HKUST

MATH150 Introduction to Differential Equations

Final Examination (Version A) Name:____________________

26^{th} May 2007 Student I.D.:____________________

08:30–10:30 Tutorial Section:____________________

Directions:

- Write your name, ID number, and tutorial section in the space provided above.
- DO NOT open the exam booklet until instructed to do so.
- When instructed to open the exam booklet, check that you have, in addition to this cover page, 12 pages of questions.
- Turn off all mobile phones and pagers during the examination.
- This is a closed book examination.
- You are advised to try the problems you feel more comfortable with first.
- You may write on both sides of the examination papers.
- There are 8 multiple choice questions. **DO NOT guess wildly! If you do not have confidence in your answer leave the question blank. Each incorrectly answered question will result in a 0.5 point deduction**.
- For the short and long questions, you must show the working steps of your answers in order to receive full points.
- You must not possess any written or printed papers that contains information related to this examination.
- Cheating is a serious offense. Students caught cheating are subject to a zero score as well as additional penalties.

Question No.	Points	Out of
Q. 1-8		24
Q. 9-14		38
Q. 15		17
Q. 16		21
Total Points		100

Part I: Fill in answer *in the allocated answer box* for the following 8 multiple choice questions is worth 3 points each. DO NOT guess wildly! If you do not have confidence in your answer leave the box blank. Each incorrectly answered question will result in a 0.5 point deduction.

Question	1	2	3	4	5	6	7	8	Total
Answer									

1. Match the graph of the direction field (slopefield) to the differential equation.

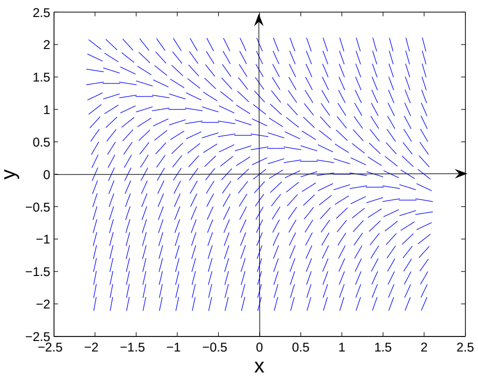
(a) $\dfrac{dy}{dx} = x - 2y$

(b) $\dfrac{dy}{dx} = x + 2y$

(c) $\dfrac{dy}{dx} = 1 - x - 2y$

(d) $\dfrac{dy}{dx} = 2 - y$

(e) $\dfrac{dy}{dx} = x - 2y + 1$



2. Which of the following differential equations is/are linear equation(s)?

(I) $\dfrac{d^3y}{dt^3} + y = e^t$

(II) $\dfrac{dy}{dt} + ty^2 = 0$

(III) $\dfrac{d^2y}{dt^2} + \sin(t + y) = \sin t$

(a) (I) (b) (II) (c) (III) (d) (I) and (II) (e) (II) and (III)

3. Determine the interval in which the solution of

$$(t^2 - 9)y' + (\ln t)y = 2t, \quad y(1) = 2$$

is certain to exist.

(a) $t < 3$ (b) $3 < t$ (c) $-3 < t < 3$ (d) $0 \le t \le 3$ (e) $0 < t < 3$

4. Consider the differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0. \qquad (*)$$

Which of the following is/are sufficient condition(s) for $(*)$ to be exact.

(I) $\dfrac{\partial M}{\partial x} = \dfrac{\partial N}{\partial y}$.

(II) There exists $F(x,y)$ such that $\dfrac{\partial F}{\partial x} = M$ and $\dfrac{\partial F}{\partial y} = N$.

(III) $\dfrac{\partial M}{\partial y} = 0$ and $\dfrac{\partial N}{\partial x} = 0$.

(a) (II) only

(b) (II) and (III) only

(c) (I) and (II) only

(d) (I) and (III) only

(e) (I), (II) and (III)

5. Consider the following differential equation

$$\frac{1}{t}\frac{dy}{dt} + y = 1$$

(a) Integrating factor $= e^{\frac{t^2}{2}}$; General solution $= 1 + Ce^{-\frac{t^2}{2}}$

(b) Integrating factor $= e^{-t}$; General solution $= C$

(c) Integrating factor $= e^{-\frac{t^2}{2}}$; General solution $= 1 + Ce^{\frac{t^2}{2}}$

(d) Integrating factor $= e^{t}$; General solution $= Ce^{-\frac{t^2}{2}}$

(e) Integrating factor $= t$; General solution $= C$

6. Suppose $\{y_1, y_2\}$ is a set of fundamental solutions to the differential equation $y'' + (\sin x)y' + e^x y = 0$. Suppose $W(y_1, y_2)$ is the Wronskian of y_1 and y_2. Then $W(y_1, y_2) =$

(a) 0.

(b) $ce^{\cos x}$ for some constant c.

(c) ce^{-e^x} for some constant c.

(d) $c_1 \sin x + c_2 e^x$ for some constants c_1 and c_2.

(e) None of the above.

7. Find the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$ of the solution of the given initial value problem :

$$2y'' + 4y' + 5y = e^{-2t}\sin 3t, \qquad y(0) = 2, \quad y'(0) = -1 \ .$$

(a) $Y(s) = \dfrac{2s+4}{2s^2+4s+5} + \dfrac{3}{(2s^2+4s+5)(s^2+4s+13)}$

(b) $Y(s) = \dfrac{2s-6}{2s^2+4s+5} + \dfrac{s+2}{(2s^2+4s+5)(s^2+4s+13)}$

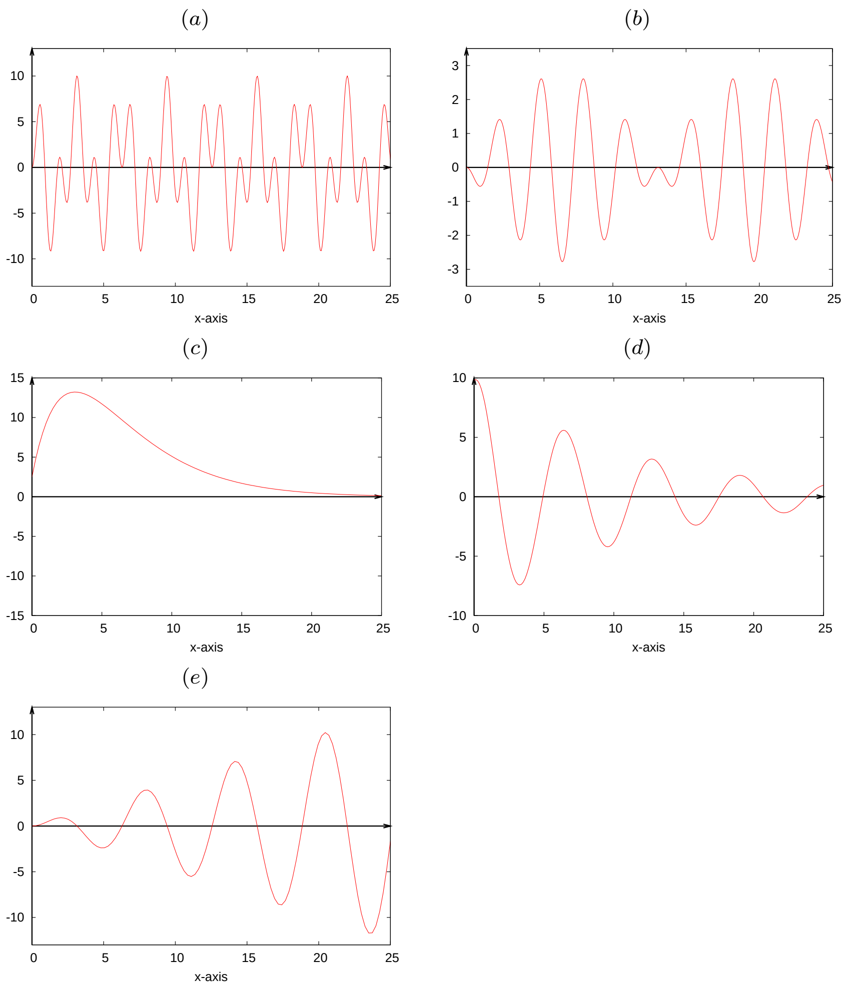
(c) $Y(s) = \dfrac{4s-2}{2s^2+4s+5} + \dfrac{3}{(2s^2+4s+5)(s^2+4s+13)}$

(d) $Y(s) = \dfrac{4s+6}{2s^2+4s+5} + \dfrac{3}{(2s^2+4s+5)(s^2+4s+13)}$

(e) $Y(s) = \dfrac{4s-6}{2s^2+4s+5} + \dfrac{3}{(2s^2+4s+5)(s^2+4s+13)}$

8. Which figure below best represents the solution to the initial value problem:

$$y'' + y = \cos t, \quad y(0) = 0, \ y'(0) = 0.$$



Part II: Answer each of the following 6 short questions. ***Show all your work for full credit.***

Question	9	10	11	12	13	14	**Total**
Points	/8	/6	/6	/6	/6	/6	/38

9. Suppose that the logistic equation
$$\frac{dx}{dt} = kx(M - x),$$
where k and M are positive constants, models a population $x(t)$ of fish in a lake after t months during which no fishing occurs. Now suppose that, because of fishing, fishes are removed from the lake at the rate of hx fishes per month, where h is a positive constant.

   (a) [*3 pts*] If $0 < h < kM$, show that the new fish population (with fishes removed at the above rate) still satisfy a logistic equation and hence find two equilibrium solutions when fishing occurs.

   Answer: ______________________________

   (b) [*2 pts*] Classify each equilibrium solution in (a) as asymptotically stable or unstable.

   Answer: ______________________________

   (c) [*3 pts*] If $h = kM$, show that $x(t) \to 0$ as $t \to \infty$ i.e. the lake is eventually fished out.

   Answer: ______________________________

10. [*6 pts*] Find the *general solution* of the differential equation

$$y'' - 2y = e^{-x}\sin x.$$

Answer: ____ .

11. Consider a piecewise continuous function

$$g(t) = \begin{cases} t - \frac{\pi}{2} & \text{if } 0 \le t < \frac{\pi}{2}; \\ \cos t & \text{if } \frac{\pi}{2} \le t < \pi; \\ 0 & \text{if } t \ge \pi. \end{cases}$$

(a) [*3 pts*] Express $g(t)$ in terms of unit step functions $u_c(t)$ and the functions appearing in the definition of $g(t)$.

Answer: $g(t)$ = ____

(b) [*3 pts*] Find the Laplace transform of $g(t)$.

Answer: $\mathcal{L}\{g(t)\}$ = ____

12. [*6 pts*] Use the method of Laplace transform to solve

$$f(t) = \sin 3t + \int_0^t f(s) \sin 3(t-s)\, ds.$$

Answer: ____ .

13. [*6 pts*] Find the Fourier series of

$$f(x) = \begin{cases} x, & -2 \le x < 0; \\ -x, & 0 \le x \le 2. \end{cases}$$

Answer: ____ .

14. [*6 pts*] Given that $y(x) = x$ is a solution of

$$x^2y'' - x(x+2)y' + (x+2)y = 0.$$

Find the general solution of the above differential equation.

Answer: ____ .

Part III: Answer the following two long questions.

15. A mass of 2 kg stretches a hanging spring downward by 0.25 metres. The mass is then acted on by an external force $20\cos(8t)$ Newtons, and is damped by a force which is opposite to the direction of motion and proportional to the magnitude of velocity. It is known that the damping force is 2 Newtons when the velocity is 0.25 metres/second. Let us assume that the acceleration due to gravity is 10 metres/second^2.

(a) [*2 pt*] Find the spring constant k; so, force $= k \cdot$ stretch.

Answer: $k =$ ____________________

(b) [*2 pt*] Find the damping constant γ; so, force $= \gamma \cdot$ velocity.

Answer: $\gamma =$ ____________________

(c) [*3 pt*] Let u denote the distance of the mass from its equilibrium position. Write down a 2nd order differential equation for u.

Answer: ____________________

(d) [*6 pt*] Suppose the mass is initially at rest at its equilibrium position. Formulate the initial value problem for u and hence solve for u.

.

Answer: $u =$ ____

(e) [*4 pt*] Express the steady-state solution in the form $R\cos(\omega t - \delta)$ and hence find the amplitude and phase of the motion.

Answer: ____

16. (A) Suppose an elastic string of length 50cm is being fixed at both ends in a horizontal position. Let $u(x,\, t)$ be the vertical displacement of the string at x centermeters from one end and at time t. Suppose the displacement function satisfies

$$\begin{aligned} &4u_{xx} = u_{tt}, && t > 0; \\ &u(0,\, t) = 0 = u(50,\, t), && t \geq 0; \\ &u_t(x,\, 0) = 0, \ \ u(x,\, 0) = f(x), && 0 \leq x \leq 50, \end{aligned} \tag{1}$$

where

$$f(x) = \begin{cases} -2x/50, & 0 \leq x \leq 25; \\ (x-50)/25, & 25 < x \leq 50. \end{cases}$$

(a) [*2 pts*] Sketch the initial position of the string given by $f(x)$:

Answer: The sketch is ____________________________________ .

(b) [*1 pt*] Is the string released initially at rest ____________ (*answer "yes" or "no"*) ?

(c) [*6 pts*] Suppose a solution to (1) is in the form

$$u(x,\, t) = \sum_{n=1}^{\infty} c_n u_n(x,\, t) \tag{2}$$

where $u_n(x,\, t) = X_n(x)T_n(t)$ for each $n = 1,\, 2,\, 3,\, \cdots$. Derive with reason the boundary value problem satisfied by $X_n(x)$. Then solve the problem.

Answers ____ .

(e) [*3 pts*] Derive with reason the boundary value problem satisfied by $T_n(t)$. Then solve the problem.

Answers: ____ .

(f) [*3 pts*] Explain how one can determine the unknown coefficients c_n. Write down a formula for c_n without actually evaluating the integrals.

Answer: ____ .

(B) Suppose the problem (A) above is modified so that the displacement function $u(x, t)$ still satisfy (1) but with new conditions $u(x, 0) = 0$ and $u_t(x, 0) = g(x), \quad 0 \le x \le 50$ instead, where $g(x)$ is some given function. We again assume that a solution $u(x, t)$ takes the form (2) for some different c_n.

(g) [*2 pts*] Is the string released at rest initially ____________ (*answer "yes" or "no"*) ?

(h) [*4 pts*] Determine with details the $u_n(x, t), \; n = 1, 2, 3, \cdots$.

Answer: ____ .

Table 1: Laplace transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	1	$\frac{1}{s}, \qquad s > 0$
2	e^{at}	$\frac{1}{s-a}, \qquad s > a$
3	$t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \qquad s > 0$
4	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \qquad s > 0$
5	$\sin at$	$\frac{a}{s^2+a^2}, \qquad s > 0$
6	$\cos at$	$\frac{s}{s^2+a^2}, \qquad s > 0$
7	$\sinh at$	$\frac{a}{s^2-a^2}, \qquad s > \|a\|$
8	$\cosh at$	$\frac{s}{s^2-a^2}, \qquad s > \|a\|$
9	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \qquad s > a$
10	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \qquad s > a$
11	$t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \qquad s > 0$
12	$u_c(t)$	$\frac{e^{-cs}}{s}, \qquad s > 0$
13	$u_c(t)\,f(t-c)$	$e^{-cs}\,F(s)$
14	$e^{ct}\,f(t)$	$F(s-c)$
15	$f(ct)$	$\frac{1}{c}\,F\left(\frac{s}{c}\right), \quad c > 0$
16	$\int_0^t f(t-\tau)\,g(\tau)\,d\tau$	$F(s)\,G(s)$
17	$\delta(t-c)$	e^{-cs}
18	$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19	$(-t)^n f(t)$	$F^{(n)}(s)$