# HKUST

## MATH150 Introduction to Differential Equations

**Final Examination** (Version A) Name:____________________

20th May 2005 Student I.D.:____________________

*16:30–18:30* Tutorial Section:____________________

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**Directions:**

- Write your name, ID number, and tutorial section in the space provided above.
- DO NOT open the exam until instructed to do so.
- When instructed to open the exam, check that you have, in addition to this cover page, 11 pages of questions.
- Turn off all mobile phones and pagers during the examination.
- This is a closed book examination.
- You are advised to try the problems you feel more comfortable with first.
- You may write on both sides of the examination papers.
- There are 8 multiple choice questions. **DO NOT guess wildly! If you do not have confidence in your answer leave the question blank. Each incorrectly answered question will result in a 0.5 point deduction**.
- For the short and long questions, you must show the working steps of your answers in order to receive full points.
- Cheating is a serious offense. Students caught cheating are subject to a zero score as well as additional penalties.

| Question No. | Points | Out of |
|---|---|---|
| **Q. 1-8** | | 32 |
| **Q. 9-15** | | 32 |
| **Q. 16** | | 18 |
| **Q. 17** | | 18 |
| **Total Points** | | 100 |

**Part I: Each correct answer *in the answer box* for the following 8 multiple choice questions is worth 4 point. DO NOT guess wildly! If you do not have confidence in your answer leave the answer box blank. Each incorrectly answered question will result in a 0.5 point deduction.**

| **Question** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | **Total** |
|---|---|---|---|---|---|---|---|---|---|
| **Answer** | | | | | | | | | |

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1. Which of the following functions is an integrating factor for the non-exact equation

$$(x^3 + y) + (x \ln x + 2xy)\frac{dy}{dx} = 0 \ ?$$

is:

(a) $x$ (b) $y$ (c) $xy$ (d) $\frac{1}{x}$ (e) $\frac{1}{y}$

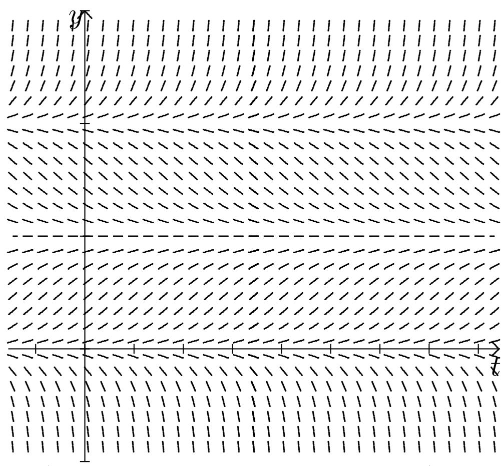
2. Using Euler's method with step size $h = 0.1$, the approximate value at $t = 0.3$ of the solution of the initial value problem

$$\frac{dy}{dt} = ty^2 + y, \qquad y(0) = 1$$

can be found as:

(a) 1.112 (b) 1.256 (c) 1.374 (d) 1.420 (e) 1.537

3. Given that $y$ satisfies $\frac{dy}{dt} = y(y-1)(y-2)$, which of the equilibrium solutions is/are stable solution(s)?




(direction field of the equation)

(a) $y = 0$ (b) $y = 1$ (c) $y = 2$ (d) $y = 0$ and $y = 1$ (e) None of the previous

4. If an external force $F(t) = 3\cos(2\omega t)$ is applied to a spring-mass system so that the equation of motion is

$$u'' + 9u = 3\cos(3\omega t)\ ,$$

at what $\omega$ will unbounded oscillation happen?

(a) 1 (b) 2 (c) 3 (d) 6 (e) 9

5. The Wronskian of the pair of functions $y_1 = e^t \sin t$, $y_2 = e^t \cos t$ can be found as:

(a) $e^{2t}(\cos 2t - 1)$ (b) $e^{2t}(\cos 2t + 1)$ (c) $-e^{2t}$ (d) $e^{2t}$ (e) $e^{2t}(\cos^2 t - \sin^2 t)$

6. Find a particular solution of $y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$.

(a) $-\frac{1}{2}e^{-x} + \frac{5}{13}\cos 3x + \frac{9}{13}\sin 3x$

(b) $\frac{1}{2}e^{-x} - \frac{7}{13}\cos 3x + \frac{3}{13}\sin 3x$

(c) $\frac{1}{3}e^{-x} + \frac{5}{13}\cos 3x - \frac{9}{13}\sin 3x$

(d) $\frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x$

(e) none of the previous

7. Find the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$ of the solution of the given initial value problem :

$$2y'' + 3y' + 2y = e^{-3t}\sin 4t, \qquad y(0) = -2, \quad y'(0) = 1\ .$$

(a) $\frac{-4s+8}{2s^2+3s+2} + \frac{4}{(2s^2+3s+2)(s^2+6s+25)}$

(b) $\frac{-4s-6}{2s^2+3s+2} + \frac{4}{(2s^2+3s+2)(s^2+6s+25)}$

(c) $\frac{-4s-4}{2s^2+3s+2} + \frac{4}{(2s^2+3s+2)(s^2+6s+25)}$

(d) $\frac{-4s+3}{2s^2+3s+2} + \frac{4}{(2s^2+3s+2)(s^2+6s+25)}$

(e) $\frac{4s-4}{2s^2+3s+2} + \frac{s+3}{(2s^2+3s+2)(s^2+6s+25)}$

8. Find the Laplace transform of

$$g(t) = \begin{cases} t+1, & 0 \le t < 6, \\ 0, & 6 \le t. \end{cases}$$

(a) $\dfrac{e^{-6s}(1+s)}{s^2}$ (b) $\dfrac{e^{-6s}(1+5s)}{s^2}$ (c) $\dfrac{e^{-6s}(1+7s)}{s^2}$

(d) $\dfrac{e^{-6s}(5+s)}{s^2}$ (e) $\dfrac{e^{-6s}(1-7s)}{s^2}$

**Part II: Answer each of the following 6 short answer questions.** ***Show all your work for full credit.***

| **Question** | 9 | 10 | 11 | 12 | 13 | 14 | 15 | **Total** |
|---|---|---|---|---|---|---|---|---|
| **Points** | | | | | | | | |

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9. A tank contains 80 gallons of pure water initially. A salt solution with 2 kg of salt per gallon is pumped into the tank at a rate of 3 gal/min, and the well-stirred mixture is pumped out at a rate of 4 gal/min. Given that the amount of salt $Q(t)$ at time $t$ in the tank satisfies the differential equation

$$\frac{dQ}{dt} = -\frac{4Q}{80-t} + 6, \quad Q(0) = 0.$$

(a) Solve the initial value problem.

Answer: $Q(t) =$ ____________________

(b) When the mixture in the tank is 40 gal, what is the salt **concentration** of the mixture in the tank.

Answer: The concentration is: ____________ kg/gal.

10. The rate of change of a rabbit population $P(t)$ at time $t$ (months) is proportional to the square root of $P(t)$. That is

$$\frac{dP}{dt} = k\sqrt{P},$$

for some constant $k > 0$.

(a) At time $t = 0$ the population numbers 100 rabbits and it is increasing at the rate of 21 rabbits per month. Determine $k$.

Answer: $k =$ ______________________________________.

(b) How many rabbits will there be one year later?

Answer: The number of rabbits after one year is __________________________.

11. (a) Find a fundamental set of solutions $\{y_1(t), y_2(t)\}$ to the differential equation:

$$y'' + \mu^2 y = 0,$$

where $\mu$ is a positive constant.

Answer: A fundamental set is ______________________________________ . [2 pts]

(b) Justify why your solution to (a) are linearly independent;

Answer: A reason is ____________________________________________ . [2]

(c) Determine all $a$, $b$, $c$, $d$ so that

$$\begin{aligned} f_1(t) &= ay_1 + by_2, \\ f_2(t) &= cy_1 + dy_2 \end{aligned}$$

becomes a fundamental set of solutions for each set of $\{a,\ b,\ c,\ d\}$ to the differential equation in (a). *Justify your answer for full credit.*

Answer: The condition is ________________________________________ . [4]

12. Given that $y_1(x) = x$ and $y_2(x) = 1 + x^2$ are solution of

$$(x^2 - 1)y'' - 2xy' + 2y = 0.$$

(a) Find the Wronskian of $y_1$ and $y_2$.

Answer: The Wronskian is ________________________________________ .

(b) Then find the particular solution the nonhomogeneous equation,

$$(x^2 - 1)y'' - 2xy' + 2y = x^2 - 1.$$

Answer: The particular solution is ________________________________________.

13. Let $\mathcal{L}(f(t)) = \dfrac{1}{(s^2+1)^2}$. Using convolution integral to find $f(t)$.

Answer: $f(t)$ is ________________________________

14. Consider a vibrating system described by the initial value problem

$$u'' + \frac{1}{4}u' + 2u = 2\cos\omega t, \quad u(0) = 0, \quad u'(0) = 2.$$

(a) Determine the steady-state part of the solution of this problem.

Answer: $U(t)$ is ________________________________________ .

(b) Find the amplitude $R$ of the steady-state solution in terms of $\omega$.

Answer: $R$ is ________________________________________

15. Extend the following function $f(t)$ to a periodic function of period $2L$ with $f(t+2L) = f(t)$ so that it can be expanded into a Fourier "**sine**" series:

$$f(t) = \begin{cases} 1, & 0 < t < L/4\,; \\ 4t, & L/4 < t < L\,. \end{cases}$$

(a) The formula of the extended part on the interval $-L < t < 0$ is : [2]

$$f(t) = \begin{cases} \\ \\ \end{cases}$$

(b) The value to which the above Fourier sine converges at $t = -L/4$ is : ____________ [1]

**Part III: Answer the two long questions.**

16. [*15 pts*] A impluse force is applied to a spring-mass system at time $t = 5$ *initially at rest at the equilibrium postion.* Suppose the displacement from equilibrium position of the mass at time $t$ is the solution of the following equation:

$$y'' + 6y' + 13y = 3\delta(t - 5).$$

(a) Find $y(t)$ by solving the given equation with appropriate initial values $y(0)$ and $y'(0)$. [*5 pts*]

(b) If an additional impluse force $b\delta(t - 5 - \pi)$ is also applied to the system at $t = 5 + \pi$, where $b$ is a constant, what should be the equation of motion of the mass? [*2 pts*]

(c) To bring the system to rest at $t = 5 + \pi$, i.e., $y(t) = 0$, if $t \geq 5 + \pi$, what impluse $b$ should one choose? [*8 pts*]

17. Suppose a metal rod of length 25 cm with *thermal diffusivity* $\alpha^2 = 1/4$. Let $u(x,\, t)$ be the temperature distribution function at $x$, $0 \le x \le 25$ and time $t \ge 0$. Suppose it is given that the initial temperature distribution of the metal rod is given by

$$u(x,\, 0) = \begin{cases} 5, & 0 < x < 10; \\ x, & 10 < x < 25. \end{cases}$$

It is known that the function $u(x,\, t)$ satisfies the heat equation

$$\alpha^2 u_{xx} = u_t, \quad 0 \le x \le 25,\ t > 0.$$

It is known that the heat equation can be solved by the method of separation of variable by assuming that $u(x,\, t) = X(x)T(t)$.

(a) If the temperatures at both ends of the metal rod are at 0°C. Write down a boundary value problem for $X(x)$. [2]

(b) Write down all possible solutions of $X(x)$. [2]

(c) Write down a differential equation for $T(t)$. [1]

(d) Solve for all possible $T(t)$ from the above equation. [2]

(e) Write down a series solution to the above heat equation in part (a) which involves an infinite number of coefficients. [2]

(f) Calculate the coefficients in the infinite expansions of functions above. [3]

(g) Suppose we now raise the temperature of one end of the metal rod to 20° C, that is, $u(25,\, t) = 20$, $t > 0$, and we define

$$v(x) = \lim_{t\to\infty} u(x,\, t), \quad x,\, 0 < x < 25,$$

Use the original heat equation in (a) above to write down a boundary value problem for $v(x)$. [3].

(h) Solve the corresponding boundary value problem for $v(x)$ above. [1]

(i) Write down a series solution to the above heat equation, leaving the coefficients in an integral forms. Do **no** need to evaluate the integrals [4]

(j) Find $\lim_{t\to\infty} u(x,\, t)$ for each $x$, $0 < x < 25$, as $t \to \infty$ under the above assumption. Does it match with the $v(x)$ found above? [2]

Table 1: Laplace transforms

| | $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|---|
| 1 | $1$ | $\frac{1}{s}, \quad s > 0$ |
| 2 | $e^{at}$ | $\frac{1}{s-a}, \quad s > a$ |
| 3 | $t^n, \quad n =$ positive integer | $\frac{n!}{s^{n+1}}, \quad s > 0$ |
| 4 | $t^p, \quad p > -1$ | $\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$ |
| 5 | $\sin at$ | $\frac{a}{s^2+a^2}, \quad s > 0$ |
| 6 | $\cos at$ | $\frac{s}{s^2+a^2}, \quad s > 0$ |
| 7 | $\sinh at$ | $\frac{a}{s^2-a^2}, \quad s > \|a\|$ |
| 8 | $\cosh at$ | $\frac{s}{s^2-a^2}, \quad s > \|a\|$ |
| 9 | $e^{at}\sin bt$ | $\frac{b}{(s-a)^2+b^2}, \quad s > a$ |
| 10 | $e^{at}\cos bt$ | $\frac{s-a}{(s-a)^2+b^2}, \quad s > a$ |
| 11 | $t^n e^{at}, \quad n =$ positive integer | $\frac{n!}{(s-a)^{n+1}}, \quad s > 0$ |
| 12 | $u_c(t)$ | $\frac{e^{-cs}}{s}, \quad s > 0$ |
| 13 | $u_c(t)\, f(t-c)$ | $e^{-cs}\, F(s)$ |
| 14 | $e^{ct}\, f(t)$ | $F(s-c)$ |
| 15 | $f(ct)$ | $\frac{1}{c}\, F\left(\frac{s}{c}\right), \quad c > 0$ |
| 16 | $\int_0^t f(t-\tau)\, g(\tau)\, d\tau$ | $F(s)\, G(s)$ |
| 17 | $\delta(t-c)$ | $e^{-cs}$ |
| 18 | $f^{(n)}(t)$ | $s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$ |
| 19 | $(-t)^n f(t)$ | $F^{(n)}(s)$ |