Math150 Introduction to Differential Equations

Prerequisite: Work on Your Integration/Differentiation Skills!!!

Indefinite Integrals (or Antiderivatives)

- **Definition**

$$\int f(x)dx = F(x) + C \overset{means}{\Longleftrightarrow} F'(x) = f(x) \Longleftrightarrow y = F(x) + C \text{ solves } \frac{dy}{dx} = f(x)$$

integrating = anti-differentiating (reversing differentiation) = solving certain differential equation

- **Basic Techniques in Finding Indefinite Integrals**

1. Directly from formulas
2. Substitution: $\int f(u(x))u'(x)dx = \int f(u)du$ (Let $u = u(x)$, hence $du = u'(x)dx$)
3. Integration by parts: $\int udv = uv - \int vdu$ (From Product Rule: $[uv]' = uv' + vu'$)
4. Partial Fractions: If $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, try writing it as a combination of a polynomial (by division algorithm, if $\deg p(x) \geq \deg q(x)$) and other simple fractions like $\dfrac{A}{(ax+b)^k}$, $\dfrac{Bx+C}{(ax^2+bx+c)^k}$. For examples,

$$\frac{x+3}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{1}{x+2}$$

$$\frac{x^2-4x+7}{(x-1)(x-2)^2} = \frac{2}{x-1} - \frac{1}{x-2} + \frac{1}{(x-2)^2}$$

$$\frac{x+3}{(x-1)(x^2+1)} = \frac{2}{x-1} - \frac{2x+1}{x^2+1}$$

Basic techniques involved: solving systems of linear equations; e.g.,

$$\frac{x+3}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \Longleftrightarrow A(x^2+1) + (Bx+C)(x-1) = x+3$$

$$\Longleftrightarrow \begin{cases} A + B = 0 \\ -B + C = 1 \\ A - C = 3 \end{cases} \longrightarrow \begin{cases} A = \_\_\_\_\_ \\ B = \_\_\_\_\_ \\ C = \_\_\_\_\_ \end{cases}$$

Sometimes one can find some of these coefficients by putting in some suitable values of x; e.g., by putting in $x = 1$, A can be found easily from $A(1^2+1) + (B+C)(1-1) = 1+3$.

- **Some Often Seen Substitutions**

$$\sqrt{a^2 - x^2} \quad \longleftrightarrow \quad x = a\cos t \quad \text{or} \quad x = a\sin t$$

$$\sqrt{a^2 + x^2} \quad \longleftrightarrow \quad x = a\tan t \quad \text{or} \quad x = a\sinh t \overset{def}{=} \tfrac{a}{2}\left(e^t - e^{-t}\right)$$

$$\sqrt{x^2 - a^2} \quad \longleftrightarrow \quad x = a\sec t \quad \text{or} \quad x = a\cosh t \overset{def}{=} \tfrac{a}{2}\left(e^t + e^{-t}\right)$$

- **Derivative-Integral Formulas**

Basic Formulas			Chain Rule Version
$\frac{d(\text{constant})}{dx}=0$			
$\frac{dx^p}{dx}=px^{p-1}$	$\overset{if\ p\neq -1}{\longleftrightarrow}$	$\int x^p dx=\frac{1}{p+1}x^{p+1}+C$	$\frac{d[f(x)]^p}{dx}=p[f(x)]^{p-1}$
$\frac{d\ln x}{dx}=\frac{1}{x}$	$\longleftrightarrow$	$\int \frac{1}{x}dx=\ln\|x\|+C$	$\frac{d\ln f(x)}{dx}=\frac{f'(x)}{f(x)}$
$\frac{de^x}{dx}=e^x$	$\longleftrightarrow$	$\int e^x dx=e^x+C$	$\frac{de^{f(x)}}{dx}=f'(x)e^{f(x)}$
$\frac{d\sin x}{dx}=\cos x$	$\longleftrightarrow$	$\int \cos x dx=\sin x+C$	$\frac{d\sin f(x)}{dx}=f'(x)\cos f(x)$
$\frac{d\cos x}{dx}=-\sin x$	$\longleftrightarrow$	$\int \sin x dx=-\cos x+C$	$\frac{d\cos f(x)}{dx}=-f'(x)\sin f(x)$
$\frac{d\tan x}{dx}=\sec^2 x$	$\longleftrightarrow$	$\int \sec^2 x dx=\tan x+C$	$\frac{d\tan f(x)}{dx}=f'(x)\sec^2 f(x)$

- **Some Basic Reduction Formulas from Integration by Parts**

$$\int p(x)e^{ax}dx=\frac{1}{a}\int p(x)de^{ax}=\frac{1}{a}p(x)e^{ax}-\frac{1}{a}\int p'(x)e^{ax}dx$$

$$\int p(x)\sin(ax)dx=-\frac{1}{a}\int p(x)d\cos ax=-\frac{1}{a}p(x)\cos(ax)+\frac{1}{a}\int p'(x)\cos(ax)dx$$

$$\int p(x)\cos(ax)dx=\frac{1}{a}\int p(x)d\sin ax=\frac{1}{a}p(x)\sin(ax)-\frac{1}{a}\int p'(x)\sin(ax)dx$$

e.g. try working out the reduction formulas for :

$$\int \sin^n axdx,\quad \int \cos^n axdx,\quad \int \tan^n axdx,\quad \int \sec^n axdx$$

- **Others** $\int \tan axdx=\frac{1}{a}\ln|\sec ax|+C,\quad \int \sec axdx=\frac{1}{a}\ln|\sec ax+\tan ax|+C$

Most Often Seen Differentiation in Math150

$$\frac{de^{kx}}{dx}=ke^{kx},\qquad \frac{d^2e^{kx}}{dx^2}=k^2e^{kx}\quad (k=\text{ any real or complex number})$$

$$\frac{d[u(x)v(x)]}{dx}=u'(x)v(x)+u(x)v'(x),\qquad \frac{d^2[u(x)v(x)]}{dx^2}=u''(x)v(x)+2u'(x)v'(x)+u(x)v''(x)$$

Fill in the following blanks:

$$\frac{d}{dx}\left[p(x)e^{kx}\right]=\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\frac{d}{dx}\left[p(x)e^{kx}\sin bx\right]=\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\frac{d}{dx}\left[p(x)e^{kx}\cos bx\right]=\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

Find also their 2nd order derivatives.

Review Exercise

Find the following indefinite integrals. *You may check you answers by differentiating them and see if you get the integrand (the function to be integrated) back!*

$$\int f(x)dx = F(x) + C \Longleftrightarrow \frac{dF(x)}{dx} = f(x)$$

1. $\int \left(x^3 - x^{-\frac{3}{2}}\right) dx =$
2. $\int \frac{x^3 + 2x - 1}{x^2} dx =$
3. $\int \frac{2}{3x - 2} dx =$
4. $\int \frac{3}{2 - 5x} dx =$
5. $\int \frac{2}{(3x - 2)^2} dx =$
6. $\int \frac{1}{(x - 2)(x - 3)} dx =$
7. $\int \frac{x + 1}{(x - 2)^2(x - 3)} dx =$
8. $\int e^{-3x} dx =$
9. $\int xe^{-x^2} dx =$
10. $\int \frac{x}{\sqrt{x^2 + 1}} dx =$
11. $\int \frac{1}{\sqrt{4 - x^2}} dx =$
12. $\int \frac{1}{3x^2 + 4} dx =$
13. $\int xe^{2x} dx =$
14. $\int xe^{-2x} dx =$
15. $\int x^2 e^{2x} dx =$
16. $\int x^2 e^{-2x} dx =$
17. $\int x \sin 2x dx$
18. $\int x^2 \cos 2x dx =$
19. $\int \ln x dx =$
20. $\int x \ln x dx =$