Math150 Introduction to Ordinary Differential Equations, Spring 05
Final Examination Solution: Version A
Part I: Multiple Choice Question.

Question	1	2	3	4	5	6	7	8	Total
Answer	d	c	b	a	c	d	c	c	

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1. Which of the following functions is an integrating factor for the non-exact equation

$$(x^3 + y) + (x \ln x + 2xy)\frac{dy}{dx} = 0 \ ?$$

(a) x (b) y (c) xy (d) $\frac{1}{x}$ (e) $\frac{1}{y}$

Solution The answer is (d), since after multiplying the equation by $\frac{1}{x}$, we have

$$\frac{1}{x}(x^3 + y) + \frac{1}{x}(x \ln x + 2xy)\frac{dy}{dx} = (x^2 + \frac{y}{x}) + (\ln x + 2y)\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^3}{3} + y \ln x + y^2\right] = 0$$

2. Using Euler's method with step size $h = 0.1$, the approximate value at $t = 0.3$ of the solution of the initial value problem

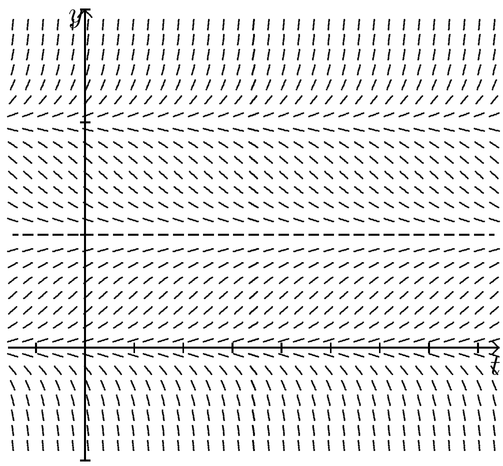
$$\frac{dy}{dt} = ty^2 + y, \qquad y(0) = 1$$

can be found as:

(a) 1.112 (b) 1.256 (c) 1.374 (d) 1.420 (e) 1.537

Solution The answer is (c). $y_1 = 1 + 0.1(0 \cdot 1^2 + 1) = 1.1$, $y_2 = 1.1 + 0.1(0.1 \cdot 1.1^2 + 1.1) = 1.2221$, $y(0.3) \approx y_3 = 1.2221 + 0.1(0.2 \cdot 1.2221^2 + 1.2221) = 1.3742$.

3. Given that y satisfies $\frac{dy}{dt} = y(y-1)(y-2)$, which of the equilibrium solutions is/are stable solution(s)?



(direction field of the equation)

(a) $y = 0$ (b) $y = 1$ (c) $y = 2$ (d) $y = 0$ and $y = 1$ (e) None of the previous

Solution The answer is (b). The slope field shows that solution curves are approaching $y = 1$ as $t \to \infty$.

4. If an external force is applied to a spring-mass system so that the equation of motion is

$$u'' + 9u = 3\cos(3\omega t) \ ,$$

at what ω will unbounded oscillation happen?

(a) 1 (b) 2 (c) 3 (d) 6 (e) 9

Solution The answer is (a). $\sqrt{9} = 3\omega$ implies $\omega = 1$.

5. Find the Wronskian of the pair of functions $y_1 = e^t \sin t$, $y_2 = e^t \cos t$ from the following list of functions:

(a) $e^{2t}(\cos 2t - 1)$ (b) $e^{2t}(\cos 2t + 1)$ (c) $-e^{2t}$ (d) $2e^{2t}$ (e) $e^{2t}(\cos^2 t - \sin^2 t)$

Solution The answer is (c) : $y_1 y_2' - y_2 y_1' = -e^{2t}$

6. Find a particular solution of $y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$.

(a) $-\frac{1}{2}e^{-x} + \frac{5}{13}\cos 3x + \frac{9}{13}\sin 3x$

(b) $\frac{1}{2}e^{-x} - \frac{7}{13}\cos 3x + \frac{3}{13}\sin 3x$

(c) $\frac{1}{3}e^{-x} + \frac{5}{13}\cos 3x - \frac{9}{13}\sin 3x$

(d) $\frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x$

(e) none of the previous

Solution The answer is (d). Just put $\frac{A}{13}\cos 3x + \frac{B}{13}\sin 3x$ into the equation to find the answer by comparing coefficients with $-10\cos 3x$.

7. Find the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$ of the solution of the given initial value problem :

$$2y'' + 3y' + 2y = e^{-3t}\sin 4t, \qquad y(0) = -2, \quad y'(0) = 1 \ .$$

(a) $\dfrac{-4s+8}{2s^2+3s+2} + \dfrac{4}{(2s^2+3s+2)(s^2+6s+25)}$

(b) $\dfrac{-4s-6}{2s^2+3s+2} + \dfrac{4}{(2s^2+3s+2)(s^2+6s+25)}$

(c) $\dfrac{-4s-4}{2s^2+3s+2} + \dfrac{4}{(2s^2+3s+2)(s^2+6s+25)}$

(d) $\dfrac{-4s+3}{2s^2+3s+2} + \dfrac{4}{(2s^2+3s+2)(s^2+6s+25)}$

(e) $\dfrac{4s-4}{2s^2+3s+2} + \dfrac{s+3}{(2s^2+3s+2)(s^2+6s+25)}$

Solution The answer is (c).

$$2s^2\mathcal{L}\{y\} - 2sy(0) - 2y'(0) + 3s\mathcal{L}\{y\} - 3y(0) + 2\mathcal{L}\{y\} = \frac{4}{(s+3)^2+4^2}$$

$$(2s^2+3s+2)\mathcal{L}\{y\} = -4s - 4 + \frac{4}{s^2+6s+25}$$

8. Find the Laplace transform of

$$g(t) = \begin{cases} 0, & 0 \le t < 6, \\ t+1, & 6 \le t. \end{cases}$$

(a) $\dfrac{e^{-6s}(1+s)}{s^2}$ (b) $\dfrac{e^{-6s}(1+5s)}{s^2}$ (c) $\dfrac{e^{-6s}(1+7s)}{s^2}$

(d) $\dfrac{e^{-6s}(5+s)}{s^2}$ (e) $\dfrac{e^{-6s}(1-7s)}{s^2}$

Solution The answer is (c).

$$\mathcal{L}\{u_6(t)(t+1)\} = \mathcal{L}\{u_6(t)((t-6)+7)\} = e^{-6s}\mathcal{L}\{t+7\} = e^{-6s}(\frac{1}{s^2} + \frac{7}{s})$$

Part II: Short Questions

9. A tank contains 80 gallons of pure water initially. A salt solution with 2 kg of salt per gallon is pumped into the tank at a rate of 3 gal/min, and the well-stirred mixture is pumped out at a rate of 4 gal/min. Given that the amount of salt $Q(t)$ at time t in the tank satisfies the differential equation

$$\frac{dQ}{dt} = -\frac{4Q}{80-t} + 6, \quad Q(0) = 0.$$

(a) Solve the initial value problem.

Solution Multiplying the integrating factor of the linear ode

$$e^{\int \frac{4}{80-t}dt} = e^{-4\ln(80-t)} = (80-t)^{-4}$$

to the equation, we have

$$\frac{d}{dt}\left[(80-t)^{-4}Q\right] = (80-t)^{-4}Q' + 4(80-t)^{-5}Q = 6(80-t)^{-4}$$

$$(80-t)^{-4}Q = \int 6(80-t)^{-4}dt = 2(80-t)^{-3} + C$$

Putting in $Q(0) = 0$, $C = -2(80)^{-3}$.

Answer: $Q(t) = 160 - 2t - \frac{2}{80^3}(80-t)^4$.

(b) When the mixture in the tank is 40 gal, what is the salt **concentration** of the mixture in the tank.

Solution The mixture in the tank is 40 gal, when $t = 40$ minutes. Therefore the concentration of the mixture at that time is

$$Q(40)/40 = 2 - \frac{2}{80^3}40^3 = 1.75$$

Answer: The concentration is 1.75 kg/gal.

10. The rate of change of a rabbit population $P(t)$ at time t (months) is proportional to the square root of $P(t)$. That is

$$\frac{dP}{dt} = k\sqrt{P},$$

for some constant $k > 0$.

(a) At time $t = 0$ the population numbers 100 rabbits and it is increasing at the rate of 21 rabbits per month. Determine k.
We have $P(t) = (kt/2 + C)^2$. Since $P(0) = 100$ and $P(1) = 121$, $k = 2$ and $C = 10$.
Answer: $k = \underline{2}$.

(b) How many rabbits will there be one year later?
We have $P(12) = (12 + 10)^2 = 484$
Answer: The number of rabbits after one year is $\underline{484}$.

11. Let $\mathcal{L}(f(t)) = \dfrac{1}{(s^2+1)^2}$. Using convolution integral to find $f(t)$.

By Convolution integral, we find that

$$\begin{aligned} f(t) &= \int_0^t \sin(t-\tau)\sin\tau d\tau \\ &= \int_0^t \frac{1}{2}(\cos(t-2\tau)-\cos t)d\tau \\ &= \frac{1}{2}\left(-\frac{\sin(t-2\tau)}{2}-\tau\cos t\right)\Bigg|_{\tau=0}^{\tau=t} \\ &= \frac{1}{2}(\sin t - t\cos t) \end{aligned}$$

Answer: $f(t)$ is $\underline{\frac{1}{2}(\sin t - t\cos t)}$

12. Consider a vibrating system described by the initial value problem

$$u'' + \frac{1}{4}u' + 2u = 2\cos\omega t, \quad u(0) = 0, \quad u'(0) = 2.$$

(a) Determine the steady-state solution $U(t)$ of this problem.

Solution By the method of undertermined coefficents, consider $U(t) = A\cos\omega t + B\sin\omega t$ so that

$$U'(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$
$$U''(t) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

Putting them back into the equation,

$$\big[(2-\omega^2)A + \frac{\omega}{4}B\big]\cos\omega t + \big[-\frac{\omega}{4}A + (2-\omega^2)B\big]\sin\omega t = 2\cos\omega t$$

i.e.,

$$\begin{cases} (2-\omega^2)A + \frac{\omega}{4}B = 2 \\ -\frac{\omega}{4}A + (2-\omega^2)B = 0 \end{cases}$$

Solving the system by Cramer's rule, or other methods,

$$A = \frac{\begin{vmatrix} 2 & \frac{\omega}{4} \\ 0 & 2-\omega^2 \end{vmatrix}}{\begin{vmatrix} 2-\omega^2 & \frac{\omega}{4} \\ -\frac{\omega}{4} & 2-\omega^2 \end{vmatrix}} = \frac{2(2-\omega^2)}{(2-\omega^2)^2 + \frac{\omega^2}{16}}, \quad B = \frac{\begin{vmatrix} 2-\omega^2 & 2 \\ -\frac{\omega}{4} & 0 \end{vmatrix}}{\begin{vmatrix} 2-\omega^2 & \frac{\omega}{4} \\ -\frac{\omega}{4} & 2-\omega^2 \end{vmatrix}} = \frac{\frac{1}{2}\omega}{(2-\omega^2)^2 + \frac{\omega^2}{16}}$$

Answer: $U(t)$ is $\underline{\frac{2(2-\omega^2)}{(2-\omega^2)^2+\frac{\omega^2}{16}}\cos\omega t + \frac{\frac{1}{2}\omega}{(2-\omega^2)^2+\frac{\omega^2}{16}}\sin\omega t}$ [5 pts]

(b) Find the amplitude R of the steady-state solution in terms of ω.

Answer: $\underline{R = \sqrt{A^2+B^2} = \frac{\sqrt{4(2-\omega^2)^2 - \frac{1}{4}\omega^2}}{(2-\omega^2)^2+\frac{\omega^2}{16}} = \frac{2}{\sqrt{(2-\omega^2)^2+\frac{\omega^2}{16}}}}$. [2 pts]

Or by using the formula in §3.9 directly:

$$U(t) = \frac{F_0}{\Delta}\cos(\omega t - \delta)$$

where $\cos\delta = \frac{m(\omega_o^2-\omega^2)}{\Delta}$, $\sin\delta = \frac{\gamma\omega}{\Delta}$, and $\Delta = \sqrt{m^2(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}$ with $\gamma = \frac{1}{4}$, $m = 1$ and $k = 2$, $\omega_0^2 = 2$.

Also, applying Laplace transform, one could have

$$\mathcal{L}\{u(t)\} = \frac{As + B\omega}{s^2 + \omega^2} + \frac{Cs + D}{s^2 + \frac{1}{4}s + 2}$$

for some suitable constants A, B, C, and D. The steady-state part then can be found as $U(t) = A\cos\omega t + B\sin\omega t$ from the inverse Laplace transform of the part $\frac{As+B\omega}{s^2+\omega^2}$. It is just a matter of doing messy partial fraction stuffs.

13. Let $f(t)$ be

$$f(t) = \begin{cases} 1, & 0 < t < L/4\,; \\ 4t, & L/4 < t < L\,. \end{cases}$$

(a) Extend $f(t)$ into periodic function of period $2L$ with $f(t + 2L) = f(t)$ such that it can be expanded into a Fourier "sine" series:

The formula of the extended part on the interval $-L < t < 0$ is : [2 pts]
Answer:

$$f(t) = \begin{cases} 4t, & -L < t < -L/4; \\ -1, & -L/4 < x < 0. \end{cases}$$

(b) Write down the value for which the above Fourier sine series converges at $t = -L/4$
Answer: $-\frac{L+1}{2}$ [2 pts]

14. Given that $y_1(x) = x$ and $y_2(x) = 1 + x^2$ are solution of

$$(x^2 - 1)y'' - 2xy' + 2y = 0.$$

(a) Find the Wronskian of y_1 and y_2. The Wronskian is

$$W(x) = \begin{vmatrix} x & 1 + x^2 \\ 1 & 2x \end{vmatrix} = x^2 - 1$$

Answer: The Wronskian is $x^2 - 1$.

(b) Then find the particular solution the nonhomogeneous equation,

$$(x^2 - 1)y'' - 2xy' + 2y = x^2 - 1.$$

We have

$$\begin{aligned} y_p(x) &= -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx \\ &= -x\int \frac{x^2+1}{x^2-1}dx + (1+x^2)\int \frac{x}{x^2-1}dx \\ &= -x^2 + x\ln\left|\frac{x+1}{x-1}\right| + \frac{1+x^2}{2}\ln|x^2-1| \end{aligned}$$

Answer: The particular solution is $-x^2 + x\ln\left|\frac{x+1}{x-1}\right| + \frac{1+x^2}{2}\ln|x^2 - 1|$.

15. (a) Find a fundamental set of solutions $\{y_1(t),\, y_2(t)\}$ to the differential equation:

$$y'' + \mu^2 y = 0,$$

where μ is a positive constant. [2 pts]

The characteristic roots are $\pm i\mu$. So we have two solutions $\cos\mu t$, $\sin\mu t$.

(b) Justify why your solution to (a) are linearly independent; hfill [2 pts]

Answer:
The two solutions in (a) are linearly independent since

$$W(\cos\mu t,\, \sin\mu t) = \mu\cos^2\mu t + \mu\sin^2\mu t = \mu,$$

which is never zero for all t. We conclude from Theorem 3.3.3 in the textbook that the two solutions are linearly independent.

(c) Determine all $a,\, b,\, c,\, d$ so that

$$\begin{aligned} f_1(t) &= ay_1 + by_2, \\ f_2(t) &= cy_1 + dy_2 \end{aligned}$$

becomes a fundamental set of solutions for each set of $\{a,\, b,\, c,\, d\}$ to the differential equation in (a). *Justify your answer for full credit.* [3 pts]
We can compute the determinant

$$W(f_1,\, f_2)(t) = \begin{vmatrix} a\cos\mu t + b\sin\mu t & c\cos\mu t + d\sin\mu t \\ -\mu a\sin\mu t + b\mu\cos\mu t & -c\mu\sin\mu t + d\mu\cos\mu t, \end{vmatrix} = ad - bc$$

directly, and to deduce from this and Theorem 3.3.3 that $\{f_1,\, f_2\}$ is a fundamental set of solution if and only if the $W(f_1,\, f_2)(t) = ad - bc \neq 0$.

Part III: Long questions.

16. [*15 pts*] A impluse force is applied to a spring-mass system at time $t = 5$ *initially at rest at the equilibrium postion.* Suppose the displacement from equilibrium position of the mass at time t is the solution of the following equation:

$$y'' + 6y' + 13y = 3\delta(t - 5).$$

(a) Find $y(t)$ by solving the given equation with appropriate initial values $y(0)$ and $y'(0)$. [*5 pts*]

<u>**Solution**</u> Taking the Laplace transform of the equation with $y(0) = y'(0) = 0$,

$$(s^2 + 6s + 13)\mathcal{L}\{y\} = 3e^{-5s}$$

$$\mathcal{L}\{y\} = 3e^{-5s} \cdot \frac{1}{(s+3)^2 + 2^2} = 3e^{-5s}\mathcal{L}\left\{\frac{1}{2}e^{-3t}\sin 2t\right\}$$

$$y(t) = \frac{3}{2}u_5(t)e^{-3(t-5)}\sin 2(t-5)$$

(b) If an additional impluse force $b\delta(t-5-\pi)$ is also applied to the system at $t=5+\pi$, where b is a constant, what should be the equation of motion of the mass? *[2 pts]*

Solution The equation is $y''+6y'+13y=3\delta(t-5)+b\delta(t-5-\pi)$.

(c) To bring the system to rest at $t=5+\pi$, i.e., $y(t)=0$, if $t\geq 5+\pi$, what impluse b should one choose? *[8 pts]*

Solution Taking the Laplace of the equation $y''+6y'+13y=3\delta(t-5)+b\delta(t-5-\pi)$, we have

$$(s^2+6s+13)\mathcal{L}\{y\}=3e^{-5s}+be^{-(5+\pi)s}$$

$$\mathcal{L}\{y\}=\left[3e^{-5s}+be^{-(5+\pi)s}\right]\cdot\mathcal{L}\left\{\frac{1}{2}e^{-3t}\sin 2t\right\}$$

$$y(t)=\frac{3}{2}u_5(t)e^{-3(t-5)}\sin 2(t-5)+\frac{b}{2}u_{5+\pi}(t)e^{-3(t-5-\pi)}\sin 2(t-5-\pi)$$

When $t\geq 5+\pi$, $u_5(t)=u_{5+\pi}(t)=1$, and hence

$$y(t)=\frac{3}{2}e^{-3(t-5)}\sin 2(t-5)+\frac{b}{2}e^{-3(t-5-\pi)}\sin 2(t-5-\pi)=\frac{3+be^{3\pi}}{2}e^{-3(t-5)}\sin 2(t-5)$$

To have $y(t)=0$ for all $t\geq 5+\pi$, we must have $3+be^{3\pi}=0$, i.e., $b=-3e^{-3\pi}$.

17. Suppose a metal rod of length 25cm with *thermal diffusivity* $\alpha^2=1/4$. Let $u(x,\,t)$ be the temperature distribution function at x, $0\leq x\leq 25$ and time $t\geq 0$. Suppose it is given that the initial temperature distribution of the metal rod is given by $u(x,\,0)=x/2,\ 0<x<25$. It is known that the function $u(x,\,t)$ satisfies the heat equation

$$\alpha^2 u_{xx}=u_t,\quad 0\leq x\leq 25,\ t>0. \tag{1}$$

It is known that the heat equation can be solved by the method of separation of variables by assuming that $u(x,\,t)=X(x)T(t)$.

(a) Suppose the temperatures at both ends of the metal rod are kept at 0°C.

(i) Write down a boundary value problem for $X(x)$. [2 pts]

$$X''+\lambda X=0,\quad X(0)=0=X(25).$$

(ii) Solve for all possible solutions for $X(x)$ above. [2 pts]

$$X(t)=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x.$$

The boundary conditions imply that $A=0$ and $\lambda=n^2\pi^2/25^2$. That is we have

$$X_n(x)=\sin\frac{n\pi x}{25},\quad n=1,\,2,\,\cdots.$$

(iii) Write down a differential equation for $T(t)$. [1 pt]

$$T'(t)+\frac{n^2\pi^2\alpha^2}{25^2}T(t)=0,\quad n=1,\,2,\,\cdots.$$

(iv) Solve for all possible $T(t)$ for the equation in (iii) above. [2 pts]

$$T_n(t) = e^{-n^2\pi^2\alpha^2 t/25^2}, \quad n = 1, 2, \cdots.$$

(v) Write down a series solution to the above heat equation (1) which involves an infinite number of coefficients. [2 pts]

$$u(x, t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t/4\cdot 25^2} \sin\frac{n\pi x}{25}.$$

(vi) Calculate and simplify the coefficients in the series solution in (v) above. [3 pts]
Since if we set $t = 0$ for $u(x, t)$ in the above infinite series solution, we obtain

$$\frac{x}{2} = u(x, 0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{25},$$

so it remains to calculate

$$\begin{aligned} c_n &= \frac{2}{25}\int_0^{25} \frac{x}{2}\sin\frac{n\pi x}{25}\,dx \\ &= \frac{1}{25}\left\{ -\frac{25}{n\pi} x\cos\frac{n\pi x}{25}\Big|_0^{25} + \frac{25}{n\pi}\int_0^{25}\cos\frac{n\pi x}{25}\,dx \right\} \\ &= -\frac{25}{n\pi}\cos n\pi + \frac{25}{n^2\pi^2}\sin\frac{n\pi x}{25}\Big|_0^{25} \\ &= (-1)^{n+1}\frac{25}{n\pi}, \quad n = 1, 2, \cdots. \end{aligned}$$

(b) Suppose we now raise the temperature of one end of the metal rod to 20°C, that is, $u(0, t) = 0$, $u(25, t) = 20$, $t > 0$, with the same initial temperature distribution $u(x, 0)$. Define

$$v(x) = \lim_{t\to\infty} u(x, t), \quad 0 < x < 25.$$

(i) Use the original heat equation (1) above to derive a boundary value problem for $v(x)$. [3 pts]

$$v''(x) = 0, \quad v(0) = 0,\ v(25) = 20.$$

(ii) Solve the corresponding boundary value problem for $v(x)$ above. [2 pts]
Integrating $v''(x)$ twice yields $v(x) = ax + b$ where a and b are integration two constants. The boundary condition $v(0) = 0$, $v(25) = 20$ give

$$v(x) = \frac{4}{5}x.$$

(iii) Write down a series solution to the heat conduction problem in (b), leaving the coefficients in integral forms. (Do **not** evaluate the integrals.) [4 pts]
Since $u(x, 0) - v(x) = \frac{x}{2} - \frac{4}{5}x = -\frac{3}{10}x$, so we have

$$u(x, t) = \frac{4}{5}x + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t/4\cdot 25^2}\sin\frac{n\pi x}{25},$$

where

$$b_n = \frac{2}{25}\int_0^{25} -\frac{3x}{10}\sin\frac{n\pi x}{25}\,dx,$$

$n = 1, 2, \cdots.$