Math150 Introduction to Differential Equations
Brief Review of Complex Numbers

1 Complex Numbers

No real number satisfies the equation $x^2 = -1$, since the square of any real number has to be non-negative. By introducing a new "*imaginary number*" $i = \sqrt{-1}$, which is supposed to have the property $i^2 = -1$, the real number system can be extended to a large number system, namely, the *complex number system.*

- A complex number is an expression of the form $x + iy$, where $x, y \in \mathbb{R}$ are real numbers.
- x is called the **real part** of the complex number, and y the **imaginary part**, of the complex number $x + iy$. The real part and imaginary part of a complex number are sometimes denoted respectively by $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$.
- Two complex numbers are equal, $a + ib = c + id$, where $a, b, c, d \in \mathbb{R}$ are all real numbers, if and only if $a = c$ and $b = d$.

The usual algebraic operations $+$, $-$, $\cdot$ (or $\times$), $\div$ on real numbers can then be extended to operations on complex numbers in a natural way:

$$\begin{aligned}
&\textbf{addition} && (a+ib)+(c+id) = (a+c)+i(b+d) \\
&\textbf{subtraction} && (a+ib)-(c+id) = (a-c)+i(b-d) \\
&\textbf{multiplication} && (a+ib)(c+id) = (ac-bd)+i(ad+bc) \\
&\textbf{division} && \frac{a+ib}{c+id} = \frac{ac+bd}{c^2+d^2} + i\frac{-ad+bc}{c^2+d^2}
\end{aligned}$$

Example 1.1

$$\begin{aligned}
(3+5i)+(-2+i) &= (3-2)+(5+1)i = 1+6i \\
(3+5i)(-2+i) &= (3\cdot(-2)-5\cdot 1)+(3\cdot 1+5\cdot(-2))i = -11-7i \\
\frac{3+5i}{-2+i} &= \frac{3\cdot(-2)+5\cdot 1}{(-2)^2+1^2} + \frac{-3\cdot 1+5\cdot(-2)}{(-2)^2+1^2}i = \frac{-1}{5} + \frac{-13}{5}i
\end{aligned}$$

For multiplication and division of complex numbers, instead of using their formal definitions, it is more convenient to simply carrying out the algebraic operations by following the usual algebraic rules, such as commutative law, associative law and distributive law.

Example 1.2 $i^2 = -1$, $i^3 = i^2 \cdot i = (-1)i = -i$, $i^4 = i^3 \cdot i = -i \cdot i = 1$, $i^5 = i^4 \cdot i = i$, $\cdots$.

$$i^{4n} = 1, \quad i^{4n+1} = i, \quad i^{4n+2} = -1, \quad i^{4n+3} = -i, \quad \text{where } n = 0, 1, 2, 3, \cdots$$

Example 1.3 (Multiplication)

$$\begin{aligned}
(3+5i)(-2+i) &= (3+5i)\cdot(-2)+(3+5i)\cdot i = 3\cdot(-2)+5i\cdot(-2)+3\cdot i+5i\cdot i \\
&= -6-10i+3i-5 = -11-7i
\end{aligned}$$

Example 1.4 (Division)

The fact that $(a+ib)(a-ib) = a^2+b^2$ leads to a usual trick to find the quotient of two complex numbers:

$$\frac{3+5i}{-2+i} = \frac{(3+5i)(-2-i)}{(-2+i)(-2-i)} = \frac{-6-3i-10i-5i^2}{(-2)^2-i^2} = \frac{-1}{5} + \frac{-13}{5}i$$

2 Conjugate, Modulus and Polar Representation of Complex Numbers

If $z = x + iy$, where x, y are real numbers, then its **complex conjugate** $\bar{z}$ is defined as the complex number $\bar{z} = x - iy$. It is easy to check that $\frac{1}{2}(z + \bar{z}) = x = \mathrm{Re}(z)$ and $\frac{1}{2}(z - \bar{z}) = iy = i\mathrm{Im}(z)$. Moreover, $z\bar{z}$ must be a nonnegative real number, since

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$$

The **modulus** or **absolute value** of a complex number $z = x + iy$ is defined and denoted by

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

Example 2.1 Let $z = 2 - 3i$. Then $\bar{z} = 2 + 3i$ and $z\bar{z} = 2^2 + 3^2 = 13$. Hence $|2 + 3i| = \sqrt{13}$.

Exercise Show that (a) $|z_1 z_2| = |z_1| \cdot |z_2|$, (b) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$.

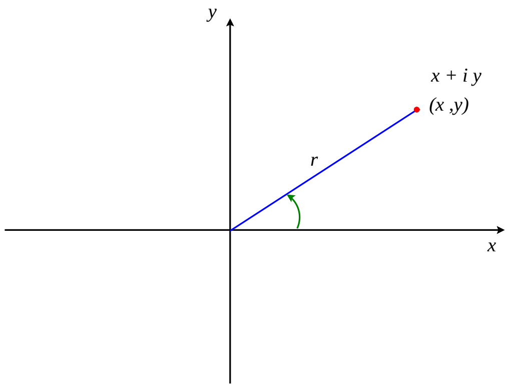
Example 2.2 (Quadratic Equations with Real Coefficients)

The roots of a quadratic equation $az^2 + bz + c = a(z + \frac{b}{2a})^2 + c - \frac{b^2}{4a} = 0$, where a, b, c are real numbers, are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \longleftrightarrow \begin{cases} \text{two distinct real roots if } b^2 - 4ac > 0 \\ \text{one repeated real roots if } b^2 - 4ac = 0 \\ \text{two distinct (conjugate) complex roots if } b^2 - 4ac < 0 \end{cases}$$

For example, $z^2 + z + 1 = 0$ has two complex roots $\dfrac{-1 + \sqrt{3}\,i}{2}$ and $\dfrac{-1 - \sqrt{3}\,i}{2}$, which are conjugate to each other.

A complex number $z = x + iy$ can be graphically represented by a point with coordinates (x, y) in the Cartesian coordinate plane.



Using polar coordinates of the plane, i.e., letting

$$x = r\cos\theta, \qquad y = r\sin\theta$$

a complex number $z = x + iy \neq 0$ can be expressed in the form

$$z = r(\cos\theta + i\sin\theta)$$

where $r = \sqrt{x^2+y^2} = |z|$ is the modulus, and θ is an **argument** of z. Since sin and cos have period 2π, the argument of z is defined up to an integral multiple of 2π:

$$\arg z = \theta + 2n\pi, \quad n = 0, \pm 1, \pm 2, \cdots$$

Example 2.3 (Multiplication and Division via Polar Representations)

Note that for $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, we have

$$\begin{aligned} z_1 \cdot z_2 &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)] \\ &= r_1 r_2 [(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \\ \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)] \end{aligned}$$

In particular, De Moivre's Theorem follows:

$$z^n = r^n(\cos n\theta + i\sin n\theta) \quad \text{for any integer } n.$$

For example, $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$, and $(\cos\theta + i\sin\theta)^{-2} = \cos(-2\theta) + i\sin(-2\theta) = \cos 2\theta - i\sin 2\theta$.

3 Complex Exponential Function

The real-valued exponential function e^x can be extended to a complex function e^z by the following definition:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

Using the complex exponential function, the polar representation of a complex number can be expressed as $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, where θ is a choice of the argument of z. It is easy to see from the definition that $e^{z+2\pi i} = e^{x+i(y+2\pi)} = e^x(\cos(2\pi+y) + i\sin(y+2\pi)) = e^z$, i.e., e^z is a complex-valued function with complex period $2\pi i$.

Example 3.1 $e^{i\pi} = \cos\pi + i\sin\pi = -1$; $e^{3+\frac{\pi}{4}i} = e^3 e^{\frac{\pi}{4}i} = e^3(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \frac{\sqrt{2}e^3}{2} + \frac{\sqrt{2}e^3}{2}i$.

Example 3.2 It is not hard to show from the definition: $e^{z_1}e^{z_2} = e^{z_1+z_2}$.

Example 3.3 The roots of the equation $z^n = a$ for any positive integer n can be found from the polar representations $z^n = a = re^{i\theta} = re^{i(2k\pi+\theta)}$, $k = 0, 1, 2, \ldots$ as

$$z = r^{\frac{1}{n}} e^{\frac{2k\pi+\theta}{n}i}, \quad k = 0, 1, 2, \ldots, n-1.$$

For example, the roots of $z^3 = 1 = e^{2k\pi i}$ are: $1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}$, i.e., $1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Example 3.4 (Complex Exponential Functions of a Real Variable)

For any real number a, b, and a *real variable* t, the complex exponential functions $e^{(a+ib)t}$ and $e^{(a-ib)t}$ of the real variable t simply mean:

$$e^{(a+ib)t} = e^{at}(\cos bt + i\sin bt)\,, \qquad e^{(a-ib)t} = e^{at}(\cos bt - i\sin bt).$$

Note that in particular,

$$e^{at}\cos bt = \mathrm{Re}(e^{(a+ib)t}) = \frac{1}{2}e^{(a+ib)t} + \frac{1}{2}e^{(a-ib)t}$$

$$e^{at}\sin bt = \mathrm{Im}(e^{(a+ib)t}) = \frac{1}{2i}e^{(a+ib)t} - \frac{1}{2i}e^{(a-ib)t}$$

For example, $e^{(2+3i)t}$ is the function $e^{2t}\cos 3t + ie^{2t}\sin 3t$.

Similarly, $e^{-t}\cos 2t$ and $e^{-t}\sin 2t$ are the real and imaginary part of the complex-valued exponential function $e^{(-1+2i)t}$ respectively.

4 Complex-valued Functions of a Real Variable

If $f(t)$ and $g(t)$ are functions of a real variable t, a complex-valued function $h(t)$ of a real variable can be defined by taking $h(t) = f(t) + ig(t)$. Then

$$f(t) = \mathrm{Re}\ h(t), \qquad g(t) = \mathrm{Im}\ h(t)$$

The derivative or integrals of $h(t)$ can be defined in a straightforward way by

$$h'(t) = f'(t) + ig'(t)$$

$$\int h(t)dt = \int f(t)dt + i\int g(t)dt, \qquad \int_a^b h(t)dt = \int_a^b f(t)dt + i\int_a^b g(t)dt$$

A useful complex-valued function for solving linear differential equation is

$$h(t) = e^{(a+ib)t} = e^{at}(\cos bt + i\sin bt)$$

Direct computation shows

$$\frac{de^{(a+ib)t}}{dt} = (a\cos bt - b\sin bt)e^{at} + i(a\sin bt + b\cos bt)e^{at} = (a+ib)e^{(a+ib)t}$$

i.e., the ordinary derivative formula for e^{at} extends to the complex case $e^{(a+ib)t}$.

Example 4.1 $\dfrac{de^{(3-2i)t}}{dt} = (3-2i)e^{(3-2i)t}$, and hence

$$\text{(I)} \qquad \int e^{(3-2i)t}dt = \frac{1}{3-2i}e^{(3-2i)t} + c = \frac{3+2i}{13}e^{3t}(\cos 2t - i\sin 2t) + c,$$

where c is an arbitrary complex integration constant. Now, since

$$\text{(II)} \qquad \int e^{(3-2i)t}dt = \int e^{3t}\cos(-2)tdt + i\int e^{3t}\sin(-2)tdt = \int e^{3t}\cos 2tdt - i\int e^{3t}\sin 2tdt,$$

by a comparison of real and imaginary parts on the right sides of (I) and (II), one gets back the real integrals:

$$\int e^{3t}\cos 2tdt = e^{3t}\big(\frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t\big) + C, \qquad \int e^{3t}\sin 2tdt = e^{3t}\big(\frac{3}{13}\sin 2t - \frac{2}{13}\cos 2t\big) + C$$

Exercise

1. Express the following complex numbers in the form of $x + iy$:

   (a) $(2-5i)+(3+4i)$ (b) $(2+3i)-(3+2i)$ (c) $(3-5i)(2+i)$

   (d) $(2-3i)^3$ (e) $\dfrac{1}{3+2i}$ (f) $\dfrac{1}{5}-\dfrac{3-5i}{4+3i}$

2. Find find $|z|$, $\bar{z}$ and as least one polar representation of z for the following complex numbers:

   (a) $z = 2-2i$ (b) $z = 1-\sqrt{3}i$ (c) $z = \dfrac{1+\sqrt{3}i}{1-\sqrt{3}i}$

3. Find all the roots of the quadratic equation $z^2 - 3z + 3 = 0$.

4. Express the following complex numbers in the form of $x + iy$:

   (a) $e^{\frac{\pi i}{4}}$ (b) $e^{-\frac{3\pi i}{4}}$ (c) $e^{\frac{3\pi i}{2}}$

5. Find $\dfrac{dy}{dt}$ and $\displaystyle\int y(t)dt$ for the following functions:

   (a) $y(t) = e^{(3+2i)t}$ (b) $y(t) = e^{it}$ (c) $y(t) = e^{(-1+2i)t}$

6. Use a suitable complex exponential function to integrate $\displaystyle\int e^{-2t}\cos 3t dt$.