# HKUST

## MATH150 Introduction to Differential Equations

**15th Dec 2004 Final Examination Solution** (Version White)

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**Part I: Each correct answer *in the answer box* for the following 8 multiple choice questions is worth 4 point. DO NOT guess wildly! If you do not have confidence in your answer leave the answer box blank. Each incorrectly answered question will result in a 0.5 point deduction.**

| **Question** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | **Total** |
|---|---|---|---|---|---|---|---|---|---|
| **Answer** | b | c | c | b | e | c | a | d | |

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1. Which of the following functions can be used as an integrating factor to turn the following non-exact equation into an exact equation?

$$(3y\cos x - xy\sin x) + 2x\cos x\frac{dy}{dx} \;=\; 0\ ?$$

**Solution** Multiplying $x^2y$ to the equation, we have an exact equation

$$(3x^2y^2\cos x - x^3y^2\sin x) + 2x^3y\cos x\frac{dy}{dx} \;=\; 0,$$

since

$$\frac{\partial}{\partial y}(3x^2y^2\cos x - x^3y^2\sin x) = 6x^2y\cos x - 2x^3y\sin x = \frac{\partial}{\partial x}(2x^3y\cos x).$$

The answer is (b).

(a) $x^2$ (b) $x^2y$ (c) $y^2$ (d) $xy^2$ (e) $xy$

2. For a simple RLC series electrical circuit with $R = 1/5$ ohm, $L = 1$ henry, and $C$ farads, the differential equation for the current $I$ through the circuit is

$$C\frac{d^2I}{dt^2} + C\frac{1}{5}\frac{dI}{dt} + I = 0\ .$$

Pick the *largest* possible $C$ from the following with which the current of the circuit will keep changing its direction as $t \to \infty$.
**Solution** The current will oscillating between positive and negative values if the characteristic equation $Cr^2 + \frac{C}{5}r + 1 = 0$ has two complex roots, i.e., $\frac{C^2}{25} - 4C < 0$, and hence $C < 100$.
The answer is (c).

(a) 260 (b) 100 (c) 80 (d) 62 (e) 15

3. On which of the given intervals will the following initial value problem have a unique, continuous, solution?

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} e^{2t} & \frac{t}{t-2} \\ te^t & t \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \tan t \\ t+t^2 \end{bmatrix}, \quad \begin{bmatrix} x_1(3) \\ x_2(3) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix},$$

**Solution** All functions appearing in the system are continuous and well-defined everywhere except $\frac{t}{t-2}$ (undefined when $t = 2$) and $\tan t$ (undefined when $t = \frac{\pi}{2} + (2k+1)\pi$, $k = 0, \pm 1, \pm 2, \cdots$). Hence the largest interval containing the initial point $t = 3$ where every function in the system is well-defined and continuous is $2 < t < \frac{3\pi}{2}$, on which a unique, continuous, solution is guaranteed by the existence and uniqueness theorem.
The answer is (c).

(a) $-2 < t < 2$ (b) $\frac{\pi}{2} < t < 3$ (c) $2 < t < \frac{3\pi}{2}$ (d) $\frac{\pi}{2} < t < \frac{3\pi}{2}$ (e) $2 < t < 5$

4. Determine the inverse Laplace transform of the function $\frac{1}{(s+2)^2}$.
**Solution** $\mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2}$. The answer is (b).

(a) $t$ (b) $t\,e^{-t}$ (c) $t^2$ (d) $t^2\,e^{-t}$ (e) $t\,e^{t}$

5. Consider the nonhomogeneous equation

$$y'' + 6y' + 9y = (2t + t^4)e^{-3t}.$$

By the method of undetermined coefficients, there is a solution to the equation which is of the form $y = Q(t)\,e^{-3t}$ where $Q(t)$ is a polynomial. The degree of $Q(t)$ is :
**Solution** The characteristic equation is $r^2 + 6r + 9 = (r+3)^2 = 0$, with repeated root $r = -3$. The degree of $Q(t)$ is 6. Or, by directly putting $Qe^{-3t}$ into the equation, we have

$$y'' + 6y' + 9y = (Q'' - 6Q' + 9Q)e^{-3t} + 6(Q' - 3Q)e^{-3t} + 9Qe^{-3t} = Q''e^{-3t} = (2t + t^4)e^{-3t},$$

the degree of $Q$ must be 6.
The answer is (e).

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

6. The following intial value problem of a first order linear system

$$\begin{cases} x' = 3x - 2y, & x(0) = 1, \\ y' = -3x + 4y, & y(0) = -2 \end{cases}$$

can be converted into an initial value problem of a 2nd order differential equation for $x(t)$. It is:
**Solution** Putting $2y = 3x - x'$ into the 2nd equation, we have

$$3x' - x'' = -6x + 12x - 4x' = 0 \Longleftrightarrow x'' - 7x' + 6x = 0.$$

Moreover, $x(0) = 1$, and $x'(0) = 3x(0) - 2y(0) = 3(1) - 2(-2) = 7$. The answer is (c).

(a) $x'' - 7x' + 6x = 0,\ \ x(0) = 1,\ x'(0) = -2$

(b) $x'' - 7x' + 6x = 0,\ \ x(0) = 1,\ x'(0) = 0$

(c) $x'' - 7x' + 6x = 0,\ \ x(0) = 1,\ x'(0) = 7$

(d) $x'' - x' + 6x = 0,\ \ x(0) = 1,\ x'(0) = -2$

(e) $x'' - x' + 6x = 0,\ \ x(0) = 1,\ x'(0) = 0$

7. If $x(t)$, $y(t)$ solve the initial value problem

$$\begin{cases} x'(t) + 2x - 2y = e^{-2t}, & x(0) = 0 \\ y'(t) - 2x + 3y = 0, & y(0) = 0 \end{cases},$$

their Laplace transforms $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$ are:
**Solution** Taking the Laplace transform, we have

$$\begin{array}{l} (s+2)X(s) - 2Y(s) = \frac{1}{s+2} \\ -2X(s) + (s+3)Y(s) = 0 \end{array} \Leftrightarrow \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} s+2 & -2 \\ -2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{s+2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \frac{1}{s^2+5s+2} \begin{bmatrix} s+3 & 2 \\ 2 & s+2 \end{bmatrix} \begin{bmatrix} \frac{1}{s+2} \\ 0 \end{bmatrix} = \frac{1}{2^2+5s+2} \begin{bmatrix} \frac{s+3}{s+2} \\ \frac{2}{s+2} \end{bmatrix}$$

The answer is (a).

(a) $X(s) = \dfrac{s+3}{(s+2)(s^2+5s+2)}$, $\quad Y(s) = \dfrac{2}{(s+2)(s^2+5s+2)}$

(b) $X(s) = \dfrac{s-3}{(s+2)(s^2-5s+2)}$, $\quad Y(s) = \dfrac{-2}{(s+2)(s^2-5s+2)}$

(c) $X(s) = \dfrac{s-2}{(s+2)(s^2-5s+2)}$, $\quad Y(s) = \dfrac{s+3}{(s+2)(s^2-5s+2)}$

(d) $X(s) = \dfrac{s+3}{(s+2)(s^2+5s+2)}$, $\quad Y(s) = \dfrac{-2}{(s+2)(s^2+5s+2)}$

(e) $X(s) = \dfrac{s-2}{(s+2)(s^2+5s+2)}$, $\quad Y(s) = \dfrac{s+3}{(s+2)(s^2+5s+2)}$

8. Which of the following convolution integral is a solution of the equation $\dfrac{d^2y}{dt^2} + 4y = u_\pi(t)f(t-\pi)$, where $u_\pi(t)$ is a unit step function.
**Solution** Taking the Laplace transform with $y(0) = y'(0) = 0$, we have

$$(s^2+4)Y(s) = e^{-\pi s}\mathcal{L}\{f(t)\}$$

$$Y(s) = \frac{e^{-\pi}}{s^2+4} \cdot \mathcal{L}\{f(t)\} = \mathcal{L}\{\frac{1}{2}u_\pi(t)\sin 2(t-\pi)\} \cdot \mathcal{L}\{f(t)\} = \mathcal{L}\{\frac{1}{2}u_\pi(t)\sin 2t\} \cdot \mathcal{L}\{f(t)\}.$$

Take the convolution integral for the inverse Laplace transform:

$$\frac{1}{2}u_\pi(t)\sin 2t\} * f(t) = \int_0^t \frac{1}{2}u_\pi(t-u)\sin 2(t-u)f(u)\,du.$$

The answer is (d).

(a) $\displaystyle\int_0^t \frac{1}{2}\cos 2(t-u)f(u)\,du$

(b) $\displaystyle\int_0^t \frac{1}{2}\delta(t-u+\pi)\sin 2(t-u)f(u)\,du$

(c) $\displaystyle\int_0^t \frac{1}{2}u_\pi(t)\sin 2(t-u)f(t-u)\,du$

(d) $\displaystyle\int_0^t \frac{1}{2}u_\pi(t-u)\sin 2(t-u)f(u)\,du$

(e) $\displaystyle\int_0^t \frac{1}{2}\sin 2(t-u)f(u)\,du$

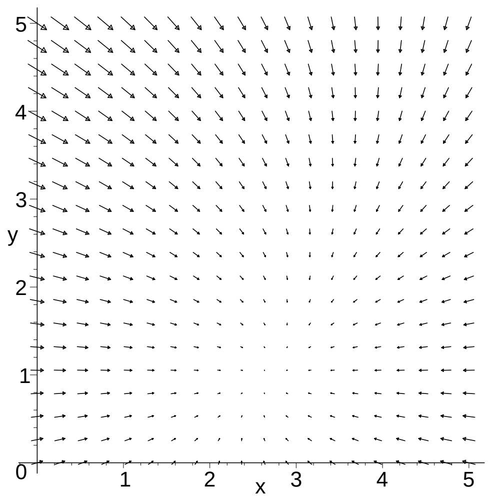
**Part II: Answer each of the following 6 short answer questions.** ***Show all your work for full credit.***

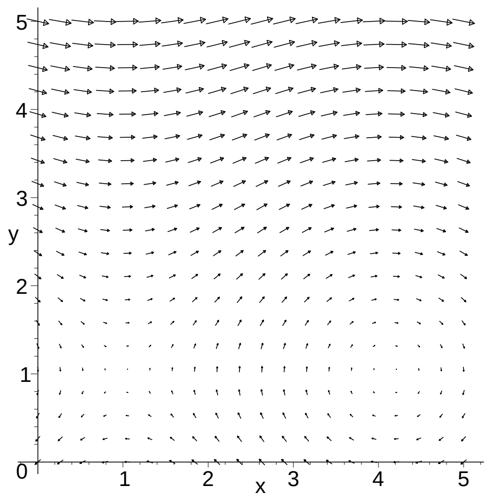
| Question | 9 | 10 | 11 | 12 | 13 | 14 | Total |
|---|---|---|---|---|---|---|---|
| Points | | | | | | | |

9. [*6 pts*] Match the systems of equations with direction-field/vector-field of the system.

$$x' = -3x + y + 7$$
$$y' = -2y + 2$$

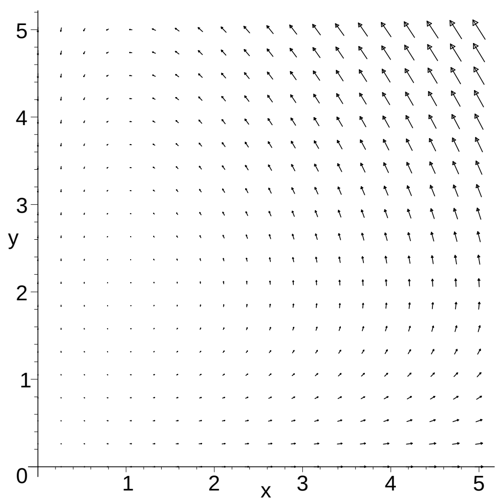
$$x' = y - 1$$
$$y' = \sin(x - 1)$$

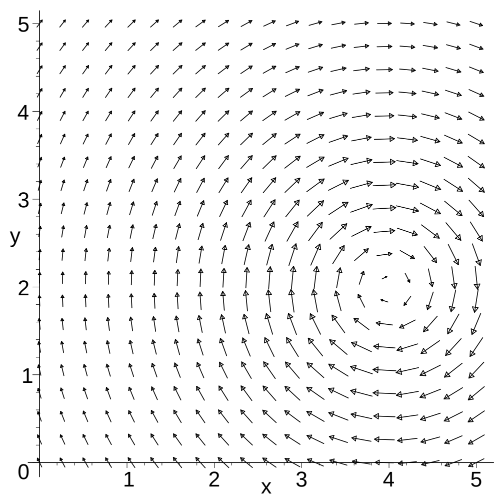



$$x' = 5x - 2.5xy$$
$$y' = 3xy - 3y$$

$$x' = \frac{y-2}{(x-4)^2+(y-2)^2+1}$$
$$y' = \frac{-(x-4)}{(x-4)^2+(y-2)^2+1}$$



10. [*8 pts*] The homogeneous equation $(1-2t)\,y'' + 4ty' - 4y = 0$ has two fundamental solutions $2t$ and $e^{2t}$. The method of variation of parameters says that, for appropriate choices of functions $u_1(t)$ and $u_2(t)$, the function $2tu_1(t) + e^{2t}u_2(t)$ is a solution of the nonhomogeneous equation

$$(1-2t)\,y'' + 4ty' - 4y = -(1-2t)^2.$$

(a) Find the 1st order linear system that $u_1(t)$ and $u_2(t)$ must satisfy. [*5 pts*]
**Solution** In standard form, the linear equation is

$$y'' + \frac{4t}{1-2t}y' - \frac{4}{1-2t}y = -1 + 2t$$

Answer: The linear system for $u_1(t)$ and $u_2(t)$ is

$$\begin{cases} 2tu_1' + e^{2t}u_2' = 0 \\ 2u_1' + 2e^{2t}u_2' = -1 + 2t \end{cases}$$

(b) From the linear system in (a), find the unknown functions $u_1(t)$, $u_2(t)$, and then also the *general solution* of the nonhomogeneous equation. (You may leave your answers in terms of some integrals.) [*3 pts*]

**Solution** From the linear system above,

$$(2-4t)u_1' = 2(2t-1) \Longrightarrow u' = -1/2 \Longrightarrow u_1(t) = -t/2$$

$$(2t-1)e^{2t}u_2' = t(2t-1) \Longrightarrow u_2' = te^{-2t} \Longrightarrow u_2(t) = \int te^{-2t}dt$$

Answer: The general solution is

$$y(t) = c_1t + c_2e^{2t} - t^2/2 + e^{2t}\int te^{-2t}dt$$

11. [*4 pts*] When a delta impulse force of 1 is applied to a certain spring-mass system initially at rest at time $t = 1$ the equation of motion of the mass is given by

$$y'' + 2y' + 10y = \delta(t-1), \quad y(0) = 0\ ,\ y'(0) = 1,$$

where $y(t)$ denotes the displacement of the mass from the equilibrium position. Solve for the Laplace transform of the solution. You only need to find the Laplace transform.
**Solution**

Take the Laplace Transform of both sides to get

$$\begin{aligned} \mathcal{L}(y'') + 2\mathcal{L}(y') + \mathcal{L}(y) &= \mathcal{L}(\delta(t-1)) \\ s^2\mathcal{L}(y) - 1 + 2s\mathcal{L}(y) + \mathcal{L}(y) &= e^{-s} \\ \mathcal{L}(y) &= \frac{e^{-s}+1}{s^2+2s+10} \end{aligned}$$

Answer: The Laplace transform is: $\mathcal{L}(y) = \frac{e^{-s}+1}{s^2+2s+10}$.

12. [*4 pts*] The solution of the vibrational differential equation

$$y'' + 2y' + (1+\pi^2)y = \pi\,\delta(t) + b\,\pi\,\delta(t-2)\ , \quad y(0) = y'(0) = 0\,,$$

($b$ is an adjustable constant) is

$$y(t) = e^{-t}\,\sin(\pi t)\ +\ b\,u_2(t)e^{-(t-2)}\sin(\pi(t-2))\ .$$

By taking an appropriate value for the constant $b$, it is possible to halt the vibration for all $t$ larger than some $t_0$. Determine $b$ and $t_0$.

**Solution**

We rewrite the solution $y(t)$ as

$$y(t)\ =\ e^{-t}\sin(\pi t)\ (1 + u_2(t)\,be^2)$$

If we make $be^2 = -1$, so $b = -e^{-2}$, then $y(t) = e^{-t}\sin(\pi t)\ (1 - u_2(t))$, so $y(t) = 0$ for $t > 2$.
Answer: $b = -e^{-2}$ . Answer: $t_0 = 2$.

13. [*4 pts*] The matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 3 \\ -2 & 2 & 1 \end{bmatrix}$ has three linearly independent eigenvectors $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix}$. Find the general solution of the $3 \times 3$ linear system $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$.

**Solution** The eigenvalues can be found easily from the product

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 3 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & -3 & 5 \\ 1 & -2 & 8 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -6 & 20 \\ -1 & -4 & 32 \\ 1 & 4 & 8 \end{bmatrix}$$

$$= \left[-\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, 2\begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}, 4\begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix}\right]$$

Answer: The general solution is

$$c_1e^{-t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_2e^{2t}\begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3e^{4t}\begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix}$$

14. [*6 pts*] The non-linear system

$$\begin{aligned} x' &= -0.7x + 0.001xy \\ y' &= -0.001xy + 0.2y \end{aligned}$$

is a model for the interaction of two populations $x$ and $y$ of predators and prey. An example is a population $x$ of snakes hunting a population $y$ of mice. If the initial populations are $x(0) = 100$, $y(0) = 500$, generalize Euler's method to the system case to estimate $x(0.2)$, and $y(0.2)$ by using two steps to go from 0 to 0.2. The two steps are 0 to 0.1 and 0.1 to 0.2. Keep 1 decimal place to your answers.

| $i$ | 0 | 1 | 2 |
|---|---|---|---|
| $t_i$ | 0.0 | 0.1 | 0.2 |
| $x_i$ | 100.0 | 98 | 96.09 |
| $y_i$ | 500.0 | 505 | 510.15 |

**Solution**
For the 1st step from 0.0 to 0.1 we have:

$$\begin{aligned}
x'(0) &= -0.7 \cdot 100 + 0.001 \cdot (100 \cdot 500) = -20 \\
y'(0) &= -0.001 \cdot (100 \cdot 500) + 0.2 \cdot 500 = 50 \\
x(0.1) &= 100 + (-20) \cdot 0.1 = 98 \\
y(0.1) &= 500 + (50) \cdot 0.1 = 505
\end{aligned}$$

For the 2nd step from 0.1 to 0.2 we have:

$$\begin{aligned}
x'(0.1) &= -0.7 \cdot 98 + 0.001 \cdot (98 \cdot 505) = -19.11 \\
y'(0.1) &= -0.001 \cdot (98 \cdot 505) + 0.2 \cdot 505 = 51.51 \\
x(0.2) &= 98 + (-19.11) \cdot 0.1 = 96.09 \\
y(0.2) &= 505 + (51.51) \cdot 0.1 = 510.15
\end{aligned}$$

Based on your calculations at $t = 0$, circle the correct statement:

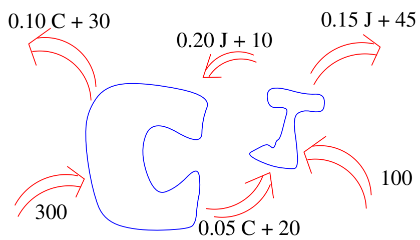
(a) The populations of both snakes and mice are increasing.
(b) The populations of both snakes and mice are decreasing.
(c) The population of snakes is increasing, but there are now less mice to eat.
(d) The number of snakes is decreasing, while the mice are growing in numbers.
(e) The two populations are in equilibrium.

**Solution**
Since $(x, y)$, goes from $(100, 500)$ to $(98, 505)$, the answer is (d).

**Part III: Answer the two long questions.**

15. [*17 pts*] Consider the flow of money into and out of two countries (country C and country J) and the rest of the world.

(a) Assume the following:




- At time $t = 0$, country C has 1 trillion (1,000 billion) dollars in assets, and country J has 2 trillion (2,000 billion) dollars in assets.
- Country J buys goods from country C at a rate of 20% of J's assets plus 10 billion dollars (per year), that is $(0.20J + 10)$.
- Country C buys goods from country J at a rate of 5% of C's assets plus 20 billion dollars, that is $(0.05C + 20)$.
- Country J also buys goods from the rest of the world at a rate 15% of its assets plus 45 billion dollars, that is $(0.15J + 45)$.
- Country C also buys goods from the rest of the world at a rate 10% of its assets plus 30 billion dollars, that is $(0.10C + 30)$.
- Country J sells goods to the rest of the world at a fixed rate of 100 billion per year.
- Country C sells goods to the rest of the world at a fixed rate of 300 billion per year.

Use the variables $C$ and $J$, and units of billions of dollars, to write the matrix differential equation, with initial conditions, which governs the system. [*6 pts*]

**Solution**

$$\begin{aligned} C' &= 300 + (0.20J + 10) - (0.10C + 30) - (0.05C + 20) \\ &= -0.15C + 0.20J + 260 \\ J' &= 100 + (0.05C + 20) - (0.20J + 10) - (0.15J + 45) \\ &= 0.05C - 0.35J + 65 \end{aligned}$$

In matrix terms

$$\begin{bmatrix} C' \\ J' \end{bmatrix} = \frac{1}{100}\begin{bmatrix} -15 & 20 \\ 5 & -35 \end{bmatrix}\begin{bmatrix} C \\ J \end{bmatrix} + \begin{bmatrix} 260 \\ 65 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C(0) \\ J(0) \end{bmatrix} = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$$

(b) For some **slightly different assumptions** than part (a), it is known that the initial value problem is

$$\begin{bmatrix} C' \\ J' \end{bmatrix} = \frac{1}{100}\begin{bmatrix} -3 & 4 \\ 4 & -9 \end{bmatrix}\begin{bmatrix} C \\ J \end{bmatrix} + \begin{bmatrix} 54 \\ 49 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C(0) \\ J(0) \end{bmatrix} = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$$

(i) Determine equilibrium solutions $C_e$ and $J_e$, and use them to convert the non-homogenous matrix differential equation into a homogeneous matrix differential equation. Hint. Take new variables $x = C - C_e$, and $y = J - J_e$. [*4 pts*]

**Solution**

We solve

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} -3 & 4 \\ 4 & -9 \end{bmatrix}\begin{bmatrix} C \\ J \end{bmatrix} + \begin{bmatrix} 54 \\ 49 \end{bmatrix}$$

to get $C = 6200$, and $J = 3300$. When we use the change of variable $x = C - C_e$, $y = J - J_e$, we get

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{100}\begin{bmatrix} -3 & 4 \\ 4 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -5200 \\ -1300 \end{bmatrix}$$

(ii) Solve the homogeneous matrix differential equation of part (i). Hint. The matrix $\begin{bmatrix} -3 & 4 \\ 4 & -9 \end{bmatrix}$ has eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with eigenvalue $-1/100$ and eigenvector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ with eigenvalue $-11/100$. *[4 pts]*

**Solution**

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-t/100} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-11t/100} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

and

$$\begin{bmatrix} -5200 \\ -1300 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

so $c_1 = -2340$, and $c_2 = -520$. Thus

$$\begin{bmatrix} C(t) \\ J(t) \end{bmatrix} = \begin{bmatrix} 6200 \\ 3300 \end{bmatrix} - 2340\, e^{-t/100} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 520\, e^{-11t/100} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

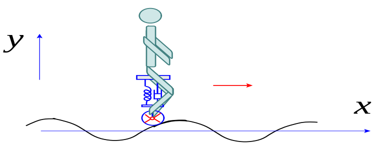
(iii) Based on your answer to part (ii), setup an equation in the time variable $t$ which can be solved to find when the assets of country C equals those of country J. *[3 pts]*

**Solution**

$$\begin{aligned} C(t) &= J(t) \\ 6200 - 4680e^{-t/100} - 520e^{-11t/100} &= 3300 - 2340e^{-t/100} + 1040e^{-11t/100} \\ 2900 - 2340e^{-t/100} - 1560e^{-11t/100} &= 0 \end{aligned}$$

$t$ is approximately 7.04 years.

16. [*18 pts*] The vibration of a single-wheel bike travelling with constant speed $v$ along a bumpy road modelled by a height function $h(x)$ can be described by the differential equation




$$my''(t) = -\gamma[y(t) - h(vt)]' - k[y(t) - h(vt)],$$

(Newton's 2nd Law: $ma = F$)

where $y(t)$ is the displacement form equilibruim position, $m = 300$ kg is the mass, $k = 12000$ N/m the spring constant, and $\gamma$ the damping constant. Note that the distance travelled in the $x$-direction after time $t$ is $x = vt$.

(a) Suppose the shock absorber has a damping constant $\gamma = 4800$ Ns/m, the speed of the bike is $v = 3$ m/s, and the height function of the bumpy road is a sine function $h(x) = 1.5\sin\frac{2\pi x}{24}$.

(i) Write down the equation of vibration for the bike and find its general solution. [*10 pts*]
(*Use the back side of this page to find your particular solution if necessary.*)
**Solution** The equation is

$$300y'' = -4800\Big[y - 1.5\sin\frac{2\pi\cdot 3t}{24}\Big]' - 12000\Big[y - 1.5\sin\frac{2\pi\cdot 3t}{24}\Big]\,,$$

$$y'' + 16y' + 40y = 6\pi\cos\frac{\pi t}{4} + 60\sin\frac{\pi t}{4}\,.$$

The characteristic equation of the associated homogeneous equation has two real roots:

$$r = \frac{-16 \pm \sqrt{16^2 - 4\cdot 40}}{2} = -8 \pm 2\sqrt{6}\,.$$

Use the method of undetermined coefficients to find a particular solution of the form $y = A\cos\frac{\pi t}{4} + B\sin\frac{\pi t}{4}$ :

$$y' = -\frac{\pi}{4}A\sin\frac{\pi t}{4} + \frac{\pi}{4}B\cos\frac{\pi t}{4}$$

$$y'' = -\frac{\pi^2}{16}A\cos\frac{\pi t}{4} - \frac{\pi^2}{16}B\sin\frac{\pi t}{4}$$

Hence from

$$y'' + 16y' + 40y = 6\pi\cos\frac{\pi t}{4} + 60\sin\frac{\pi t}{4}$$

we have

$$\begin{cases} \Big[40 - \frac{\pi^2}{16}\Big]A + 4\pi B = 6\pi & (I) \\ -4\pi A + \Big[40 - \frac{\pi^2}{16}\Big]B = 60 & (II) \end{cases}$$

(II) $\times\Big[40 - \frac{\pi^2}{16}\Big]$ − (II) $\times 4\pi$, we have $\Big[\big[40 - \frac{\pi^2}{16}\big]^2 + 16\pi^2\Big]A = -\frac{3\pi^3}{8}$.
(II) $\times\Big[40 - \frac{\pi^2}{16}\Big]$ + (I) $\times 4\pi$, we have $\Big[\big[40 - \frac{\pi^2}{16}\big]^2 + 16\pi^2\Big]B = 60\big[40 - \frac{\pi^2}{16}\big] + 24\pi^2$.

$$A = -\frac{3\pi^3}{8\big[40 - \frac{\pi^2}{16}\big]^2}, \qquad B = \frac{9600 + 81\pi^2}{4\big[40 - \frac{\pi^2}{16}\big]^2}$$

Then the general solution is: $c_1e^{(-8+2\sqrt{6})t} + c_2e^{(-8-2\sqrt{6})t} + A\cos\frac{\pi t}{4} + B\sin\frac{\pi t}{4}$.

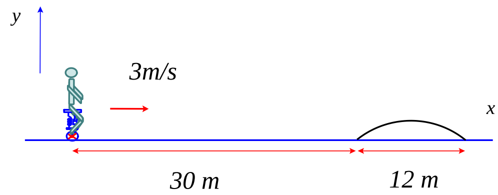
(ii) Is it possible that the vibration will eventually die down as $t \to +\infty$? Why? [*2 pts*]
**Solution** No. The steady state solution $A\cos\frac{\pi t}{4} + B\sin\frac{\pi t}{4}$ is a period function, oscillating between $\pm\sqrt{A^2 + B^2}$. Or, just because that the road is bumpy all the way.

(b) At what travelling speed $v$ would resonance occur after the shock absorber worn out (i.e., no more dampling)? *[3 pts]*

**Solution** With $\gamma = 0$, the equation of motion is $y'' + 40y = 60 \sin \frac{\pi v t}{12}$. Hence resonance occurs if

$$\sqrt{40} = \frac{\pi v}{12}, \qquad i.e., \quad v = \frac{24\sqrt{10}}{\pi} \quad (\text{m/s})$$

(c) Suppose the bike now travels again along a plain road with only one bump at 30 m from the inital position $x = 0$ m, with constant speed $v = 3$ m/s in the $x$-direction, and without any vibration in the $y$ direction initially. Suppose the height function of the bump can still be modelled by the same sine function as above, so that the maximum height is 1.5m and the length of the bump is 12 m. Write down an initial value problem for the vibration of the bike. *[3 pts]*




**Solution** The initial value problem is

$$300y'' = -4800\Big[y - (u_{10}(t) - u_{14}(t)) \sin \frac{\pi t}{4}\Big]' - 12000\Big[y - (u_{10}(t) - u_{14}(t)) \sin \frac{\pi t}{4}\Big]$$

$$y(0) = 0, \quad y'(0) = 0$$