HKUST

MATH150 Introduction to Differential Equations

Final Examination (Version White) Name:____________________

15th December 2004 Student I.D.:____________________

8:30–10:30 Tutorial Section:____________________

Directions:

- Write your name, ID number, and tutorial section in the space provided above.
- DO NOT open the exam until instructed to do so.
- When instructed to open the exam, check that you have, in addition to this cover page, 11 pages of questions.
- Turn off all mobile phones and pagers during the examination.
- This is a closed book examination.
- You are advised to try the problems you feel more comfortable with first.
- You may write on both sides of the examination papers.
- There are 8 multiple choice questions. **DO NOT guess wildly! If you do not have confidence in your answer leave the question blank. Each incorrectly answered question will result in a 0.5 point deduction**.
- For the short and long questions, you must show the working steps of your answers in order to receive full points.
- Cheating is a serious offense. Students caught cheating are subject to a zero score as well as additional penalties.

Question No.	Points	Out of
Q. 1-8		32
Q. 9-14		32
Q. 15		18
Q. 16		18
Total Points		100

Part I: Each correct answer *in the answer box* for the following 8 multiple choice questions is worth 4 point. DO NOT guess wildly! If you do not have confidence in your answer leave the answer box blank. Each incorrectly answered question will result in a 0.5 point deduction.

Question	1	2	3	4	5	6	7	8	**Total**
Answer									

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1. Which of the following functions can be used as an integrating factor to turn the following non-exact equation into an exact equation?

$$(3y\cos x - xy\sin x) + 2x\cos x\frac{dy}{dx} \;=\; 0\ ?$$

(a) x^2 (b) x^2y (c) y^2 (d) xy^2 (e) xy

2. For a simple RLC series electrical circuit with $R = 1/5$ ohm, $L = 1$ henry, and C farads, the differential equation for the current I through the circuit is

$$C\frac{d^2I}{dt^2} + \frac{C}{5}\frac{dI}{dt} + I = 0\ .$$

Pick the *largest* possible C from the following with which the current of the circuit will keep changing its direction (oscillates) as $t \to \infty$.

(a) 260 (b) 100 (c) 80 (d) 62 (e) 15

3. On which of the given intervals will the following initial value problem have a unique, continuous, solution?

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} e^{2t} & \frac{t}{t-2} \\ te^t & t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \tan t \\ t+t^2 \end{bmatrix}, \quad \begin{bmatrix} x_1(3) \\ x_2(3) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix},$$

(a) $-2 < t < 2$ (b) $\frac{\pi}{2} < t < 3$ (c) $2 < t < \frac{3\pi}{2}$ (d) $\frac{\pi}{2} < t < \frac{3\pi}{2}$ (e) $2 < t < 5$

4. Determine the inverse Laplace transform of the function $\frac{1}{(s+1)^2}$.

(a) t (b) $t\,e^{-t}$ (c) t^2 (d) $t^2\,e^{-t}$ (e) $t\,e^{t}$

5. Consider the nonhomogeneous equation

$$y'' + 6y' + 9y = (2t + t^4)e^{-3t}.$$

By the method of undetermined coefficients, there is a solution to the equation which is of the form $y = Q(t)\,e^{-3t}$ where $Q(t)$ is a polynomial. The degree of $Q(t)$ is :

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

6. The following intial value problem of a first order linear system

$$\begin{cases} x' = 3x - 2y, & x(0) = 1, \\ y' = -3x + 4y, & y(0) = -2 \end{cases}$$

can be converted into an initial value problem of a 2nd order differential equation for $x(t)$. It is

(a) $x'' - 7x' + 6x = 0, \;\; x(0) = 1, \; x'(0) = -2$

(b) $x'' - 7x' + 6x = 0, \;\; x(0) = 1, \; x'(0) = 0$

(c) $x'' - 7x' + 6x = 0, \;\; x(0) = 1, \; x'(0) = 7$

(d) $x'' - x' + 6x = 0, \;\; x(0) = 1, \; x'(0) = -2$

(e) $x'' - x' + 6x = 0, \;\; x(0) = 1, \; x'(0) = 0$

7. If $x(t)$, $y(t)$ solve the initial value problem

$$\begin{cases} x'(t) + 2x - 2y = e^{-2t}, & x(0) = 0 \\ y'(t) - 2x + 3y = 0, & y(0) = 0 \end{cases},$$

their Laplace transforms $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$ are:

(a) $X(s) = \dfrac{s+3}{(s+2)(s^2+5s+2)}$, $\quad Y(s) = \dfrac{2}{(s+2)(s^2+5s+2)}$

(b) $X(s) = \dfrac{s+3}{(s+2)(s^2+5s+2)}$, $\quad Y(s) = \dfrac{-2}{(s+2)(s^2+5s+2)}$

(c) $X(s) = \dfrac{s-2}{(s+2)(s^2+5s+2)}$, $\quad Y(s) = \dfrac{s+3}{(s+2)(s^2+5s+2)}$

(d) $X(s) = \dfrac{s-3}{(s+2)(s^2-5s+2)}$, $\quad Y(s) = \dfrac{-2}{(s+2)(s^2-5s+2)}$

(e) $X(s) = \dfrac{s-2}{(s+2)(s^2-5s+2)}$, $\quad Y(s) = \dfrac{s+3}{(s+2)(s^2-5s+2)}$

8. Which of the following convolution integral is a solution of the equation $\dfrac{d^2y}{dt^2} + 4y = u_\pi(t)f(t-\pi)$, where $u_\pi(t)$ is a unit step function.

(a) $\displaystyle\int_0^t \frac{1}{2}\sin 2(t-u)f(u)\,du$

(b) $\displaystyle\int_0^t \frac{1}{2}\cos 2(t-u)f(u)\,du$

(c) $\displaystyle\int_0^t \frac{1}{2}\delta(t-u+\pi)\sin 2(t-u)f(u)\,du$

(d) $\displaystyle\int_0^t \frac{1}{2}u_\pi(t-u)\sin 2(t-u)f(u)\,du$

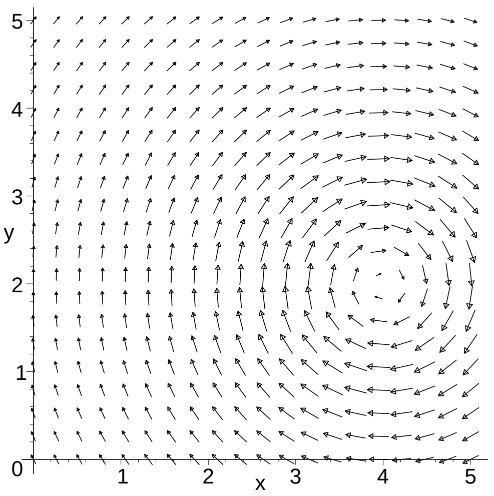
(e) $\displaystyle\int_0^t \frac{1}{2}u_\pi(t)\sin 2(t-u)f(t-u)\,du$

Part II: Answer each of the following 6 short answer questions. ***Show all your work for full credit.***

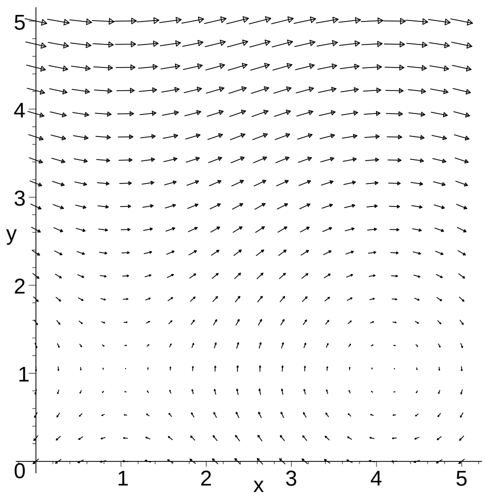
Question	9	10	11	12	13	14	**Total**
Points							

9. [*6 pts*] Match each of the **three** systems of equations with one of the **four** direction-field/vector-fields shown.

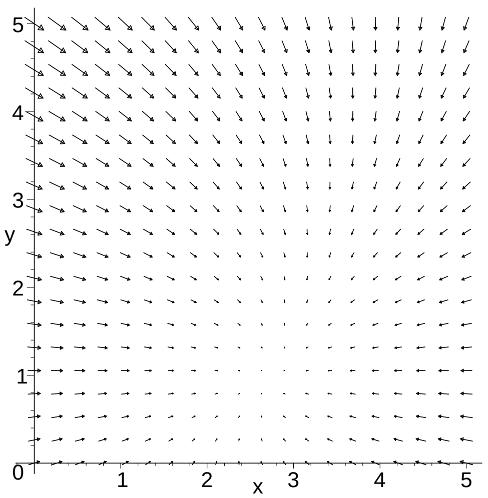
(a)



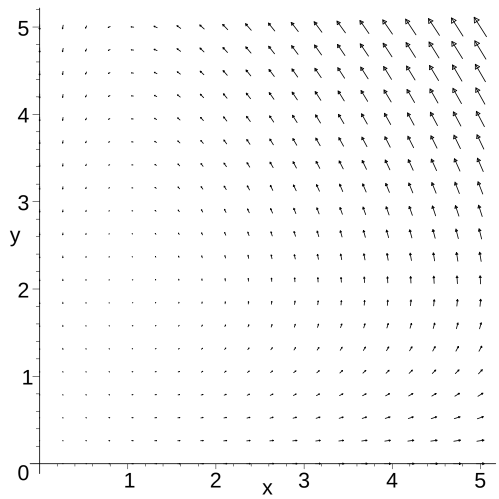
(b)



(c)



(d)



Equation	$x' = y - 1$ $y' = \sin(x-1)$	$x' = 5x - 2.5xy$ $y' = 3xy - 3y$	$x' = -3x + y + 7$ $y' = -2y + 2$
Graph			

10. [*8 pts*] The homogeneous equation $(1-2t)\,y''+4ty'-4y=0$ has two fundamental solutions $2t$ and e^{2t}. The method of variation of parameters says that, for appropriate choices of functions $u_1(t)$ and $u_2(t)$, the function $2tu_1(t)+e^{2t}u_2(t)$ is a solution of the nonhomogeneous equation

$$(1-2t)\,y''+4ty'-4y=-(1-2t)^2.$$

(a) Find the 1st order linear system that $u_1(t)$ and $u_2(t)$ must satisfy. (*Warning: the given nonhomogeneous equation is not written in the standard normal form.*) [*4 pts*]

Answer: The linear system for $u_1(t)$ and $u_2(t)$ is : ______________________________.

(b) From the linear system in (a), find the unknown functions $u_1(t)$, $u_2(t)$, and then also the *general solution* of the nonhomogeneous equation. (You may leave your answers in terms of some integrals.) [*4 pts*]

Answer: The general solution is : ____________________________________.

11. [*4 pts*] When a delta impulse force of 1 is applied to a certain spring-mass system initially at rest at time $t = 1$ the equation of motion of the mass is given by

$$y'' + 2y' + 10y = \delta(t-1), \quad y(0) = 0\ ,\ y'(0) = 1,$$

where $y(t)$ denotes the displacement of the mass from the equilibrium position. Solve for the Laplace transform of the solution. You only need to find the Laplace transform.

Answer: The Laplace transform is: $\mathcal{L}(y) =$ ________________________________.

12. [*4 pts*] The solution of the vibrational differential equation

$$y'' + 2y' + (1+\pi^2)y = \pi\,\delta(t) + b\,\pi\,\delta(t-2)\ , \quad y(0) = y'(0) = 0\,,$$

(b is an adjustable constant) is

$$y(t) = e^{-t}\sin(\pi t) \ + \ b\,u_2(t)e^{-(t-2)}\sin(\pi(t-2))\ .$$

By taking an appropriate value for the constant b, it is possible to halt the vibration for all t larger than some t_0. Determine b and t_0.

Answer: $b =$ ____________________. Answer: $t_0 =$ ____________________.

13. [*4 pts*] The matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 3 \\ -2 & 2 & 1 \end{bmatrix}$ has three linearly independent eigenvectors $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix}$. Find the general solution of the 3×3 linear system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$.

Answer: The general solution is ______________________________

14. [*6 pts*] The non-linear system

$$\begin{aligned} x' &= -0.7x + 0.001xy \\ y' &= -0.001xy + 0.2y \end{aligned}$$

is a model for the interaction of two populations x and y of predators and prey. An example is a population x of snakes hunting a population y of mice. If the initial populations are $x(0) = 100$, $y(0) = 500$, generalize Euler's method to the system case to estimate $x(0.2)$, and $y(0.2)$ by using two steps to go from 0 to 0.2. The two steps are 0 to 0.1 and 0.1 to 0.2. Keep 1 decimal place to your answers. **Remember to show your work for full credit.**

i	0	1	2
t_i	0.0	0.1	0.2
x_i	100.0		
y_i	500.0		

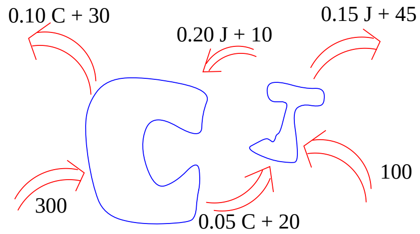
Based on your calculations at $t = 0$, circle the correct statement. At $t = 0$, the number of mice is:

increasing **decreaing** **stays the same**

Part III: Answer the two long questions.

15. [*18 pts*] Consider the flow of money into and out of two countries (country C and country J) and the rest of the world.

(a) Assume the following:



- At time $t = 0$, country C has 1 trillion (1,000 billion) dollars in assets, and country J has 2 trillion (2,000 billion) dollars in assets.
- Country J buys goods from country C at a rate of 20% of J's assets plus 10 billion dollars (per year), that is $(0.20J + 10)$.
- Country C buys goods from country J at a rate of 5% of C's assets plus 20 billion dollars, that is $(0.05C + 20)$.
- Country J also buys goods from the rest of the world at a rate 15% of its assets plus 45 billion dollars, that is $(0.15J + 45)$.
- Country C also buys goods from the rest of the world at a rate 10% of its assets plus 30 billion dollars, that is $(0.10C + 30)$.
- Country J sells goods to the rest of the world at a fixed rate of 100 billion per year.
- Country C sells goods to the rest of the world at a fixed rate of 300 billion per year.

Use the variables C and J, and units of billions of dollars, to write the matrix differential equation, with initial conditions, which governs the system. [*6 pts*]

(b) For some **slightly different assumptions** than part (a), it is known that the initial value problem is

$$\begin{bmatrix} C' \\ J' \end{bmatrix} = \frac{1}{100}\begin{bmatrix} -3 & 4 \\ 4 & -9 \end{bmatrix}\begin{bmatrix} C \\ J \end{bmatrix} + \begin{bmatrix} 54 \\ 49 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C(0) \\ J(0) \end{bmatrix} = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$$

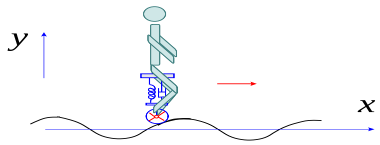
(i) Determine equilibrium solutions C_e and J_e, and use them to convert the non-homogeneous matrix differential equation into a homogeneous matrix differential equation. Hint. Take new variables $x = C - C_e$, and $y = J - J_e$. [*4 pts*]

(ii) Solve the initial value problem for x, y in part (i), and then determine the asset functions $C(t)$ and $J(t)$. (Hint. The matrix $\frac{1}{100}\begin{bmatrix} -3 & 4 \\ 4 & -9 \end{bmatrix}$ has eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with eigenvalue $-1/100$ and eigenvector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ with eigenvalue $-11/100$.) [*5 pts*]

(iii) Based on your answer to part (ii), setup an equation in the time variable t which can be solved to find when the assets of country C equals those of country J. [*3 pts*]

The equation is: $0 = \underline{\qquad\quad} \cdot 1 + \underline{\qquad\quad} \cdot e^{-t/100} + \underline{\qquad\quad} \cdot e^{-11t/100}$

16. [*18 pts*] The vibration of a single-wheel bike travelling with constant speed v along a bumpy road modelled by a height function $h(x)$ can be described by the differential equation



$$my''(t) = -\gamma[y(t) - h(vt)]' - k[y(t) - h(vt)],$$

(Newton's 2nd Law: $ma = F$)

where $y(t)$ is the displacement form equilibruim position, $m = 300$ kg is the mass, $k = 12000$ N/m the spring constant, and γ the damping constant. Note that the distance travelled in the x-direction after time t is $x = vt$.

(a) Suppose the shock absorber has a damping constant $\gamma = 4800$ Ns/m, the speed of the bike is $v = 3$ m/s, and the height function of the bumpy road is a sine function $h(x) = 1.5 \sin \frac{2\pi x}{24}$.

(i) Write down the equation of vibration for the bike and find its general solution. [*10 pts*]
(*Use the back side of this page to find your particular solution if necessary.*)

(ii) Is it possible that the vibration will eventually die down as $t \to +\infty$? Why? [*2 pts*]

(b) At what travelling speed v would resonance occur after the shock absorber worn out (i.e., no more dampling)? [*3 pts*]

(c) Suppose the bike now travels again along a plain road with only one bump at 30 m from the inital position $x = 0$ m, with constant speed $v = 3$ m/s in the x-direction, and without any vibration in the y direction initially. Suppose the height function of the bump can still be modelled by the same sine function as above, so that the maximum height is 1.5m and the length of the bump is 12 m. Write down an initial value problem for the vibration of the bike. [*3 pts*]