



# The kinetic scheme for the full-Burnett equations

Taku Ohwada <sup>a,\*</sup>, Kun Xu <sup>b</sup>

<sup>a</sup> *Department of Aeronautics and Astronautics, Graduate School of Engineering, Kyoto University, Kyoto 606-8501, Japan*

<sup>b</sup> *Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong*

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## Abstract

The present paper concerns two aspects for the Burnett equations. First, we are going to theoretically show the consistency between the traditional Chapman–Enskog expansion and the successive approximation for the BGK equation up to the super-Burnett order. Second, we will design a numerical scheme to efficiently solve the Burnett equations. The current approach is an improvement of the BGK–Burnett scheme [K. Xu, *Phys. Fluids* 15 (2003) 2077], where the locally constant collision frequency is considered. Based on the Burnett distribution function, a high order time accurate numerical flux is derived by different approaches. The resulting scheme is tested in the problem of normal shock wave and that of force-driven Poiseuille flow.

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## 1. Introduction

The distribution functions of gas molecules derived from the Chapman–Enskog expansion for the Boltzmann equation (or the BGK model equation) are often employed in the construction of kinetic schemes for gasdynamic equations. For example, the first order approximation in the expansion, the local Maxwellian, is employed for the compressible Euler equations (e.g. [15]) and the second order approximation is used for the compressible Navier–Stokes equations (e.g., [3,7,11,12,17]). When we try to incorporate higher order rarefaction effects by extending the distribution function to the Burnett or super-Burnett order (the third or fourth order approximation), we are faced with the theoretical problems, such as the ill-posedness of the Burnett and super-Burnett equations [5] and the boundary conditions [6].

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\* Corresponding author. Tel.: +81-75-753-5801; fax: +81-75-753-4947.  
*E-mail address:* [ohwada@kuaero.kyoto-u.ac.jp](mailto:ohwada@kuaero.kyoto-u.ac.jp) (T. Ohwada).

Despite the above difficulties, this approach is far from being useless. In fact, the Burnett and super-Burnett equations are employed in some numerical experiments of rarefied gas flows, e.g., [1,4,18,19,21], and the improvements over the NS results are reported. In connection with the hybrid approach, which employs the gasdynamic equations in slowly varying regions, such as the far field, and the kinetic equations in rapidly varying regions, such as the shock layer and the Knudsen layer, the numerical study of the Burnett and super-Burnett equation has been attracting attentions recently. However, it is not too much to say that the progress still remains in each specialist's skill and the clear understanding and common basis are not prevailing. For example, it is not clear whether the simple extension of the existing kinetic NS solvers is valid or not. Due to the complicated nature of the Burnett and super-Burnett terms, which usually give small effects, some approaches may be based on the equations which are not consistent with the Chapman–Enskog expansion.

The objective of the present paper is to display the common basis and the clear understanding of the construction of kinetic schemes for the gasdynamic equations beyond the NS equations. The organization of the paper is as follows. In Section 2, we briefly review the Chapman–Enskog expansion for the BGK equation. The equivalence between the standard procedure and the successive approximation has been recognized among some of the specialists in kinetic theory but its clear understanding has not yet been prevailing. It is clearly explained and demonstrated up to the super-Burnett level there. The result of the successive approximation is applied to an algebraic construction of the Burnett and super-Burnett distribution function in Section 3. The algebraic method for the compressible Navier–Stokes equations is developed in [17] and is extended to the case of the Burnett and super-Burnett equations in [19]. However, this extension is based on the assumption of locally constant collision frequency and the error caused by this simplification appears from the Burnett order. We derive the full-Burnett distribution function without using this simplification. In Section 4, a kinetic scheme for the full-Burnett equations is derived together with the information of the time step truncation error. The different derivations of the scheme, which are helpful for the clear understanding, are explained. In Section 5, we carry out the numerical test of the newly developed Burnett scheme in the problem of normal shock wave and that of force-driven Poiseuille flow. The first numerical test exhibits the robustness of the resulting scheme around the border of the application range of the Chapman–Enskog expansion. The second numerical test exhibits the non-Navier–Stokes behavior [2,9,16,19,20].

## 2. Chapman–Enskog expansion

### 2.1. Basic equation and notation

We consider the case of a monatomic gas without external force. The dimensional BGK model equation is written as

$$\frac{\partial \hat{f}}{\partial \hat{t}} + \hat{\mathbf{u}} \frac{\partial \hat{f}}{\partial \hat{\mathbf{x}}} = \hat{\nu}(\hat{g} - \hat{f}), \quad (1)$$

$$\hat{g} = \frac{\hat{\rho}}{(2\pi R\hat{T})^{3/2}} \exp \left[ -\frac{(\hat{\mathbf{u}} - \hat{\mathbf{U}})^2}{2R\hat{T}} \right], \quad (2)$$

$$\begin{pmatrix} \hat{\rho} \\ \hat{\rho}\hat{\mathbf{U}} \\ 3\hat{\rho}R\hat{T} \end{pmatrix} = \int \begin{pmatrix} 1 \\ \hat{\mathbf{u}} \\ (\hat{\mathbf{u}} - \hat{\mathbf{U}})^2 \end{pmatrix} \hat{f} \, d\hat{\mathbf{u}}, \quad (3)$$

where  $\hat{t}$  is the time;  $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$  is the space rectangular coordinate system;  $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$  is the molecular velocity;  $\hat{f}(\hat{t}, \hat{\mathbf{x}}, \hat{\mathbf{u}})$  is the distribution function of gas molecules;  $\hat{\rho}$ ,  $\hat{\mathbf{U}}$ , and  $\hat{T}$  are the density, flow velocity, and temperature of the gas, respectively;  $R$  is the specific gas constant; and  $\hat{\nu}$  is the collision frequency. The collision frequency  $\hat{\nu}$  is proportional to the density and is written as

$$\hat{\nu} = \hat{A}_c \hat{\rho}. \tag{4}$$

The coefficient  $\hat{A}_c$  is usually a constant but it may depend on the temperature of the gas. In the present paper, the explanation is for the latter general case, i.e.  $\hat{A}_c = \hat{A}_c(\hat{T})$ . In addition, vectors are sometimes expressed by using subscripts, e.g.  $\hat{U}_i$  is the component of  $\hat{\mathbf{U}}$  in the direction of  $\hat{x}_i$ .

In order to carry out the Chapman–Enskog expansion, we rewrite the BGK equation in a nondimensional form. For this purpose, we introduce the following nondimensional variables:  $\mathbf{x} = L^{-1}\hat{\mathbf{x}}$ ,  $t = (2R\hat{T}_0)^{1/2}L^{-1}\hat{t}$ ,  $\mathbf{u} = (2R\hat{T}_0)^{-1/2}\hat{\mathbf{u}}$ ,  $f = (2R\hat{T}_0)^{3/2}\hat{\rho}_0^{-1}\hat{f}$ ,  $\rho = \hat{\rho}_0^{-1}\hat{\rho}$ ,  $\mathbf{U} = (2R\hat{T}_0)^{-1/2}\hat{\mathbf{U}}$ ,  $T = \hat{T}_0^{-1}\hat{T}$ , where  $L$ ,  $\hat{\rho}_0$ , and  $\hat{T}_0$  are the reference length, density, and temperature of the system under consideration, respectively. Then, the nondimensional BGK equation is written as

$$Df = \frac{(g - f)}{\tau}, \tag{5}$$

$$g = \frac{\rho}{(\pi T)^{3/2}} \exp\left(-\frac{(\mathbf{u} - \mathbf{U})^2}{T}\right), \tag{6}$$

where  $D = \partial_t + u_i \partial_{x_i}$ ,  $\tau = \epsilon \rho^{-1} S(T)^{-1}$ ,  $S(T) = \hat{A}_c(\hat{T}_0 T) / \hat{A}_c(\hat{T}_0)$  [ $S(1) = 1$ ],  $\epsilon = \sqrt{\pi} l_0 / (2L)$ , and  $l_0$  is the mean free path of gas molecules in the equilibrium state at rest with the density  $\hat{\rho}_0$  and the temperature  $\hat{T}_0$ , i.e.

$$l_0 = 2\pi^{-1/2} \hat{A}_c(\hat{T}_0)^{-1} \hat{\rho}_0^{-1} (2R\hat{T}_0)^{1/2}.$$

The nondimensional macroscopic variables  $\mathbf{h} = (\rho, \rho U_1, \rho U_2, \rho U_3, 3\rho T/2 + \rho[U_1^2 + U_2^2 + U_3^2])$  are given by the moments of  $f$

$$\mathbf{h} = \int \boldsymbol{\psi} f \, d\mathbf{u}, \tag{7}$$

where  $\boldsymbol{\psi} = (\psi_0, \psi_1, \psi_2, \psi_3, \psi_4) = (1, u_1, u_2, u_3, u_1^2 + u_2^2 + u_3^2)$  and the domain of the integration with respect to the molecular velocity is the whole velocity space (this is applied to all the following integrals with respect to the velocity variables).

### 2.2. Standard procedure of Chapman–Enskog expansion

In this subsection, we briefly review the Chapman–Enskog expansion for the BGK equation according to [8].

In the Chapman–Enskog expansion, the situation where  $\epsilon \ll 1$  is considered. The following functional form of the solution is the postulation of this expansion

$$f = f(\mathbf{h}, \mathcal{D}\mathbf{h}, \mathbf{u}, \epsilon), \tag{8}$$

where  $\mathcal{D}$  represents the differential operators with respect to the space coordinates. Eq. (8) means that the distribution function  $f$  depends on  $t$  and  $\mathbf{x}$  only through the macroscopic variables  $\mathbf{h}$  and their space derivatives  $\mathcal{D}\mathbf{h}$ . Multiplying both hand sides of the BGK equation (5) by  $\boldsymbol{\psi}$ , carrying out the integration over the whole velocity space, and substituting Eq. (8) into the result, we have the conservation equations in the form:

$$\frac{\partial \mathbf{h}}{\partial t} = - \int \boldsymbol{\psi} \left( \mathbf{u} \frac{\partial f}{\partial \mathbf{x}} \right) d\mathbf{u} = \boldsymbol{\Phi}(\mathbf{h}, \mathcal{D}\mathbf{h}, \epsilon). \quad (9)$$

The  $f$  and  $\boldsymbol{\Phi}$  are formally expanded into the power series of  $\epsilon$ :

$$f(\mathbf{h}, \mathcal{D}\mathbf{h}, \mathbf{u}, \epsilon) = f_0(\mathbf{h}, \mathcal{D}\mathbf{h}, \mathbf{u}) + \epsilon f_1(\mathbf{h}, \mathcal{D}\mathbf{h}, \mathbf{u}) + \epsilon^2 f_2(\mathbf{h}, \mathcal{D}\mathbf{h}, \mathbf{u}) + \dots, \quad (10)$$

$$\boldsymbol{\Phi}(\mathbf{h}, \mathcal{D}\mathbf{h}, \epsilon) = \boldsymbol{\Phi}_0(\mathbf{h}, \mathcal{D}\mathbf{h}) + \epsilon \boldsymbol{\Phi}_1(\mathbf{h}, \mathcal{D}\mathbf{h}) + \epsilon^2 \boldsymbol{\Phi}_2(\mathbf{h}, \mathcal{D}\mathbf{h}) + \dots. \quad (11)$$

Substituting Eqs. (10) and (11) into the BGK Eq. (5) and arranging the results in the order of power of  $\epsilon$  formally, we have

$$f_0 = g, \quad (12)$$

$$S(T)\rho f_1 = - \frac{\partial g}{\partial \mathbf{h}} \boldsymbol{\Phi}_0 - \mathbf{u} \frac{\partial g}{\partial \mathbf{x}}, \quad (13)$$

$$S(T)\rho f_2 = - \frac{\partial g}{\partial \mathbf{h}} \boldsymbol{\Phi}_1 - \frac{\partial f_1}{\partial \mathbf{h}} \boldsymbol{\Phi}_0 - \frac{\partial f_1}{\partial \nabla \mathbf{h}} \nabla \boldsymbol{\Phi}_0 - \mathbf{u} \frac{\partial f_1}{\partial \mathbf{x}}, \quad (14)$$

$$S(T)\rho f_3 = - \frac{\partial g}{\partial \mathbf{h}} \boldsymbol{\Phi}_2 - \frac{\partial f_1}{\partial \mathbf{h}} \boldsymbol{\Phi}_1 - \frac{\partial f_1}{\partial \nabla \mathbf{h}} \nabla \boldsymbol{\Phi}_1 - \frac{\partial f_2}{\partial \mathbf{h}} \boldsymbol{\Phi}_0 - \frac{\partial f_2}{\partial \nabla \mathbf{h}} \nabla \boldsymbol{\Phi}_0 - \frac{\partial f_2}{\partial \nabla^2 \mathbf{h}} \nabla^2 \boldsymbol{\Phi}_0 - \mathbf{u} \frac{\partial f_2}{\partial \mathbf{x}}, \quad (15)$$

where  $\nabla^n$  is the abbreviation of the  $n$ th order differential operators with respect to space. Since  $f_0$  is the local Maxwellian,  $f_k$  ( $k = 1, 2, 3, \dots$ ) should satisfy the orthogonality condition

$$\int \boldsymbol{\psi} f_k d\mathbf{u} = 0 \quad (k = 1, 2, 3, \dots), \quad (16)$$

which is reduced to

$$\boldsymbol{\Phi}_k = - \int \boldsymbol{\psi} \left( \mathbf{u} \frac{\partial f_k}{\partial \mathbf{x}} \right) d\mathbf{u} \quad (k = 0, 1, 2, \dots). \quad (17)$$

Once  $\boldsymbol{\Phi}_k$  is determined, we immediately obtain  $f_{k+1}$ . The truncated conservation equations

$$\frac{\partial \mathbf{h}}{\partial t} = \sum_{k=0}^m \epsilon^k \boldsymbol{\Phi}_k \quad (18)$$

become the compressible Euler equations for  $m = 0$ , the compressible Navier–Stokes equations for  $m = 1$ , the Burnett equations for  $m = 2$ , and the super-Burnett equations for  $m = 3$ .

Incidentally, Eqs. (18) are rewritten as the evolutionary equations for the primitive variables  $\bar{\mathbf{h}} = {}^t(\rho, U_i, T)$ , i.e.

$$\frac{\partial \bar{\mathbf{h}}}{\partial t} = \sum_{k=0}^m \epsilon^k \bar{\boldsymbol{\Phi}}_k. \quad (19)$$

We can replace  $[\mathbf{h}, \boldsymbol{\Phi}_k, \nabla \boldsymbol{\Phi}_k]$  by  $[\bar{\mathbf{h}}, \bar{\boldsymbol{\Phi}}_k, \nabla \bar{\boldsymbol{\Phi}}_k]$  in Eqs. (13)–(15); the replacement makes the explicit computation of  $f_k$  easier.

Finally, we show the explicit form of  $f_1$  and the compressible NS equations  $\partial_t \mathbf{h} = \boldsymbol{\Phi}_0 + \epsilon \boldsymbol{\Phi}_1$  for reference. The  $f_1$  is given by

$$f_1 = - \left[ 2 \left( \xi_i \xi_j - \frac{\xi^2}{3} \delta_{ij} \right) \frac{\partial U_i}{\partial x_j} + \frac{\xi_i}{T^{1/2}} \left( \xi^2 - \frac{5}{2} \right) \frac{\partial T}{\partial x_i} \right] \rho^{-1} S(T)^{-1} f_0, \tag{20}$$

where  $\xi_i = (u_i - U_i)/T^{1/2}$  and  $\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$ . The corresponding NS equations are

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho U_i \\ \rho [\frac{3}{2} T + U_k^2] \end{pmatrix} + \frac{\partial}{\partial x_j} \begin{pmatrix} \rho U_j \\ \rho U_i U_j + \frac{1}{2} P_{ij} \\ \rho [\frac{3}{2} T + U_k^2] U_j + P_{kj} U_k + Q_j \end{pmatrix} = 0, \tag{21}$$

where

$$P_{ij} = \rho T \delta_{ij} - \epsilon [T/S(T)] \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} - \frac{2}{3} \frac{\partial U_k}{\partial x_k} \delta_{ij} \right), \tag{22}$$

$$Q_i = - \frac{5}{4} \epsilon [T/S(T)] \frac{\partial T}{\partial x_i}. \tag{23}$$

Then, we notice that the viscosity and thermal conductivity are proportional to  $T/S(T)$ . So, the temperature dependence of these coefficients can be controlled by  $S(T)$ . However, the Prandtl number is equal to unity irrespective of  $S(T)$ , while it is around 2/3 for real monatomic gases. This is a well-known drawback of the BGK model.

### 2.3. Successive approximation of the BGK solution

The BGK equation (5) is rewritten in the form:

$$f = g - \tau Df. \tag{24}$$

Then, we notice

$$f = g - \tau D(g - \tau Df) = g - \tau Dg + \tau D[\tau D(g - \tau Df)].$$

So, we have the solution of the BGK equation in the form of the successive approximation

$$f = g - \tau Dg + \tau D(\tau Dg) - \tau D[\tau D(\tau Dg)] + \dots \tag{25}$$

The successive approximation (25) implies

$$f = g + \tau g_1(\mathbf{h}, \mathcal{D}\mathbf{h}, \mathbf{u}) + \tau^2 g_2(\mathbf{h}, \mathcal{D}\mathbf{h}, \mathbf{u}) + \tau^3 g_3(\mathbf{h}, \mathcal{D}\mathbf{h}, \mathbf{u}) + \dots = f(\mathbf{h}, \mathcal{D}\mathbf{h}, \mathbf{u}, \tau), \tag{26}$$

$$\frac{\partial \mathbf{h}}{\partial t} = \tilde{\Phi}_0(\mathbf{h}, \mathcal{D}\mathbf{h}) + \tau \tilde{\Phi}_1(\mathbf{h}, \mathcal{D}\mathbf{h}) + \tau^2 \tilde{\Phi}_2(\mathbf{h}, \mathcal{D}\mathbf{h}) + \tau^3 \tilde{\Phi}_3(\mathbf{h}, \mathcal{D}\mathbf{h}) + \dots = \tilde{\Phi}(\mathbf{h}, \mathcal{D}\mathbf{h}, \tau). \tag{27}$$

Thus, the successive approximation (25) is consistent with the postulation of the Chapman–Enskog expansion. Therefore, we should, in principle, obtain the same result as that of the Chapman–Enskog expansion, i.e.  $\tau^n \tilde{\Phi}_n = \epsilon^n \Phi_n$  and  $\tau^n g_n = \epsilon^n f_n$  for  $n = 0, 1, 2, 3, \dots$ . We will explicitly demonstrate this up to the super-Burnett level.

Since  $\mathbf{h}$  is computed from  $g$ ,  $g_i$  ( $i = 1, 2, 3, \dots$ ) should satisfy the orthogonality condition

$$\int \psi g_i d\mathbf{u} = 0 \quad (i = 1, 2, 3, \dots). \tag{28}$$

The term  $Dg$  is expressed as

$$Dg = \frac{\partial g}{\partial \mathbf{h}} \left( \tilde{\Phi}_0 + \tau \tilde{\Phi}_1 + \tau^2 \tilde{\Phi}_2 + \dots \right) + \mathbf{u} \frac{\partial g}{\partial \mathbf{x}}. \quad (29)$$

Then,  $\tau g_1$  in Eq. (26) is given by

$$\tau g_1 = -\tau \left( \frac{\partial g}{\partial \mathbf{h}} \tilde{\Phi}_0 + \mathbf{u} \frac{\partial g}{\partial \mathbf{x}} \right). \quad (30)$$

The orthogonality condition for  $g_1$  is reduced to  $\tilde{\Phi}_0 = -\int \psi [u_i (\partial g / \partial x_i)] d\mathbf{u}$  since  $\int \psi_i (\partial g / \partial h_j) d\mathbf{u} = \delta_{ij}$ . Then, we have

$$\tilde{\Phi}_0 = \Phi_0, \quad (31)$$

i.e. we have the compressible Euler equation system. At the same time, we have

$$\tau g_1 = -\epsilon \rho^{-1} S(T)^{-1} \left( \frac{\partial g}{\partial \mathbf{h}} \Phi_0 + \mathbf{u} \frac{\partial g}{\partial \mathbf{x}} \right) = \epsilon f_1, \quad (32)$$

i.e. we have the same NS distribution function.

The term  $\tau Dg$  is expressed as

$$\tau Dg = -\epsilon f_1 + \frac{\partial g}{\partial \mathbf{h}} \left( \tau^2 \tilde{\Phi}_1 + \tau^3 \tilde{\Phi}_2 + \dots \right). \quad (33)$$

So, the term  $\tau D(\tau Dg)$  is expressed as

$$\begin{aligned} \tau D(\tau Dg) &= -\tau \epsilon \frac{\partial f_1}{\partial \mathbf{h}} (\Phi_0 + \tau \tilde{\Phi}_1 + \dots) - \tau \epsilon \frac{\partial f_1}{\partial \nabla \mathbf{h}} [\nabla \Phi_0 + \nabla (\tau \tilde{\Phi}_1 + \dots)] \\ &\quad + \tau D \left[ \frac{\partial g}{\partial \mathbf{h}} (\tau^2 \tilde{\Phi}_1 + \dots) \right] - \tau \epsilon \mathbf{u} \frac{\partial f_1}{\partial \mathbf{x}}. \end{aligned} \quad (34)$$

By extracting the terms of the order  $\epsilon^2$  from Eq. (25), we have

$$\tau^2 g_2 = -\tau^2 \frac{\partial g}{\partial \mathbf{h}} \tilde{\Phi}_1 - \tau \epsilon \left( \frac{\partial f_1}{\partial \mathbf{h}} \Phi_0 + \frac{\partial f_1}{\partial \nabla \mathbf{h}} \nabla \Phi_0 \right) - \tau \epsilon \mathbf{u} \frac{\partial f_1}{\partial \mathbf{x}}. \quad (35)$$

From the orthogonality condition for  $g_2$ , we have

$$\tau \tilde{\Phi}_1 = \epsilon \Phi_1, \quad (36)$$

i.e. we have the same compressible NS system. At the same time, we have

$$\tau^2 g_2 = \epsilon^2 f_2, \quad (37)$$

i.e. we have the same Burnett distribution function.

The term  $\tau D(\tau Dg)$  is rewritten in the form

$$\tau D(\tau Dg) = \tau^2 g_2 + \tau^2 \frac{\partial g}{\partial \mathbf{h}} \tilde{\Phi}_1 + \tau D \left[ \frac{\partial g}{\partial \mathbf{h}} (\tau^2 \tilde{\Phi}_1 + \dots) \right] - \tau \epsilon \frac{\partial f_1}{\partial \mathbf{h}} (\epsilon \Phi_1 + \dots) - \tau \epsilon \frac{\partial f_1}{\partial \nabla \mathbf{h}} (\epsilon \nabla \Phi_1 + \dots). \quad (38)$$

So, we have

$$\tau D[\tau D(\tau Dg)] = \tau \epsilon^2 \left( \frac{\partial f_2}{\partial \mathbf{h}} \Phi_0 + \frac{\partial f_2}{\partial \nabla \mathbf{h}} \nabla \Phi_0 + \frac{\partial f_2}{\partial \nabla^2 \mathbf{h}} \nabla^2 \Phi_0 \right) + \tau \epsilon^2 \mathbf{u} \frac{\partial f_2}{\partial \mathbf{x}} + \tau D \left[ \tau^2 \frac{\partial g}{\partial \mathbf{h}} \tilde{\Phi}_1 \right] + \dots. \quad (39)$$

Thus,  $\tau^3 g_3$  is given by

$$\tau^3 g_3 = -\tau^3 \frac{\partial g}{\partial \mathbf{h}} \tilde{\Phi}_2 - \tau \epsilon^2 \left( \frac{\partial f_1}{\partial \mathbf{h}} \Phi_1 + \frac{\partial f_1}{\partial \nabla \mathbf{h}} \nabla \Phi_1 \right) - \tau \epsilon^2 \left( \frac{\partial f_2}{\partial \mathbf{h}} \Phi_0 + \frac{\partial f_2}{\partial \nabla \mathbf{h}} \nabla \Phi_0 + \frac{\partial f_2}{\partial \nabla^2 \mathbf{h}} \nabla^2 \Phi_0 \right) - \tau \epsilon^2 \mathbf{u} \frac{\partial f_2}{\partial \mathbf{x}}. \tag{40}$$

Both of Eqs. (38) and (39) have the term  $\tau D[\tau^2(\partial g/\partial \mathbf{h})\tilde{\Phi}_1]$ . This term does not appear in Eq. (40) because of the cancellation. From the orthogonality condition for  $g_3$ , we have

$$\tau^2 \tilde{\Phi}_2 = \epsilon^2 \Phi_2, \tag{41}$$

i.e. we have the same Burnett equation system. At the same time, we have

$$\tau^3 g_3 = \epsilon^3 f_3, \tag{42}$$

i.e. we have the same super-Burnett distribution function.

### 3. Algebraic construction of distribution functions

In the gas-kinetic BGK scheme for the compressible NS equations [17], the distribution function  $f_1$  is computed without using the formulas of  $\tilde{\Phi}_0$  and  $\tilde{\Phi}_1$ , which are derived from the orthogonality condition. Instead, the orthogonality condition is employed in the actual numerical computation and is reduced to the computation of the linear algebra. In this section, we explain the extension of this approach to the case of Burnett and super-Burnett distribution functions.

#### 3.1. NS distribution function

The distribution function  $g_1$  is given by

$$g_1 = - \left( \frac{\partial g}{\partial t} \right)_0 - \mathbf{u} \frac{\partial g}{\partial \mathbf{x}}, \tag{43}$$

where  $(\partial/\partial t)_0$  is the time derivative evaluated by using

$$\frac{\partial \mathbf{h}}{\partial t} = \tilde{\Phi}_0. \tag{44}$$

We express  $(\partial g/\partial t)_0$  and  $\partial g/\partial x_k$  as  $A g$  and  $a_k g$ , respectively, where  $A$  and  $a_k$  are the linear combinations of  $\psi_i$  ( $i = 0, 1, 2, 3, 4$ ).

The  $a_k$  is computed from the finite difference approximation of  $\nabla \mathbf{h}$  in the following way. The  $a_k$  is expressed as

$$a_k = \sum_{i=0}^4 a_k^i \psi_i. \tag{45}$$

Then, the vectors  $\mathbf{a} = {}^t(a_k^0, a_k^1, a_k^2, a_k^3, a_k^4)$  and  $\mathbf{d} = \partial \mathbf{h}/\partial x_k$  satisfy  $\mathbf{M} \mathbf{a} = \mathbf{d}$ , where  $\mathbf{M}$  is the matrix the element of which is defined by  $M_{ij} = \int \psi_i \psi_j g \, d\mathbf{u}$ . We can easily obtain the inversion of  $\mathbf{M}$  (see Appendix of [17]), and therefore, we have  $\mathbf{a}$  from the finite difference approximation of  $\mathbf{d}$ . Similarly,  $A$  is obtained from

$$\langle A + u_k a_k \rangle = 0, \tag{46}$$

where  $\langle \cdot \rangle = \int \boldsymbol{\psi} \cdot \mathbf{g} \, d\mathbf{u}$ . Since Eq. (46) is nothing more than the orthogonality condition for  $g_1$ ,  $A$  so obtained satisfies  $Ag = (\partial g / \partial \mathbf{h}) \tilde{\boldsymbol{\Phi}}_0$ . The explicit computation of  $A$  is reduced to the inversion of  $\mathbf{M}$  as in the case of  $a_k$ . Thus, we have

$$g_1 = -(A + u_k a_k)g. \quad (47)$$

In the actual computation, the data of  $\int u_k \psi_i \psi_j g \, d\mathbf{u}$  are computed for each cell interface at the beginning of each time step. These data are employed not only in the construction of the distribution function but also in the computation of the numerical flux. Then, the computation of the numerical flux is reduced to the manipulation of the coefficients of the polynomials.

### 3.2. Burnett distribution function

The sum of two terms in the bracket on the right-hand side of Eq. (35) is equal to  $\partial f_1 / \partial t$  with Eq. (44). By using Eq. (32), Eq. (35) is rewritten as

$$\tau^2 g_2 = \tau^2 \bar{g}_2 - \tau \left[ \left( \frac{\partial \tau}{\partial t} \right)_0 + \mathbf{u} \frac{\partial \tau}{\partial \mathbf{x}} \right] g_1, \quad (48)$$

$$\bar{g}_2 = -\frac{\partial g}{\partial \mathbf{h}} \tilde{\boldsymbol{\Phi}}_1 - \left( \frac{\partial g_1}{\partial t} \right)_0 - \mathbf{u} \frac{\partial g_1}{\partial \mathbf{x}}, \quad (49)$$

where

$$\left( \frac{\partial g_1}{\partial t} \right)_0 = -[B' + A^2 + u_k(Aa_k + C_k)]g, \quad (50)$$

$(\partial g / \partial t)_0 = Ag$ ,  $B' = (\partial A / \partial t)_0$ , and  $C_k = (\partial a_k / \partial t)_0$ . Since  $\partial^2 g / \partial t \partial x_k = \partial^2 g / \partial x_k \partial t$ ,  $C_k$  is given by  $C_k = \partial A / \partial x_k$ . We can obtain  $C_i$  as the finite difference approximation of  $\partial A / \partial x_i$  or by making use of the orthogonality of  $\partial g_1 / \partial x_i$ , which is reduced to

$$\langle C_i + Aa_i + u_j(a_i a_j + b_{ij}) \rangle = 0, \quad (51)$$

where  $b_{ij} = \partial a_i / \partial x_j = \partial a_j / \partial x_i$  (the symmetry of  $b_{ij}$  follows from  $\partial^2 g / \partial x_i \partial x_j = \partial^2 g / \partial x_j \partial x_i$ ) and  $b_{ij}$  is computed from  $\nabla^2 \mathbf{h}$  in the same way as the computation of  $a_k$ . The term  $(\partial g / \partial \mathbf{h}) \tilde{\boldsymbol{\Phi}}_1$  in Eq. (49) is written as  $-B''g$ , where  $B''$  is a linear combination of  $\psi_i$  ( $i = 0, 1, 2, 3, 4$ ). Then,  $\bar{g}_2$  is expressed as

$$\bar{g}_2 = [B + A^2 + 2u_k(Aa_k + C_k) + u_i u_j(a_i a_j + b_{ij})]g, \quad (52)$$

where  $B = B' + B''$ . The  $B$  is determined by the orthogonality condition for  $g_2$ , which is reduced to

$$\tau^2 \langle B + A^2 + 2u_k(Aa_k + C_k) + u_i u_j(a_i a_j + b_{ij}) \rangle + \tau \langle u_k \frac{\partial \tau}{\partial x_k} (A + u_k a_k) \rangle = 0. \quad (53)$$

Thus, we have the Burnett distribution function:

$$\tau^2 g_2 = \tau^2 [B + A^2 + 2u_k(Aa_k + C_k) + u_i u_j(a_i a_j + b_{ij})]g + \tau \left[ \left( \frac{\partial \tau}{\partial t} \right)_0 + u_j \frac{\partial \tau}{\partial x_j} \right] (A + u_k a_k)g. \quad (54)$$

We have obtained  $B$  without computing  $B'$ . The  $B'$  will be employed in the construction of kinetic scheme for the Burnett equation and  $B'' = B - B'$  will be employed in the computation of super-Burnett distribution function. The  $B'$  is obtained from the orthogonality of  $(\partial g_1 / \partial t)_0$ , which is reduced to



$$\langle B' + A^2 + u_i(Aa_i + C_i) \rangle = 0. \tag{55}$$

Incidentally, in the case where  $\partial_t \tau$  and  $\partial_{x_k} \tau$  are much smaller than  $\tau$  itself, i.e.

$$\frac{\partial \tau}{\partial \alpha} \ll \tau \quad (\alpha = t, x_k), \tag{56}$$

which is equivalent to  $\partial_x[\rho S(T)] \ll \rho S(T)$ , we can handle  $\tau$  as a local constant. Then, we can put  $g_2 = \bar{g}_2$  and compute  $B$  from

$$\langle B + A^2 + 2u_k(Aa_k + C_k) + u_i u_j (a_i a_j + b_{ij}) \rangle = 0. \tag{57}$$

### 3.3. Super-Burnett distribution function

For the super-Burnett distribution function, we consider the simple case where Eq. (56) holds. In this case, we can handle  $\tau$  as a local constant.

From Eqs. (32), (36), and (37), the super-Burnett distribution function, Eq. (40), is rewritten as

$$\begin{aligned} \tau^3 g_3 = & -\tau^3 \frac{\partial g}{\partial \mathbf{h}} \tilde{\Phi}_2 - \tau \left[ \frac{\partial \tau g_1}{\partial \mathbf{h}} \tau \tilde{\Phi}_1 + \frac{\partial \tau g_1}{\partial \nabla \mathbf{h}} \nabla (\tau \tilde{\Phi}_1) \right] \\ & - \tau \left[ \frac{\partial \tau^2 g_2}{\partial \mathbf{h}} \tilde{\Phi}_0 + \frac{\partial \tau^2 g_2}{\partial \nabla \mathbf{h}} \nabla \tilde{\Phi}_0 + \frac{\partial \tau^2 g_2}{\partial \nabla^2 \mathbf{h}} \nabla^2 \tilde{\Phi}_0 \right] - \tau \mathbf{u} \frac{\partial \tau^2 g_2}{\partial \mathbf{x}}. \end{aligned} \tag{58}$$

The sum of the two terms in the first bracket on the right-hand side is equal to  $\partial \tau g_1 / \partial t$  with

$$\frac{\partial \mathbf{h}}{\partial t} = \tau \tilde{\Phi}_1, \tag{59}$$

and the sum of three terms in the second bracket is equal to  $\partial \tau^2 g_2 / \partial t$  with Eq. (44). Then,  $g_3$  is expressed as

$$\tau^3 g_3 = -\tau^3 \frac{\partial g}{\partial \mathbf{h}} \tilde{\Phi}_2 - \tau \left[ \tau \left( \frac{\partial g_1}{\partial t} \right)_1 + \left( \frac{\partial \tau}{\partial t} \right)_1 g_1 \right] - \tau \left[ \tau^2 \left( \frac{\partial g_2}{\partial t} \right)_0 + 2\tau \left( \frac{\partial \tau}{\partial t} \right)_0 g_2 \right] - \tau \mathbf{u} \left[ \tau^2 \frac{\partial g_2}{\partial \mathbf{x}} + \frac{\partial \tau^2}{\partial \mathbf{x}} g_2 \right], \tag{60}$$

where  $(\partial / \partial t)_1$  is the time derivatives evaluated by using Eq. (59).

Up to now, the computation is exact. From now on, we will consider the case where Eq. (56) holds, i.e.  $\tau$  is treated as a local constant. Under this simplification, we have

$$\tau^3 g_3 \sim -\tau^3 \frac{\partial g}{\partial \mathbf{h}} \tilde{\Phi}_2 - \tau^2 \left( \frac{\partial g_1}{\partial t} \right)_1 - \tau^3 \left( \frac{\partial g_2}{\partial t} \right)_0 - \tau^3 \mathbf{u} \frac{\partial g_2}{\partial \mathbf{x}}, \tag{61}$$

where  $g_2 = \bar{g}_2$  with  $B$  determined by Eq. (57). We evaluate each term in Eq. (61) below.

First, we consider the time derivative  $(\partial g_2 / \partial t)_0$ . It is expressed as

$$\begin{aligned} \left( \frac{\partial g_2}{\partial t} \right)_0 = & \left[ \left( \frac{\partial B}{\partial t} \right)_0 + 2AB' + 2u_k \left[ B'a_k + AC_k + \left( \frac{\partial C_k}{\partial t} \right)_0 \right] + u_i u_j \left( C_i a_j + a_i C_j + \frac{\partial C_i}{\partial x_j} \right) \right. \\ & \left. + A(B + A^2) + 2u_k(Aa_k + C_k) + u_i u_j (a_i a_j + b_{ij}) \right] g. \end{aligned} \tag{62}$$

The  $(\partial C_i / \partial x_j)$  is computed as a finite difference approximation of  $C_i$  or from the orthogonality condition for  $(\partial^2 g_1 / \partial x_i \partial x_j)$ , which is reduced to

$$\left\langle \frac{\partial C_i}{\partial x_j} + a_i C_j + a_j C_i + A(b_{ij} + a_i a_j) + u_k d_{ijk} \right\rangle = 0, \quad (63)$$

where

$$d_{ijk} = a_i a_j a_k + a_i b_{jk} + a_j b_{ki} + a_k b_{ij} + \frac{\partial^2 a_i}{\partial x_j \partial x_k}. \quad (64)$$

The  $\partial^2 a_i / \partial x_j \partial x_k$  is computed from the finite difference approximation of  $\nabla^3 \mathbf{h}$ . The  $(\partial C_i / \partial t)_0$  is computed from the orthogonality condition for  $(\partial(\partial g_1 / \partial x_i) / \partial t)_0$ , which is reduced to

$$\left\langle \left( \frac{\partial C_i}{\partial t} \right)_0 + (A^2 + B') a_i + 2AC_i + u_j \left[ \frac{\partial C_i}{\partial x_j} + a_i C_j + a_j C_i + A(a_i a_j + b_{ij}) \right] \right\rangle = 0. \quad (65)$$

Thus, only  $(\partial B / \partial t)_0$  is unknown in  $(\partial g_2 / \partial t)_0$ . We express this unknown quantity as  $E$ , i.e.  $E = (\partial B / \partial t)_0$ , which is a linear combination of  $\psi_k$  ( $k = 0, 1, 2, 3, 4$ ).

Next, we consider the  $(\partial g_1 / \partial t)_1$ . It is expressed as

$$\left( \frac{\partial g_1}{\partial t} \right)_1 = - \left[ \left( \frac{\partial A}{\partial t} \right)_1 + u_k \left( \frac{\partial a_k}{\partial t} \right)_1 - \tau B'' (A + u_k a_k) \right] g. \quad (66)$$

From  $\partial^2 / \partial x_k \partial t = \partial^2 / \partial t \partial x_k$ , we have  $(\partial a_k / \partial t)_1 = -\tau (\partial B'' / \partial x_k)$ . Similar to the derivation of  $(\partial a_k / \partial t)_1$ , we have  $\partial B' / \partial x_k = (\partial C_k / \partial t)_0$ . The  $\partial B / \partial x_k$  is computed from the orthogonality condition for  $\partial g_2 / \partial x_k$ , which is reduced to

$$\left\langle \frac{\partial g_2}{\partial x_k} \right\rangle = \left\langle \frac{\partial B}{\partial x_k} + 2AC_k + a_k (A^2 + B) + 2u_i \left[ a_i C_k + a_k C_i + A(b_{ik} + a_i a_k) + \frac{\partial C_i}{\partial x_k} \right] + u_i u_j d_{ijk} \right\rangle = 0. \quad (67)$$

Since  $B = B' + B''$ , we have  $\partial B'' / \partial x_k = \partial B / \partial x_k - \partial B' / \partial x_k$ . Then, only  $(\partial A / \partial t)_1$  is unknown in  $(\partial g_1 / \partial t)_1$ . We express this unknown quantity as  $-\tau F$ , i.e.  $\tau F = -(\partial A / \partial t)_1$ , where  $F$  is a linear combination of  $\psi_k$  ( $k = 0, 1, 2, 3, 4$ ). Incidentally, the space derivatives of  $B$  and those of  $B'$  can also be computed as their finite difference approximation.

The term  $(\partial g / \partial \mathbf{h}) \Phi_2$  is expressed as  $Gg$ , where  $G$  is a linear combination of  $\psi_k$  ( $k = 0, 1, 2, 3, 4$ ). The explicit form of the term  $\mathbf{u}(\partial g_2 / \partial \mathbf{x})$  is readily obtained from Eq. (67).

Summarizing the above results, the unknown quantity left in  $g_3$  is the sum  $J = E + F + G$ , which is a linear combination of  $\psi_k$  ( $k = 0, 1, 2, 3, 4$ ) and is determined by the orthogonality condition for  $g_3$ . Thus, the recipe of the construction of the super-Burnett distribution function is completed for the case where the variation of  $\tau$  with respect to time and space variables is much smaller than  $\tau$  itself. Although  $\tau$  is treated as a local constant in the construction of the distribution functions, it is varied from cell to cell and from time step to time step in the actual computation.

#### 4. Kinetic scheme for the Burnett equations

In this section, we derive the formula of the numerical flux, which will be employed in the kinetic scheme for the Burnett equations. For simplicity, we consider the spatially one dimensional case, where the physical quantities of the gas are independent of  $x_2$  and  $x_3$ ; the extension to the multidimensional case can be done straightforwardly. For the simple expression, we will omit the subscripts in  $x_1$  and  $u_1$ , i.e.  $x = x_1$  and  $u = u_1$  [the bold symbol  $\mathbf{u}$  still means the vector  $(u_1, u_2, u_3)$ ].

The kinetic scheme for the spatially one-dimensional case is written in the form:

$$\mathbf{h}_j(\Delta t) = \mathbf{h}_j(0) - \frac{1}{\Delta x} [\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}], \tag{68}$$

where  $\mathbf{h}_j(t)$  is the average of  $\mathbf{h}(x, t)$  over the cell ( $s_{j-1/2} < x < s_{j+1/2}$ ),  $\Delta x = s_{j+1/2} - s_{j-1/2}$ , and  $\mathbf{F}_{j+1/2}$  is the numerical flux at  $x_1 = s_{j+1/2}$ , which is defined by

$$\mathbf{F}_{j+1/2} = \int_0^{\Delta t} \int \psi u f(s_{j+1/2}, t, \mathbf{u}) d\mathbf{u} dt. \tag{69}$$

Once the formula of  $f(s_{j+1/2}, t, \mathbf{u})$  is specified, we can readily construct the scheme. We will derive the formula of  $f$  that yields a high order accurate numerical flux for the Burnett equations by different methods.

#### 4.1. Derivation by the railroad method

In the conventional approach, the collisionless Boltzmann equation is employed to derive the formula of the numerical flux. For example, in Pullin’s scheme for the compressible Euler equation, the distribution function employed in the computation of the numerical flux is the solution of the Cauchy problem for the collisionless Boltzmann equation from the initial data in the form of the local Maxwellian. The theory of kinetic equation for the numerical flux is studied in [11,12,14] and the role of the collision effect is clarified there. In this subsection, we apply the theory, which we call the railroad method, to the case of the Burnett equations and derive the formula of the numerical flux. The information of the time step truncation error of the numerical flux is also presented.

Consider the Burnett distribution function

$$f = g + \epsilon f_1 + \epsilon^2 f_2 \tag{70}$$

the macroscopic variables of which satisfy the Burnett equations

$$\frac{\partial \mathbf{h}}{\partial t} = \mathbf{\Phi}_0 + \epsilon \mathbf{\Phi}_1 + \epsilon^2 \mathbf{\Phi}_2. \tag{71}$$

The desired kinetic equation is derived by substituting Eq. (70) into  $(\partial_t + u\partial_x)f$ , converting the time derivative  $\partial_t f$  by Eq. (71), and equating the result to  $(\partial_t + u\partial_x)f$ . Making use of Eqs. (13) and (14), we can rewrite the resulting kinetic equation in the form:

$$\frac{\partial f}{\partial t} + \zeta_i \frac{\partial f}{\partial x_i} = -S(T)\rho[f_1 + \epsilon f_2] + Res, \tag{72}$$

where  $Res$  is  $O(\epsilon^2)$  and is orthogonal to  $\psi$ . The solution of the Burnett equations can be obtained as the moments of solution of the Cauchy problem for the kinetic equation (72) from the initial data in the form of Eq. (70). If the simplified equation

$$\frac{\partial f}{\partial t} + \zeta_i \frac{\partial f}{\partial x_i} = -S(T)\rho[f_1 + \epsilon f_2], \tag{73}$$

is employed instead of the exact kinetic equation, the error at  $t = \Delta t$  is at most  $O(\epsilon^2 \Delta t^2)$  because of the orthogonality of  $Res$ .

The approximate solution of the Cauchy problem for the simplified kinetic equation (73) from the initial data in the form of Eq. (70) is employed in the derivation of the numerical flux. For simplicity, we consider

the case where the cell interface is located at  $x = 0$ , i.e.  $s_{j+1/2} = 0$ . As mentioned before, the difference between the solution of the exact kinetic equation (72) and that of the simplified kinetic equation (73) is  $O(\epsilon^2 \Delta t^2)$ . So, it is sufficient to compute the numerical flux for the simplified equation (73) within the error of  $O(\epsilon^2 \Delta t^2)$ . This gives the guideline of the approximation of the solution of the Cauchy problem, i.e. the error of the approximation for  $f(0, t, \mathbf{u})$  should be at most  $O(\epsilon^2 t)$ . The solution of Eq. (73) is expressed as the sum of two terms; one is the contribution from the initial data and the other is the integral of the right-hand side of Eq. (73) along its characteristics. For the case where  $\epsilon \sim \Delta t$ , the accuracy of the integration should be higher order. Otherwise, the error of the integration exceeds the intrinsic error, i.e.  $\epsilon^2 t \ll t^2$ . So, we employ the formula of higher order time integration of the kinetic equation [13]:

$$f = f_0(-ut, \mathbf{u}) - tQ(-ut, 0, \mathbf{u}) - \frac{t^2}{2} D_1 Q(0, 0, \mathbf{u}) + O(t^3), \quad (74)$$

where

$$Q(x, t, \mathbf{u}) = [S(T)\rho(f_1 + \epsilon f_2)](x, t, \mathbf{u}), \quad (75)$$

and  $D_1 = \partial_t + u\partial_x$ . Under the guideline for the accuracy, we can approximate each term as follows:

$$f_0(-ut, \mathbf{u}) = g - utg_x + \frac{(ut)^2}{2} g_{xx} + \epsilon f_1 - \epsilon ut\partial_x f_1 + \epsilon^2 f_2 + O(\epsilon t^2, \epsilon^2 t, t^3), \quad (76)$$

$$Q(-ut, 0, \mathbf{u}) = S(T)\rho f_1 - ut\partial_x[S(T)\rho f_1] + \epsilon S(T)\rho f_2 + O(t^2, \epsilon t), \quad (77)$$

$$D_1 Q = (\partial_t + u\partial_x)[S(T)\rho f_1] + O(\epsilon), \quad (78)$$

where each term on the right-hand sides of Eqs. (76)–(78) is evaluated at  $(x, t) = (0, 0)$ . Then, we have

$$f(0, t, \mathbf{u}) = g + \epsilon f_1 + \epsilon^2 f_2 - t[ug_x + S(T)\rho f_1] - \epsilon t[u(f_1)_x + S(T)\rho f_2] + \frac{t^2}{2} [g_{xx} - (S(T)\rho f_1)_t + u(S(T)\rho f_1)_x] + O(\epsilon t^2, \epsilon^2 t, t^3). \quad (79)$$

The time derivative in Eq. (79) is determined from the moment equation of Eq. (73) and the initial data; the time derivatives of the macroscopic variables are given by the Burnett equations. However, the guideline assures that we can employ the time derivatives evaluated by the compressible Euler equations,

$$\frac{\partial \mathbf{h}}{\partial t} = \Phi_0. \quad (80)$$

Next, we express Eq. (79) using the notation employed in the algebraic construction of distribution function (Section 3). In the present one-dimensional case, we omit the subscripts in  $a_i$ ,  $b_{ij}$ , and  $C_i$ . Recalling

$$\epsilon = S(T)\rho\tau, \quad \epsilon^n f_n = \tau^n g_n, \quad Q = g_1 + \tau g_2.$$

$$(g_1)_x = -[C + aA + (a^2 + b)u]g, \quad (g_1)_t = -[B' + A^2 + (C + Aa)u]g,$$

$B = B' + B''$ , and the formula of  $g_2$ , we find that Eq. (79) is rewritten as

$$f(0, t, \mathbf{u}) = g[1 - \tau au + (-\tau + t)A + (\tau^2 - \tau t)(A^2 + B) + (t^2/2)(A^2 + B') + (2\tau^2 - \tau t)(aA + C)u + \tau^2(a^2 + b)u^2 + (au + A)[(\tau - t)\tau_t + \tau u\tau_x]] + O(\epsilon t^2, \epsilon^2 t, t^3), \quad (81)$$

where  $\tau_t$  is evaluated by using the compressible Euler equations (80). The formula of  $f(0, t, \mathbf{u})$  for the Burnett equations in [19], which is derived without taking account of the local variation of  $\tau$ , is now modified.

#### 4.2. Derivation by Cauchy–Kowalevskaya technique

In this subsection, we derive the result of the previous subsection by a different method from the same starting point, i.e. the Burnett distribution function the macroscopic variables of which satisfy the Burnett equations. We consider the time evolution of such  $f$  at the cell interface  $x = 0$ . Making use of the Taylor expansion, we have the formula:

$$f(0, t, \mathbf{u}) = g + tg_t + \frac{t^2}{2} g_{tt} + \epsilon[f_1 + t(f_1)_t] + \epsilon^2 f_2 + \mathcal{O}(t^3, \epsilon t^2, \epsilon^2 t). \tag{82}$$

Then, we employ

$$g_t = \frac{\partial g}{\partial \mathbf{h}}(\Phi_0 + \epsilon \Phi_1) + \mathcal{O}(\epsilon^2), \tag{83}$$

$$(f_1)_t = \frac{\partial f_1}{\partial \mathbf{h}} \Phi_0 + \frac{\partial f_1}{\partial \nabla \mathbf{h}} \nabla \Phi_0 + \mathcal{O}(\epsilon), \tag{84}$$

and  $g_{tt}$  is evaluated by using the compressible Euler equation (80). The above computation is nothing more than the Cauchy–Kowalevskaya procedure at the level of the distribution function. From the derivation of Chapman–Enskog expansion, we recall

$$\frac{\partial g}{\partial \mathbf{h}} \Phi_0 = -ug_x - S(T)\rho f_1,$$

$$\frac{\partial g}{\partial \mathbf{h}} \Phi_1 + \frac{\partial f_1}{\partial \mathbf{h}} \Phi_0 + \frac{\partial f_1}{\partial \nabla \mathbf{h}} \nabla \Phi_0 = -u(f_1)_x - S(T)\rho f_2.$$

The  $g_{tt}$  for Eq. (80) is given by

$$g_{tt} = -(S(T)\rho f_1)_t + u(S(T)\rho f_1)_x + u^2 g_{xx}, \tag{85}$$

where the time derivatives are computed by Eq. (80). After a simple computation, we will notice that Eq. (82) with the above formulas is identical to Eq. (79).

#### 4.3. Application of Cauchy–Kowalevskaya technique to the macroscopic level

Substituting Eq. (79) into Eq. (69), we have the formula of the numerical flux for the Burnett equations. For the smooth reconstruction of the initial data, we can directly apply the Cauchy–Kowalevskaya technique to the numerical flux. The Burnett equations are written in the form

$$\frac{\partial \mathbf{h}}{\partial t} = \frac{\partial \Psi}{\partial x}, \tag{86}$$

where

$$\Psi = \Psi_0 + \epsilon \Psi_1 + \epsilon^2 \Psi_2, \quad \partial_x \Psi_i = \Phi_i. \tag{87}$$

The time integration of  $\Psi$  gives the formula of the numerical flux. Applying the Cauchy–Kowalevskaya technique to  $\Psi$ , we have

$$\Psi(0, t) = \Psi_0 + t\partial_t\Psi_0 + (t^2/2)\partial_{tt}\Psi_0 + \epsilon[\Psi_1 + t\partial_t\Psi_1] + \epsilon^2\Psi_2 + O(t^3, \epsilon t^2, \epsilon^2 t), \quad (88)$$

where each term on the right-hand side is evaluated at  $(x, t) = (0, 0)$ ,  $\partial_t\Psi_0$  is evaluated by the compressible NS equations, and  $\partial_t\Psi_1$  and  $\partial_{tt}\Psi_0$  are done by the compressible Euler equations. The resulting numerical flux is the same as that obtained in the previous subsections. This is similar to the case of the compressible Euler equations; the well-known Lax–Wendroff scheme, which is nothing more than the product of the Cauchy–Kowalevskaya technique at the macroscopic level, is recovered in the kinetic derivation when the continuous piecewise linear reconstruction with smoothness at cell interfaces is employed (see, e.g. [12]). In the case of the discontinuous reconstruction, which is employed in various shock capturing schemes, the Cauchy–Kowalevskaya technique cannot be employed directly. In this case, we can compute the numerical flux as a certain special average of the fluxes computed at both sides of the cell interface. In the kinetic scheme, the averaging can be done naturally at the level of the distribution function. For the management of discontinuous reconstruction in kinetic schemes, we refer the reader to [14].

## 5. Numerical test

In this section, we carry out the numerical test of the kinetic scheme for the Burnett equations.

The first problem is the structure of the stationary normal shock wave for the upstream Mach number equal to 2 ( $M = 2$ ). Fig. 1 shows the distributions of the nondimensional density  $\rho/\rho_1$  and nondimensional temperature  $T/T_1$  ( $\rho_1$ ,  $T_1$ , and  $\lambda_1$  are, respectively, the density, temperature, and mean free path at upstream condition). Fig. 2 shows the distributions of stress and heat flux ( $\tau_{xx} = P_{11} - p$ ,  $q_x = Q_1$ , where the flow is in the  $x_1$  direction). The results for the Burnett equations are shown together with those for the Navier–Stokes equations. The present kinetic schemes are modified by introducing the variable collision frequency and the

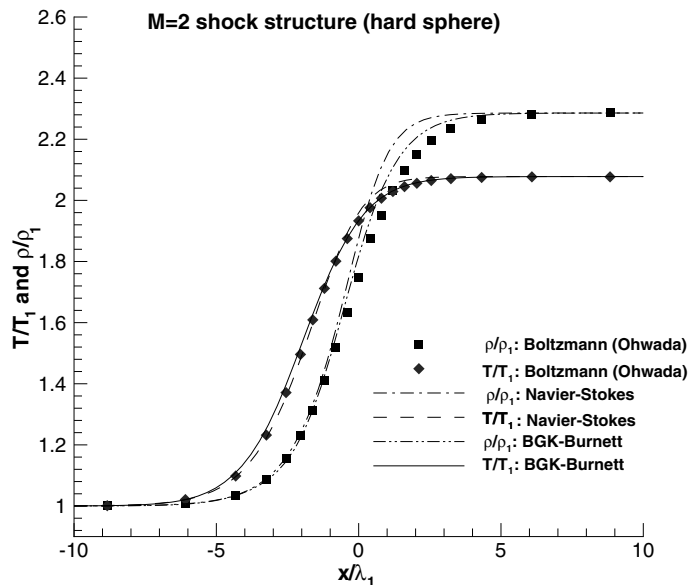


Fig. 1. Shock structures for  $M = 2$ . Density and temperature distributions.

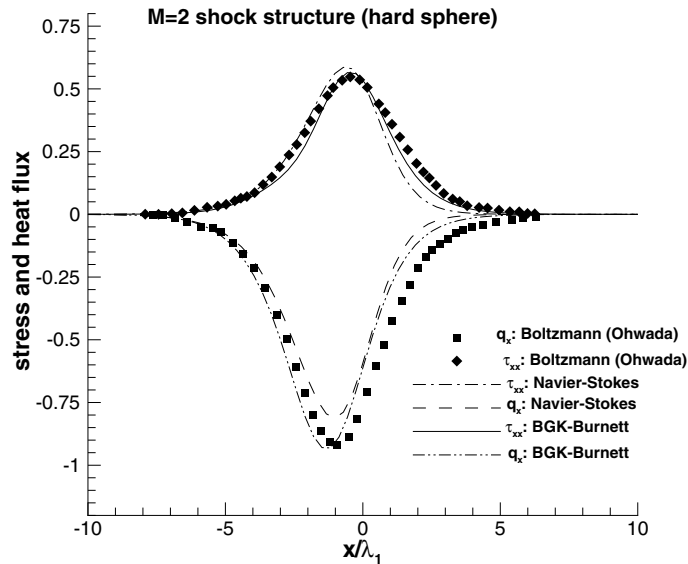


Fig. 2. Shock structure for  $M = 2$ . Stress and heat flow distributions.

Prandtl number fix technique of [17] so that the stress and the heat flux at the NS level are converted to those for hard-sphere molecules; the NS results correspond to hard-sphere molecules exactly. The results of the Boltzmann equations for hard-sphere molecules in [10] are also presented as the standard. These two figures show the improvement of the Burnett solution over the NS. Without the conversion of the stress and heat flux, the agreement with the standard solution becomes poor. On the other hand, as seen in Fig. 2 of [10], the deviation of the Boltzmann solution from the local Maxwellian is appreciable even for  $M = 2$ ; the Boltzmann solution exhibits the bimodal distribution of gas molecules for  $3 \lesssim M$ . Small deviation from the local Maxwellian is one of the assumptions of the Chapman–Enskog expansion and the present results demonstrate the robustness of the Burnett scheme around the border of the application range of the Chapman–Enskog expansion. The present Burnett scheme is not stable for  $M \sim 4$ , where the assumption is no longer valid.

The second problem is the force-driven Poiseuille flow between two parallel plates. Since the appropriate boundary condition for the kinetic Burnett solver has not yet been established, we adopt the diffuse reflection as a plausible one. In this problem, the nonuniform pressure profile and the bimodal temperature distribution with the central minimum are exhibited as higher order rarefaction effects, which are not predicted by the Navier–Stokes system (see, e.g. [2,9,16,19,20]). According to [2], where the systematic asymptotic analysis of this problem for small Knudsen numbers is carried out on the basis of the BGK equation, the former characteristic is attributed to the Burnett approximation and the latter is done to the non-Navier–Stokes stress and heat flow beyond the Burnett approximation. These non-Navier–Stokes terms beyond the Burnett approximation are treated together with the second order slip boundary conditions. Therefore, the latter characteristic is not expected in the present framework of the Burnett equations and the diffuse reflection. The purpose of the present computation is the confirmation whether the BGK-Burnett scheme really works and captures the non-Navier–Stokes behavior of slightly rarefied gas flow to some extent. The set up of the problem is given in [20]; both of the plates are at rest and have the same uniform temperature, which is taken as the reference temperature  $\hat{T}_0$ ; the average density is taken as the reference density  $\hat{\rho}_0$ ; the external force acting on the unit mass is  $0.128 \times (2RT_0/L)$ , where  $L$  is the

distance between the plates; the Knudsen number  $Kn_{HS} = l_{HS}/L$ , where  $l_{HS}$  is the mean free path of hard-sphere molecular gas in the equilibrium state at rest with  $\hat{\rho} = \hat{\rho}_0$ , is equal to 0.1. The Burnett equations employed in the numerical test correspond to  $S(T) = 1$  and  $Kn_{HS} = 0.1$  corresponds to  $\epsilon = 0.098$  (the mean free path is related to the viscosity and the conversion of  $l_{HS}$  to that for the BGK equation is done through the viscosity at  $\hat{T} = \hat{T}_0$ ; no conversion of the heat flux is introduced). The external force is treated as the simple additional term. The results of the present computation are shown in Fig. 3 together with the DSMC results for hard-sphere molecules [20] [ $p(= \rho T)$  is the nondimensional pressure]. Despite the difference of the molecular model, i.e.  $S(T) = 1$  and  $Pr = 1$  for the present BGK-Burnett and  $S(T) \sim T^{1/2}$  and  $Pr \approx 2/3$  for hard-sphere molecular gas, where  $Pr$  is the Prandtl number, the agreement with the DSMC result is fair. The curved pressure distribution is well captured around the center but the temperature minimum is not observed in the present computation as expected. The discrepancy is remarkable near the plates, which is considered to be due to the lack of the Knudsen-layer corrections and the employment of the diffuse reflection boundary condition in the present computation. In [20], the NS solution under the first order slip boundary condition (the velocity slip due to shear stress and the temperature jump due to the heat flow normal to the boundary) without the corresponding Knudsen-layer corrections is compared with the DSMC result. For the density  $\rho$  and flow speed  $U_1$ , the agreement with the DSMC is as good as that of the Burnett result. For the pressure  $p$ , however, the error is larger than that of the Burnett result; the pressure  $p$  is constant ( $p = 1.034$ ). As for the temperature  $T$ , the shape of the profile is similar to that of the Burnett solution; it takes the maximum value 1.038 at  $x = 0$ . Incidentally, the previous BGK-Burnett scheme [19], which does not take account of the local variation of  $\tau$ , yields almost the identical result to that of the full-Burnett scheme.

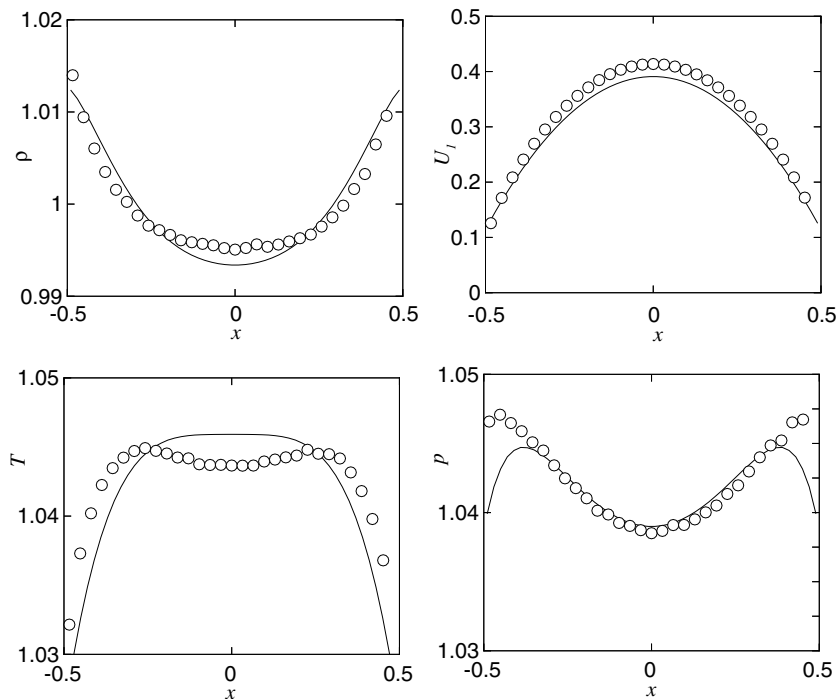


Fig. 3. Flow distribution in the cross-stream direction. Circles: DSMC results [20], solid lines: BGK-Burnett solution.



Finally, we mention the computational performance. We have prepared two codes; one is based on Eq. (81) and the other is based on Eq. (88) [Eq. (79)]. The explicit formula of the numerical flux for Eq. (88) [Eq. (79)] is easily obtained if the computer algebra, such as MATHEMATICA, is employed. These two codes are theoretically identical and the numerical agreement is confirmed within the range of the round-off error. The difference is observed only in the computational cost; the code for Eq. (88) [Eq. (79)] is 2.5 times faster than that for Eq. (81) [these codes are not optimized]. This is inferred from the case of the compressible Euler solver that employs smooth reconstruction of initial data; the Lax-Wendroff scheme is faster than the kinetic scheme with the algebraic computation of the numerical flux. The algebraic method becomes useful when the discontinuous reconstruction of the initial data is considered. As mentioned before, this reconstruction is employed in shock capturing schemes. The extension of the present scheme to the case of discontinuous reconstruction is in preparation.

## 6. Conclusion

In this paper, we clearly show the consistency between the Chapman–Enskog expansion and the successive approximation for the BGK solution, where the gas distribution function up to the super-Burnett order is obtained. Then, a high order numerical scheme for the Burnett equations is constructed by using different methods and the resulting scheme is validated in the shock wave problem and the force-driven Poiseuille flow problem. This paper provides a useful numerical tool for the Burnett equations, which can be used in the study of abnormal phenomena for the Navier–Stokes equations, even in the continuum flow regime.

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## References

- [1] R.K. Agarwal, K.Y. Yun, R. Balakrishnan, Beyond the Navier–Stokes: Burnett equations for flows in the continuum-transition regime, *Phys. Fluids* 13 (2001) 3061.
- [2] K. Aoki, S. Takata, T. Nakanishi, Poiseuille-type flow of a rarefied gas between two parallel plates driven by a uniform external force, *Phys. Rev. E* 65 (2002) 026315.
- [3] V.V. Aristov, F.G. Tcheremissine, The kinetic numerical method for rarefied and continuum gas flows, in: O.M. Belotserkovskii, M.N. Kogan, S.S. Kutateladze, A.K. Rebrov (Eds.), *Rarefied Gas Dynamics*, Plenum Press, New York, 1985, pp. 269–276.
- [4] R. Balakrishnan, R.K. Agarwal, Higher-order distribution functions, BGK–Burnett equations and Boltzmann H-theorem, in: M. Hafez, K. Oshima (Eds.), *CFD Review*, vol. 2, 1997, pp. 794–832.
- [5] A.V. Bobylev, The Chapman–Enskog and Grad methods for solving the Boltzmann equation, *Sov. Phys. Dokl.* 27 (1982) 29–31.
- [6] C. Cercignani, *The Boltzmann Equation and Its Applications*, Springer, Berlin, 1988.
- [7] S.Y. Chou, D. Baganoff, Kinetic flux-vector splitting for the Navier–Stokes equations, *J. Comput. Phys.* 130 (1997) 217–230.
- [8] H. Grad, Principles of the kinetic theory of gases, in: S. Flügge (Ed.), *Handbuch der Physik*, vol. XII, Springer, Berlin, 1958, pp. 205–294.
- [9] M.M. Malek, F. Baras, A.L. Garcia, On the validity of hydrodynamics in plane Poiseuille flows, *Physica A* 240 (1997) 255–267.
- [10] T. Ohwada, Structure of normal shock waves: direct numerical analysis of the Boltzmann equation for hard-sphere molecules, *Phys. Fluids A* 5 (1993) 217–234.

- [11] T. Ohwada, Boltzmann schemes for the compressible Navier–Stokes equations, in: T.J. Bartel and M. Gallis (Eds.), *Rarefied Gas Dynamics, Proceedings of the International Symposium on Rarefied Gas Dynamics, 2000*, Sydney; AIP Conference Proceedings 585, American Institute of Physics, 2001, pp. 321–328.
- [12] T. Ohwada, On the construction of kinetic schemes, *J. Comput. Phys.* 177 (2002) 156–175.
- [13] T. Ohwada, Higher order approximation methods for the Boltzmann equation, *J. Comput. Phys.* 139 (1998) 1–14.
- [14] T. Ohwada, S. Kobayashi, Management of the discontinuous reconstruction in kinetic schemes, *J. Comput. Phys.* 197 (2004) 116–138.
- [15] D.I. Pullin, Direct simulation methods for compressible inviscid ideal gas flow, *J. Comput. Phys.* 34 (1980) 231–244.
- [16] M. Tij, A. Santos, Perturbation analysis of a stationary nonequilibrium flow generated by an external force, *J. Stat. Phys.* 76 (1994) 1399–1414.
- [17] K. Xu, A gas-kinetic BGK scheme for the Navier–Stokes equations and its connection with artificial dissipation and Godunov method, *J. Comput. Phys.* 171 (2001) 289–335.
- [18] K. Xu, Regularization of the Chapman–Enskog expansion and its description of shock structure, *Phys. Fluids* 14 (2002) 17–20.
- [19] K. Xu, Super-Burnett solutions for Poiseuille flow, *Phys. Fluids* 15 (2003) 2077–2080.
- [20] Y. Zheng, A.L. Garcia, B.J. Alder, Comparison of kinetic theory and hydrodynamics for Poiseuille flow, in: A.D. Ketsdever, E.P. Muntz (Eds.), *Rarefied Gas Dynamics, Proceedings of the International Symposium on Rarefied Gas Dynamics, 2002*, Whistler; AIP Conference Proceedings 663, American Institute of Physics, 2003, pp. 149–156.
- [21] X. Zhong, R.W. MacCormack, D.R. Chapman, Stabilization of the Burnett equations and application to hypersonic flows, *AIAA J.* 31 (1993) 1036.