $F(P, n)$  and  $F(P^0, n)$  to all integers  $n$ .

**Theorem 4.2** *Let  $P$  be a rational polyhedron in  $\mathbf{R}^N$  whose underlying topological space is an  $m$ -dimensional manifold (possibly with boundary). Then for any integer  $n$ ,*

$$F(P, -n) = (-1)^m F(-P^0, n). \quad (15)$$

In the case of a convex lattice polytope, this theorem was obtained by Brion [1]. His proof is based on the theory of toric varieties and the Serre duality. Our proof of Theorem 4.2 is quite elementary. It is based on a reduction to simplices provided by the following lemma, cf. [7].

**Lemma 4.3** *Let  $\psi_1, \psi_2$  be two finitely additive measures on the class of bounded subsets of  $\mathbf{R}^N$  with values in an Abelian group. If  $\psi_1(\Delta^0) = (-1)^{\dim(\Delta)}\psi_2(\Delta)$  for any rational simplex  $\Delta$  then  $\psi_1(P^0) = (-1)^m\psi_2(P)$  for any rational polyhedron  $P$  whose underlying topological space is an  $m$ -dimensional manifold.*

*Proof* By additivity, we have for any closed simplex  $\Delta$  and  $i = 1, 2$ ,

$$\psi_i(\Delta) = \sum_{\tau \leq \Delta} \psi_i(\tau^0)$$

where  $\tau$  runs over all closed faces of  $\Delta$  and  $\tau^0$  is the interior of  $\tau$ . Using the inclusion-exclusion arguments we obtain that

$$\psi_i(\Delta^0) = \sum_{\tau \leq \Delta} (-1)^{|\Delta| - |\tau|} \psi_i(\tau)$$

where  $|\tau|$  denotes the dimension of  $\tau$ .

Now, consider a rational polyhedron  $P \subset \mathbf{R}^N$  whose underlying topological space is an  $m$ -dimensional manifold. Fix a triangulation  $X$  of  $P$  consisting of (closed) rational simplices. By additivity,

$$\psi_1(P) = \sum_{\Delta \in X} \psi_1(\Delta^0) = \sum_{\Delta \in X} \sum_{\tau \leq \Delta} (-1)^{|\Delta| - |\tau|} \psi_1(\tau) = \sum_{\tau \in X} \psi_1(\tau) \left( \sum_{\Delta \in X, \Delta \geq \tau} (-1)^{|\Delta| - |\tau|} \right).$$

It is clear that for a given simplex  $\tau \in X$ ,

$$\sum_{\Delta \in X, \Delta \geq \tau} (-1)^{|\Delta| - |\tau|} = 1 + \sum_{\Delta \in X, \Delta > \tau} (-1)^{|\Delta| - |\tau|} = 1 - \chi(lk_P(\tau))$$

where  $\chi$  is the Euler characteristic and  $lk_P(\tau)$  is the link of the simplex  $\tau$  in the polyhedron  $P$ . If  $\tau \subset \partial P$ , then the link  $lk_P(\tau)$  is homeomorphic to a closed ball of dimension  $m - |\tau| - 1$  and  $1 - \chi(lk_P(\tau)) = 0$ . If  $\tau$  does not lie on the boundary of  $P$ , then the link of  $\tau$  is homeomorphic to an  $(m - |\tau| - 1)$ -dimensional sphere and

$$1 - \chi(lk_P(\tau)) = 1 - (1 + (-1)^{m - |\tau| - 1}) = (-1)^{m - |\tau|}.$$

Hence,

$$\psi_1(P) = \sum_{\tau \in X, \tau \cap P^0 \neq \emptyset} (-1)^{m - |\tau|} \psi_1(\tau).$$

By assumption and the additivity,

$$\sum_{\tau \in X, \tau \cap P^0 \neq \emptyset} (-1)^{m - |\tau|} \psi_1(\tau) = (-1)^m \sum_{\tau \in X, \tau \cap P^0 \neq \emptyset} \psi_2(\tau^0) = (-1)^m \psi_2(P^0)$$

which proves the claim of the lemma.



## 4.1 Proof of Theorem 4.2

By the previous lemma, it suffices to prove the equality  $F(P, -n) = (-1)^m F(-P^0, n)$  for any  $m$ -dimensional rational simplex  $P$  with vertices  $\alpha_1, \dots, \alpha_{m+1}$ . The same arguments as in the proof of Formulas (7) and (11) imply that

$$F(P^0, n) = F(nP^0) = \sum_{\substack{u \in D^0 \\ d \mid (n-|u|)}} \left( x^{\hat{u}} \sum_{k=1}^{m+1} x^{(n-|u|+dm)\alpha_k} \prod_{\substack{i=1 \\ i \neq k}}^{m+1} \frac{1}{x^{d\alpha_k} - x^{d\alpha_i}} \right).$$

As we have already observed, each vector  $u \in D^0$  can be uniquely written as  $d(\beta_1 + \dots + \beta_{m+1}) - v$  with  $v \in D(P)$ , and vice versa. Then  $\hat{u} = de - \hat{v}$  where  $e = \alpha_1 + \dots + \alpha_{m+1} \in \mathbf{R}^N$ . Note that  $|u| = d(m+1) - |v|$ . Therefore,

$$F(P^0, n) = \sum_{\substack{v \in D(P) \\ d \mid (n+|v|)}} \left( x^{de - \hat{v}} \sum_{k=1}^{m+1} x^{(n+|v|-d)\alpha_k} \prod_{\substack{i=1 \\ i \neq k}}^{m+1} \frac{1}{x^{d\alpha_k} - x^{d\alpha_i}} \right).$$

Thus,

$$F(P^0, -n) = \sum_{\substack{v \in D(P) \\ d \mid (n-|v|)}} \left( x^{de - \hat{v}} \sum_{k=1}^{m+1} x^{(-n+|v|-d)\alpha_k} \prod_{\substack{i=1 \\ i \neq k}}^{m+1} \frac{1}{x^{d\alpha_k} - x^{d\alpha_i}} \right).$$

Applying the conjugation, we obtain

$$\begin{aligned} F(-P^0, -n) &= \sum_{\substack{v \in D(-P) \\ d \mid (n-|v|)}} \left( x^{de - \hat{v}} \sum_{k=1}^{m+1} x^{(-n+|v|-d)\alpha_k} \prod_{\substack{i=1 \\ i \neq k}}^{m+1} \frac{1}{x^{d\alpha_k} - x^{d\alpha_i}} \right) \\ &= \sum_{\substack{v \in D(P) \\ d \mid (n-|v|)}} \left( x^{\hat{v} - de} \sum_{k=1}^{m+1} x^{(n+d-|v|)\alpha_k} \prod_{\substack{i=1 \\ i \neq k}}^{m+1} \frac{-x^{d\alpha_k + d\alpha_i}}{x^{d\alpha_k} - x^{d\alpha_i}} \right) \\ &= (-1)^m \sum_{\substack{v \in D(P) \\ d \mid (n-|v|)}} \left( x^{\hat{v}} \sum_{k=1}^{m+1} x^{(n-|v|+dm)\alpha_k} \prod_{\substack{i=1 \\ i \neq k}}^{m+1} \frac{1}{x^{d\alpha_k} - x^{d\alpha_i}} \right) = (-1)^m F(P, n) \end{aligned}$$

where the last equality follows from Formula (11).

## 5 Miscellaneous

(1). The elements  $P_\alpha \in \mathbf{Z}(\Lambda)$  appearing in Formula (1) have an interesting geometric interpretation. Let  $C_\alpha \subset \mathbf{R}^N$  be the tangent cone to  $P$  at its vertex  $\alpha$ , i.e., the convex cone generated by half-lines  $\mathbf{R}_+(p - \alpha)$  where  $p$  runs over all points of  $P$ . Clearly, the set  $C_\alpha \cap \Lambda$  is infinite so that the formal sum  $F(C_\alpha) = \sum_{\beta \in C_\alpha \cap \Lambda} x^\beta$  is not an element of  $\mathbf{Z}[\Lambda]$ . However,  $F(C_\alpha)$  can be viewed as a formal Laurent series representing a rational function,

i.e., an element of  $\mathbf{Z}(\Lambda)$ . Brion [1] proved that for any vertex  $\alpha$  of  $P$ , we have  $P_\alpha = F(C_\alpha)$ . There is a natural generalization of this formula to rational convex polytopes. Namely,

$$P_\alpha(n) = \sum_{\beta \in C_\alpha, \beta + n\alpha \in \Lambda} x^\beta.$$

By additivity, it suffices to check this formula for closed rational simplices where it can be directly deduced from Formula (13).

(2). There is another generalization of the Ehrhart polynomial obtained by counting lattice points with numerical weights. For a function  $\varphi : \Lambda \rightarrow \mathbf{R}$ , set

$$L_\varphi(P, n) = \sum_{\alpha \in nP \cap \Lambda} \varphi(\alpha).$$

Brion and Vergne [3] proved that if  $\varphi$  is a homogeneous polynomial in the coordinates in  $\mathbf{R}^N$  of degree  $k$ , then for any convex lattice polytope  $P$ , the number  $L_\varphi(P, n)$  is a polynomial of  $n$  of degree  $\dim(P) + k$ . Here is a generalized version of their results.

**Theorem 5.1** *Let  $\varphi : \mathbf{R}^N \rightarrow \mathbf{R}$  be a homogeneous polynomial function of degree  $k$ . If  $P$  is a rational polyhedron in  $\mathbf{R}^N$  then  $n \mapsto L_\varphi(P, n)$  is a quasi-polynomial of degree  $\dim(P) + k$  and period  $d_P$ . If the underlying topological space of  $P$  is an  $m$ -dimensional manifold (possibly with boundary), then  $n \mapsto L_\varphi(P^0, n)$  is a quasi-polynomial of degree  $m + k$  and for any  $n$ ,*

$$L_\varphi(P, -n) = (-1)^{m+k} L_\varphi(P^0, n) \tag{16}$$

This can be deduced from Theorems 1.1, 1.2, 4.2. The main ingredient is the observation that  $L_\varphi(P, n)$  is determined by a finite truncation of the Taylor power series of  $F(P, n)$ .

(3). The computations of Section 2 suggest a geometric interpretation of the coefficients of the Ehrhart polynomial of a closed  $m$ -dimensional lattice simplex  $P$ . Let us write  $L(P, n)$  as

$$L(P, n) = c_m(P)n^m + c_{m-1}(P)n^{m-1} + \cdots + c_1(P)n + c_0(P).$$

It is clear that each  $c_i(P)$  is a *valuation*, i.e., a finitely additive measure on the class of lattice polytopes. These valuations are the lattice analogues of the intrinsic volumes  $\mu_i$ ,  $0 \leq i \leq N$ . For a rectangular box  $B$  with side length  $x_1, x_2, \dots, x_N$ ,  $\mu_i$  has the explicit geometric interpretation

$$\mu_i(B) = s_i(x_1, \dots, x_N),$$

where  $s_i(x_1, \dots, x_N)$  is the  $i$ -th elementary symmetric polynomial of  $x_1, \dots, x_N$ , and  $s_0 = 1$  by convention. With our formulas (4) and (5), the coefficients  $c_i$ 's obtain a similar geometric interpretations. Write each term of (4) as

$$\binom{n+m-|u|}{m} = (n+m-|u|)(n+m-1-|u|) \cdots (n+1-|u|)/m!.$$

Then coefficient of  $n^i$  is the  $(m-i)$ -th symmetric polynomial of the variables  $m-|u|, \dots, 1-|u|$ , divided by  $m!$ , i.e.,

$$\frac{1}{m!} s_{m-i}(m-|u|, \dots, 1-|u|).$$

Thus,

$$c_i(P) = \frac{1}{m!} \sum_{u \in D(P)} s_{m-i}(m - |u|, \dots, 1 - |u|).$$

Recall the geometric interpretation of  $c_m$  and  $c_{m-1}$  (see for instance [2] or [8]):

$$c_m(P) = \text{vol}(P), c_{m-1}(P) = \text{vol}(\partial P).$$

This implies  $\text{card}(D(P)) = m! \text{vol}(P)$  and

$$\sum_{u \in D(P)} |u| = (m-1)! \left( \frac{m(m+1)}{2} \text{vol}(P) - \text{vol}(\partial P) \right).$$

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