PRICING JUMP RISK WITH UTILITY INDIFFERENCE

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Abstract. In an incomplete market, option prices depend on investors' utility functions. In this paper, we establish the connection between risk preference and optimal hedging strategy, and price options according to the principle of *utility indifference*. Taking the exponential utility function, we completely characterize the *risk-neutral valuation* for jump-diffusion processes. By using a recent result of duality by Delbaen *et. al.* (2000) we prove that pricing measure for the risk neutral valuation is just the equivalent minimal entropy martingale measure. We show that risk aversion contributes a price spread from the risk neutral price. We also show that, however, risk-neutral valuation does not correspond to any practical hedging strategy. Minimal variance hedging strategy is discussed. Parallel analysis is carried over to discrete setting with multi-nomial random walks, and efficient numerical methods are developed. Numerical examples show that our model reproduces "crash-o-phobia" and other features of market prices of options.

Key words: Utility maximization, utility indifference prices, minimal entropy martingale measure, jump-diffusion processes, risk-neutral valuation.

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1. Introduction

The researches of option pricing in recent years have largely been driven by two major empirical results. The first is the *leptokurtic feature* in the actual return distribution of asset prices, which means that, compared with (log-)normal marginal distribution, the actual distribution of many asset returns are typically skewed to the left, has higher peak and fatter tails. The second is the presence of *volatility smiles or skews*, meaning the smile shaped or skewed implied volatility curves, as contrast to the flat implied volatility curve for the classic Black-Scholes-Merton's model (Black-Scholes (1973); Merton (1973)). To accommodate these empirical facts, researchers have generalized the Black-Scholes-Merton's model to more general underlying driving dynamics, including lognormal processes with determinitic volatility, lognormal processes with stochastic volatility, and mixed jump diffusion processes. For comprehensive discussions of models and surveys of literatures we refer readers to Kou (1999) and Andersen and Andreasen (2000). These authors argued favorably for jump-diffusion model and showed the capacity of the model to reconcile with the empirical results.

Jump-diffusion option pricing model was pioneered by Merton (1976). Since market driven by both jump and diffusion is dynamically incomplete, option pricing theories, explicitly or implicitly, were developed under the equilibrium framework. Under the equilibrium framework, specifically, the Capital Asset Pricing Model (CAPM), an option price depends on the correlation between the underlying asset and the market portfolio, in addition to other factors. Several well-known pricing theories for jump-diffusion processes, including Naik and Lee (1991), Bates (1991), Wilmott (1999) and more recently Andersen and Andeasen (2000) were developed particularly for options on S&P500 index, which is considered the index for market portfolio. Recently Wu (2002) has extended the theory of Bates (1991) to options on individual stocks. Nevertheless, pricing stock options using CAPM is considered too complicated for practical use and may not necessarily be advantageous. In this paper, we take an alternative approach, namely, *utility indifference pricing* for options on asset driven by jump-diffusion process.

Utility indifference pricing was first introduced by Hodges and Neuberger (1989) and extended by Davis *et. al.* (1994) in the context of option pricing with transaction costs. Frittelli (2000a) studied the general properties of utility indifference prices. Rouge and Karoui (2000) considered especially the utility indifference pricing in an incomplete market driven by diffusion. The current paper carries the utility pricing over to a jump-diffusion economy by taking the approach of stochastic optimal control. In our approach, we assume that an option writer delta-hedges his liability with the optimal strategy that maximizes the utility of the terminal wealth of the hedged portfolio. The option premium is defined as the initial cash endowment with which the writer can achieve the same utility as that of a "pure" optimal investment: the one without cash endowment and the liability. Measured by utility, the premium of an option can be regarded as the "cost of hedging". From practical point of view utility indifference pricing has two advantages: it does not refer to the market portfolio, and it generates optimal hedging strategy together with an option price.

In this paper we only consider the exponential utility function. In principle, other utility functions can also be used. However, as it will become apparent later that, except for exponential utility, option premiums will depend on the initial wealth of the writer, this would violate the price universality. With the duality between exponetail utility and relative entropy obtained recently be Delbaen *et. al.* (2000), we justify the non-arbitrageability of the price. By letting the risk aversion tend to zero, we define and then characterize the "risk neutral measure" for a jump-diffusion process. It turns out that the risk neutral measure is just the *minimal entropy martingale measure*, which has been studied by Miyahara (1996), Frittelli (2000b); and Delbaen *et. al.* (2000) in various contexts. Such derived process plays an important central role in option pricing, as we show that risk aversion only generates spread from the price obtained by "risk neutral valuation". For the purpose of practical implementation, we have also extend our theory in continuous time to discrete time, where hedging can only take place in discrete moments. Numerical studies show that our model generates rather naturally "crash-o-phobia" and other features in the market price of options.

This paper will be organized as follows. In § 2 we will introduce utility pricing methodology in a general setting. Properties of utility indifference price are discussed. In § 3, for the jump-diffusion processes, we derive the Hamilton-Jacobi-Bellman (HJB) type governing equation for the utility indifference prices. Moreover, we give a detail characterization of the risk neutralized jump-diffusion process. In § 4, we develop the theory of utility pricing in discrete time, approximate the continuous-time jump-diffusion process by a multi-nomial discrete random walk, and construct a numerical scheme which converges to the HJB equation. § 5 contains numerical results where we will examine how option prices vary in response to the changes in the major model parameters. Interestingly, we will witness that a volatility smile simply corresponds to zero mean jump size, while volatility skew corresponds to a negative mean jump size. We extend the utility pricing to American options in § 6. Finally in § 7 we conclude the paper.

2. Properties of utility indifference prices

In this section, we will establish quantitatively a precise link between risk attitude and option prices. For clarity, we will work with the forward prices (for delivery at an option's maturity) of all involving assets. In formalism, this is equivalent to taking risk-free interest rate $r = 0$ and the discount factor $\beta \equiv 1$. As we shall see, different risk attitude will lead to different hedging strategies, which ensure proper risk exposure, and consequently, different option premiums to finance the hedging strategies.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete probability space such that $(\mathcal{F}_t)_{t\geq 0}$ is generated by independent Brownian diffusion and jump processes to be specified later. We consider the pricing of a European option from the viewpoint of an option seller. After receiving the premium, the seller has to hedge to reduce his risk exposure. Typically, the entire investment operation to the seller consists of three steps:

1. selling the option at time 0 for the future price v (for delivery at T);
2. paying out to the buyer the payoff $V(S_T)$ at time T;
3. accumulating the profits and losses arising from the self-financing hedging strategy $\{\Delta_t\}$ with the underlying asset.

Note that a strategy is a non-anticipative stochastic process. The terminal profit or loss of the investment realized at time T is

$$W_T = v - V(S_T) + \int_0^T \Delta_t dS_t.$$

If the seller was risk neutral, he would choose a strategy to maximize the expected value of the terminal wealth. In reality, however, all reasonable sellers are risk averse, and he will instead follow the strategy that maximizes the expected utility of the terminal wealth:

$$\sup_{\{\Delta\}} E^P[U(W_T)],$$

where $U(x)$ is an monotonically increasing and concave function, i.e., it has the properties $U'(x) > 0$ and $U''(x) < 0$. If a seller charges a premium v that finances a dominating strategy such that $W_T \geq 0$ with probability 1, the price could be too high to have a taker. In practice sellers always bear some risk of loss, and the terminal wealth of the hedged portfolio can be both positive and negative. With this understanding in mind, we choose, among the popular choices of utility functions, the exponential function as U:

$$U(x) = -e^{-\alpha x}, \quad \alpha > 0. \tag{1}$$

Note that both *log* and *power* utility function take only positive arguments. In (1), parameter α is the measure of risk aversion. Bigger α corresponds to higher degree of risk aversion. In particular, $\alpha = +\infty$ indicates absolute risk aversion while $\alpha = 0$ corresponds to risk neutrality.

To the seller, once the option is sold, he will follow a hedging strategy, namely, $\{\Delta_t\}$ shares of the underlying asset, so as to maximize the expected utility:

$$J(v, V_T) \equiv \sup_{\{\Delta_t\}} E^P[U(W_T)] = \sup_{\{\Delta_t\}} E^P\left[-exp\left\{-\alpha(v + \int_0^T \Delta_t dS_t - V(S_T))\right\}\right].$$

The fair price of the option, v, is defined implicitly by the principle of *utility indifference* (Davis et. al, 1993):

$$\sup_{\{\Delta_t\}} E^P\left[-exp\left(-\alpha(v + \int_0^T \Delta_t dS_t - V(S_T))\right)\right] = \sup_{\{\Delta_t\}} E^P\left[-exp\left(-\alpha\int_0^T \Delta_t dS_t\right)\right].$$

The above equation means that the hedger will not mind writing the option for the premium that allows him to achieve the same expected utility as that of writing no option. Due to the desirable separability of the exponential function, we have the following explicit expression for the option value

$$\begin{aligned} v &= \frac{1}{\alpha}\ln \inf_{\{\Delta_t\}} E^P\left[exp\left(-\alpha(\int_0^T \Delta_t dS_t - V_T)\right)\right] - \frac{1}{\alpha}\ln \inf_{\{\Delta_t\}} E^P\left[exp\left(-\alpha\int_0^T \Delta_t dS_t\right)\right] \\ &= \frac{1}{\alpha}\ln \inf_{\{\tilde{\Delta}_t\}} E^P\left[exp\left(\int_0^T \tilde{\Delta}_t dS_t + \alpha V_T)\right)\right] - \frac{1}{\alpha}\ln \inf_{\{\tilde{\Delta}_t\}} E^P\left[exp\left(\int_0^T \tilde{\Delta}_t dS_t\right)\right] \end{aligned} \tag{2}$$

In general, for a given underlying price process of S_t, the two terms in (2) can at least be evaluated by numerical methods. The $\{\tilde{\Delta}_t\}$ in either term of (2) is called optimal control, which relates to optimal investment strategy by

$$\Delta_t = -\tilde{\Delta}_t/\alpha. \tag{3}$$

Maximizing the expected utility of exponential function has an interesting dual problem. Recently Delbaen *et. al.* (2000) obtained

$$\sup_{\{\Delta_t\}} E^P\left[-exp\left(-\alpha[v + \int_0^T \Delta_t dS_t - V_T]\right)\right] = -\,exp\left(\alpha \sup_{Q\in P_{a,e}}[E^Q[V_T] - v - \frac{1}{\alpha}H(Q|P)]\right), \tag{4}$$

for rather general stochastic processes. Here, S_t is a locally bounded semimarytingale, Δ_τ is restricted such that $\int_0^t(\alpha\Delta_\tau)dS_\tau$ is bounded, $P_{a,e}$ stands for the set of martingale measures which are *absolutely continuous and locally equivalent* to P, and $H(Q|P)$ is the relative entropy of Q with respect to P, defined by

$$H(Q|P) := \begin{cases} E^P\left[\frac{dQ}{dP}\ln\frac{dQ}{dP}\right] & \text{if } Q \ll P, \\ +\infty & \text{otherwise.} \end{cases}$$

From the duality we obtain an alternative interpretation of expected utility maximization: making optimal investment (with an open short-option position) is equivalent to finding the equivalent martingale measure that maximizes the expected value of the option under an entropic penalty term. Note that (4) can be proved with Legendre transformation (Rockafellar, 1970). The duality for the special case $v = V_T = 0$ was obtained earlier by Grandits and Rheinländer (1999) and Frittelli (2000a).

From the duality we obtain an alternative expression of utility indifference price

$$v = \sup_{Q\in P_{a,e}}\left[E^Q[V_T] - \frac{1}{\alpha}H(Q|P)\right] - \sup_{Q\in P_{a,e}}\left[-\frac{1}{\alpha}H(Q|P)\right]. \tag{5}$$

Note that Rouge and Karoui (2000) had obtained the above expression of option price for geometric Brownian underlying process using the ideas of dynamic programming. Expression (5) carries a great advantage: it can serve to justify the no-arbitrage condition for utility indifference price (El Karoui and Quenez (1991); Cvitanić and Karatzas (1993))

$$\min_{Q\in P_{a,e}} E^Q[V] \le v \le \max_{Q\in P_{a,e}} E^Q[V].$$

For later reference we denote by Q_0 the minimal entropy martingale measure (MEMM) such that

$$H(Q_0|P) = \inf_{Q\in P_{a,e}} H(Q|P).$$

Similar to Rouge and Karoui (2000), we establish the following two propositions on the properties of utility indifference prices and the Value-at-Risk (VaR) measure of the hedged portfolio.

Proposition 2.1. *Let $v(\alpha, V)$ stands for the price for a contingent claim V under risk aversion α. $v(\alpha, V)$ has the following properties.*

1. $v(\alpha_2, V) \ge v(\alpha_1, V)$ *for* $\alpha_2 \ge \alpha_1 \ge 0$.
2. $v(\alpha, V_2) \ge v(\alpha, V_1)$ *for* $V_2 \ge V_1$.
3. *Convexity: for* $\mu \in (0, 1)$,

$$v(\alpha, \mu V_1 + (1-\mu)V_2) \le \mu v(\alpha, V_1) + (1-\mu)v(\alpha, V_2).$$

PROOF: The first and the second conclusions are obvious. We will only proceed to prove 3. Let $\{\tilde{\Delta}_t^1\}$ and $\{\tilde{\Delta}_t^2\}$ be the optimal controls for V_1 and V_2, respectively. By the Hölder's inequality,

$$\begin{aligned}
&\inf_{\{\tilde{\Delta}_t\}} E^P\left[exp\left(\int_0^T \tilde{\Delta}_t dS_t + \alpha\mu V_1 + \alpha(1-\mu)V_2\right)\right] \\
\le& E^P\left[exp\left(\int_0^T (\mu\tilde{\Delta}_t^1 + (1-\mu)\tilde{\Delta}_t^2)dS_t + \alpha\mu V_1 + \alpha(1-\mu)V_2\right)\right] \\
=& E^P\left[exp\left(\mu(\int_0^T \tilde{\Delta}_t^1 dS_t + \alpha V_1) + (1-\mu)(\int_0^T \tilde{\Delta}_t^2 dS_t + \alpha V_2)\right)\right] \\
\le& \left(E^P\left[exp\left(\int_0^T \tilde{\Delta}_t^1 dS_t + \alpha V_1\right)\right]\right)^{\mu} \cdot \left(E^P\left[exp\left(\int_0^T \tilde{\Delta}_t^2 dS_t + \alpha V_2\right)\right]\right)^{1-\mu} \\
=& \inf_{\{\tilde{\Delta}_t\}}\left(E^P\left[exp\left(\int_0^T \tilde{\Delta}_t dS_t + \alpha V_1\right)\right]\right)^{\mu} \cdot \inf_{\{\tilde{\Delta}_t\}}\left(E^P\left[exp\left(\int_0^T \tilde{\Delta}_t dS_t + \alpha V_2\right)\right]\right)^{1-\mu}.
\end{aligned}$$

The desired inequality then follows from the price definition (2) □

Unlike the delta hedge in the Black-Scholes-Merton's model, the optimal investment strategy defined in (3) is not for eliminating uncertainty, no matter the market is complete or not. We can derive an upper bound for the VaR.

Proposition 2.2. *For a risk aversion parameter α, we denote the optimal trading and utility indifference price for a contingent claim V_T by $\Delta^{*,\alpha}$ and v_α, respectively, then for any premium v we have*

$$P(v + \int_0^T \Delta_t^{*,\alpha} dS_t - V_T \le -k) \le e^{-\alpha(k+v-v_\alpha) - H(Q_0|P)}.$$

When $v = v_\alpha$,

$$P(v_\alpha + \int_0^T \Delta_t^{*,\alpha} dS_t - V_T \le -k) \le e^{-\alpha k - H(Q_0|P)}.$$

PROOF: Using sequentially the Chebyshev's inequality and the definition of utility indifference price and duality, we have

$$\begin{aligned}
P(v + \int_0^T \Delta_t^{*,\alpha} dS_t - V_T \le -k) &= P(e^{-\alpha(v+\int_0^T \Delta_t^{*,\alpha} dS_t - V_T)} \ge e^{\alpha k}) \\
&\le e^{-\alpha k} E^P\left[e^{-\alpha(v+\int_0^T \Delta_t dS_t - V_T)}\right] \\
&= e^{-\alpha(k+v-v_\alpha)} E^P\left[e^{-\alpha(v_\alpha+\int_0^T \Delta_t dS_t - V_T)}\right] \\
&= e^{-\alpha(k+v-v_\alpha)} \inf_{\{\Delta_t\}} E^P\left[e^{\int_0^T \Delta_t dS_t}\right] \\
&= e^{-\alpha(k+v-v_\alpha)} e^{H(Q_0|P)}.
\end{aligned}$$

The proposition is thus proved $\square$

For the utility indifference premium v_α, a lower bound to, for example, the 95% VaR is

$$-k \leq [\ln(0.05) + H(Q_0|P)]/\alpha.$$

The martingale measures that achieve the two supremums in equations (5) can be expressed in terms of optimal controls. Let $\{\tilde{\Delta}_t^*\}$ be the optimal control such that

$$\begin{aligned} J^{(0)}(S_t, t) &= \inf_{\{\tilde{\Delta}_\tau\}} E_t^P[exp\{\int_t^T \tilde{\Delta}_\tau dS_\tau\}] \\ &= E_t^P[exp\{\int_t^T \tilde{\Delta}_\tau^* dS_\tau\}], \end{aligned}$$

then the Radon-Nikodym derivative of the equivalent martingale measure Q_0 is given by (Frittelli, 2000a)

$$\frac{dQ_0}{dP} = \frac{exp\{\int_0^T \tilde{\Delta}_t^* dS_t)\}}{J^{(0)}(S_0, 0)}. \tag{6}$$

Extending the arguments of Frittelli we can obtain the Radon-Nikodym derivative for the equivalent martingale measure corresponding to $J^{(1)}(S_t, t)$:

$$\frac{dQ_1}{dP} = \frac{exp\{\int_0^T \tilde{\Delta}_t^{*,V} dS_t + \alpha V_T)\}}{J^{(1)}(S_0, 0)},$$

where $\tilde{\Delta}_t^{*,V}$ is the optimal control such that

$$\begin{aligned} J^{(1)}(S_t, t) &= \inf_{\{\tilde{\Delta}_\tau\}} E_t^P[exp\{\int_t^T \tilde{\Delta}_\tau dS_\tau + \alpha V_T)\}] \\ &= E_t^P[exp\{\int_t^T \tilde{\Delta}_\tau^{*,V} dS_\tau + \alpha V_T)\}]. \end{aligned}$$

The expression can be made more explicit in term of $\tilde{\Delta}_t^{*,V}$ for concrete asset price processes.

3. Pricing with Jump-diffusion process in continuous time

In this section we consider option pricing in the incomplete market represented by, specifically, a jump-diffusion process

$$\frac{dS}{S} = \mu_t dt + \sigma_t dZ_t + (Y-1)dN_t \tag{7}$$

for the underlying asset. Here, Z_t is a standard Wiener process under the objective measure P, N_t is the Poisson process with arrival intensity λ, $Y-1$ is the random jump size, μ_t and σ_t are deterministic functions of time. In this paper we assume the independence between the Wiener process and the Poisson process. The percentage jump size $Y-1$ can take any distributions and the usual ones will be introduced later.

For an option with payoff V_T at time T, the utility indifference price is defined in (2), where it requires the evaluation of two minimized expected utility functions

$$J^{(0)}(S_t,t) = \inf_{\{\tilde{\Delta}_\tau\}} E_t^P[exp\{\int_t^T \tilde{\Delta}_\tau dS_\tau\}],$$
$$J^{(1)}(S_t,t) = \inf_{\{\tilde{\Delta}_\tau\}} E_t^P[exp\{\int_t^T \tilde{\Delta}_\tau dS_\tau + \alpha V_T\}].$$

For the jump-diffusion process, we can derive Hamilton-Jacobi-Bellman like equations for $J^{(l)}(S,t), l = 0,1$.

Proposition 3.1. *Assume that $J^{(l)}(S,t) \in C^2$ under the jump-diffusion driving dynamics, $l = 0,1$, then they satisfy*

$$\inf_{\{\tilde{\Delta}_t\}} \left\{ J_t^{(l)} + \left[\frac{1}{2}\sigma^2 S^2 \left(\frac{\partial}{\partial S} + \tilde{\Delta}_t I \right)^2 + \mu S \left(\frac{\partial}{\partial S} + \tilde{\Delta}_t I \right) \right] J^{(l)} \right.$$
$$\left. + \lambda_t E_Y [\exp\{\tilde{\Delta}_t S(Y-1)\} J^{(l)}(SY,t) - J^{(l)}(S,t)] \right\} = 0, \quad (8)$$

with the terminal conditions

$$J^{(0)}(S,T) = 1, \quad \textit{and} \quad J^{(1)}(S,T) = e^{\alpha V_T(S)}. \quad (9)$$

To prove the above proposition we need the following theorem of *change of measure* for the jump-diffusion process (see Björk *et. al.*, 1997).

Theorem 3.1 (Girsonov)**.** *Let Γ be a non-anticipative process and $\Phi(Y)$ be a strictly positive measurable function such that for finite t*

$$\int_0^t \|\Gamma_s\|^2 ds < \infty, \qquad E_Y[|\Phi(Y)|] < \infty.$$

Define a process L_t by

$$dL_t = L_{t^-} \left(\Gamma_t dZ_t + (\Phi(Y) - 1)dN_t - \lambda_t E_Y[\Phi(Y) - 1]dt \right), \quad L_0 = 1,$$

or, equivalently,

$$\ln L_t = \int_0^t \Gamma_s dZ_s - \frac{1}{2}\int_0^t \|\Gamma_s\|^2 ds + \int_0^t \ln \Phi(Y) dN_s - \int_0^t \lambda(t) E_Y[\Phi(Y) - 1]dt.$$

Then L_t is an martingale under P:

$$E^P[L_t] = 1.$$

Define a new measure

$$dQ_t = L_t dP_t,$$

then, Q is locally equivalent to P, and

(i) $d\tilde{Z}_t = dZ_t - \Gamma_t dt$ *is a Q-Wiener process.*

(ii) *Let $f(Y)$ be the density function of Y. The Poisson process N_t has a Q-intensity given by*

$$\lambda^*(t, dY) = \lambda(t)\Phi(Y)dY.$$

Now we are ready to prove Proposition 3.1.

PROOF OF PROPOSITION 3.1: Define a martingale

$$L_t = exp\left(\int_0^T -\left(\frac{1}{2}(\tilde{\Delta}_t\sigma_t S_t)^2 + \lambda_t E_Y[e^{\tilde{\Delta}_t S_t(Y-1)} - 1]\right) dt \right.$$
$$\left. + \tilde{\Delta}_t\sigma_t S_t dZ_t + \tilde{\Delta}_t S_t(Y-1)dN_t\right),$$

and a new measure Q_t:

$$dQ_t = L_t dP_t,$$

we then have

$$\begin{aligned} J(S_0, 0) &= \inf_{\{\tilde{\Delta}_t\}} E^P\left[exp\left(\int_0^T \tilde{\Delta}_t dS_t + \alpha V_T\right)\right] \\ &= \inf_{\{\tilde{\Delta}_t\}} E^P\left[L_t exp\left(\int_0^T \left(\tilde{\Delta}_t\mu_t S_t + \frac{1}{2}(\tilde{\Delta}_t\sigma_t S_t)^2 + \lambda_t E_Y[e^{\tilde{\Delta}_t S_t(Y-1)} - 1]\right) dt + \alpha V_T\right)\right] \\ &= \inf_{\{\tilde{\Delta}_t\}} E^Q\left[exp\left(\int_0^T \left(\tilde{\Delta}_t\mu_t S_t + \frac{1}{2}(\tilde{\Delta}_t\sigma_t S_t)^2 + \lambda_t E_Y[e^{\tilde{\Delta}_t S_t(Y-1)} - 1]\right) dt + \alpha V_T\right)\right]. \end{aligned} \tag{10}$$

The last term poses a standard optimal control problem whose value satisfies a Hamilton-Jacobi-Bellman type equation. Under the new measure the asset price process becomes

$$dS_t = (\mu S_t + \tilde{\Delta}_t\sigma_t^2 S_t^2)dt + \sigma_t S_t d\tilde{Z}_t + S_t(Y-1)dN_t^*, \tag{11}$$

where

$$d\tilde{Z}_t = dZ_t - \tilde{\Delta}_t\sigma_t S_t dt$$

is a Q-Wiener process, and N_t^* is a Poisson process with Q-intensity

$$\lambda^*(t, dY) = \lambda(t)e^{\tilde{\Delta}_t S_t(Y-1)}dY. \tag{12}$$

The HJB equation for the optimal control problem is

$$\inf_{\tilde{\Delta}_t}\left\{\frac{\partial J}{\partial t} + \frac{1}{2}\sigma_t^2 S^2\frac{\partial^2 J}{\partial S^2} + (\mu_t S + \tilde{\Delta}_t\sigma_t^2 S^2)\frac{\partial J}{\partial S} + E_Y^Q\left[J(SY, t) - J(S, t)\right]\right.$$
$$\left. + \left(\tilde{\Delta}_t\mu_t S + \frac{1}{2}(\tilde{\Delta}_t\sigma_t S)^2 + \lambda_t E_Y[e^{\tilde{\Delta}_t S(Y-1)} - 1]\right) J\right\} = 0, \tag{13}$$

where $E_Y^Q[\cdot]$ means the expectation with respect to the Poisson process under measure Q. Using (12) we have

$$E_Y^Q[J(SY, t) - J(S, t)] = \lambda_t E_Y[e^{\tilde{\Delta}_t S(Y-1)}(J(SY, t) - J(S, t))]. \tag{14}$$

Put (14) back to (13) and notice some cancellation we then arrive at

$$\inf_{\tilde{\Delta}_t}\left\{\frac{\partial J}{\partial t}+\frac{1}{2}\sigma_t^2S^2\frac{\partial^2 J}{\partial S^2}+(\mu_t S+\tilde{\Delta}_t\sigma_t^2S^2)\frac{\partial J}{\partial S}+\lambda_t E_Y\left[e^{\tilde{\Delta}_t S(Y-1)}J(SY,t)-J(S,t)\right]\right.$$
$$\left.+\left(\tilde{\Delta}_t\mu_t S+\frac{1}{2}(\tilde{\Delta}_t\sigma_t S)^2\right)J\right\}=0. \quad (15)$$

Recombining the terms we finally end up with equation (8) $\square$.

The optimal control $\{\tilde{\Delta}_t^*\}$ has to be solved from equation (8). The first order condition with respect to $\tilde{\Delta}_t$ is

$$\mu_t+\sigma_t^2 S\left(\frac{J_S^{(l)}}{J^{(l)}}+\tilde{\Delta}_t\right)+\frac{\lambda_t E_Y[(Y-1)e^{\tilde{\Delta}_t S(Y-1)}J^{(l)}(SY,t)]}{J^{(l)}(S,t)}=0. \quad (16)$$

Apparently $\tilde{\Delta}_t^*$ can be solved numerically once $J^{(l)}$ and $J_S^{(l)}$ have been obtained. Yet for $l=0$, the special case corresponding to $\alpha=0$, $\tilde{\Delta}_t^*$ can be obtained without going through the solution procedure for (8). Notice that for $\alpha=0$ equation (8) admits a unique and state-independent solution, i.e., $J^{(0)}(S,t)=J^{(0)}(t)$. Consequently, equation (16) reduces to

$$\mu_t+\sigma_t^2 S\tilde{\Delta}_t+\lambda_t E_Y[(Y-1)e^{\tilde{\Delta}_t S(Y-1)}]=0. \quad (17)$$

The left-hand side of (17) is a monotonic function of $\tilde{\Delta}_t$, and the unique solution is in the form

$$S\tilde{\Delta}_t^*=C_0(t). \quad (18)$$

C_0 will be time-independent if none of μ,σ and λ depends on t.

Function $J^{(0)}(t)$ can be expressed in terms of $C_0(t)$. Using the last equality of (10) and Girsanov Theorem we obtain

$$\begin{aligned}J^{(0)}(S_0,0)&=E^P\left[exp\left(\int_0^T C_0(t)(\mu_t dt+\sigma_t dZ_t+(Y-1)dN_t)\right)\right]\\&=exp\left(\int_0^T(\mu_t C_0(t)+\frac{1}{2}\sigma_t^2C_0^2(t)-\lambda_t E_Y[1-e^{C_0(t)(Y-1)}])dt\right)\end{aligned} \quad (19)$$

A direct implication of (19) is, by the duality (4),

$$\inf_{Q\in P_{a,e}}H(Q|P)=\int_0^T\left(\mu_t C_0(t)+\frac{1}{2}\sigma^2C_0^2(t)-\lambda_t E_Y[1-e^{C_0(t)(Y-1)}]\right)dt.$$

Due to (19) we only need to solve a single optimal control problem for the utility indifference price (2).

Equations (17) resembles the capital asset pricing model (CAPM) for jump-diffusion processes. Together with (18) they have the following implications on the relation between risk aversion and optimal investment on the risky asset.

Proposition 3.2. *Under the exponential utility function,*

1. *the allocation of wealth to risky asset is maintained to be a constant or at most a time-dependent function;*
2. *given a specific allocation to risky asset, say $S_t\Delta_t = C_1(t)$, the implied risk aversion is related to the cum-dividend excess return by*

$$\mu_t - \sigma_t^2 \alpha C_1(t) + \lambda_t E_Y[(Y-1)e^{-aC_1(t)(Y-1)}] = 0.$$

Next we consider *risk neutral valuation*, which is corresponding to the limit $\alpha \to 0$.

Proposition 3.3. *When $\alpha \to 0$, v_α tends to the limit*

$$v = E^{Q_0}[V_T]. \tag{20}$$

Under Q_0, the asset price process becomes

$$\frac{dS}{S} = -\lambda_t^* E^*[(Y-1)]dt + \sigma d\tilde{Z}_t + (Y-1)dN_t^*,$$

where $d\tilde{Z}_t$ is a Q_0-Wiener process, dN_t^* is the Q_0-Poisson process with intensity

$$\lambda_t^* = \lambda_t E_Y\left[e^{C_0(Y-1)}\right].$$

The corresponding PDE for v is

$$\frac{\partial v}{\partial t} - \lambda_t^* E_Y^*[Y-1]S\frac{\partial v}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + \lambda_t^* E_Y^*[v(SY,t) - v(S,t)] = 0, \tag{21}$$

with terminal condition

$$v(S,T) = V_T(S), \tag{22}$$

while $E_Y^*[\cdot]$ is defined by

$$E_Y^*[\mathcal{S}] = E_Y\left[\mathcal{S}\frac{e^{C_0(Y-1)}}{E_Y\left[e^{C_0(Y-1)}\right]}\right].$$

for any set of paths $\mathcal{S}$.

Proof: With the price formula (5), we can easily argue that

$$v = \lim_{\alpha\to 0} v_\alpha \geq E_{Q_0}[V_T], \tag{23}$$

that is, the risk-neutral price of an option is the lower bound of risk averse price. We want to show next that equality in (23) holds. Recall the expression of minimal equivalent martingale measure (6), we have

$$\begin{aligned}
v_\alpha &= \frac{1}{\alpha}\left(\ln \inf_{\{\tilde{\Delta}_t\}} E^P\left[exp\left(\int_0^T \tilde{\Delta}_t dS_t + \alpha V_T\right)\right] - \ln \inf_{\{\tilde{\Delta}_t\}} E^P\left[exp\left(\int_0^T \tilde{\Delta}_t dS_t\right)\right]\right) \\
&\le \frac{1}{\alpha}\left(\ln E^P\left[exp\left(\int_0^T \tilde{\Delta}_t^* dS_t + \alpha V_T\right)\right] - \ln E^P\left[exp\left(\int_0^T \tilde{\Delta}_t^* dS_t\right)\right]\right) \\
&= \frac{1}{\alpha}\ln E^P\left[\frac{exp\left(\int_0^T \tilde{\Delta}_t^* dS_t\right)}{E^P\left[exp\left(\int_0^T \tilde{\Delta}_t^* dS_t\right)\right]}\cdot exp(\alpha V_T)\right] \\
&= \frac{1}{\alpha}\ln E^{Q_0}\left[exp\left(\alpha V_T\right)\right] \longrightarrow E^{Q_0}[V_T] \quad \text{as} \quad \alpha \to 0.
\end{aligned}$$

The limit in the last term is a well-known result. Hence we also have $v \le E^{Q_0}[V_T]$ and (20) follows.

Combining (11) and (17) we conclude that, under measure Q_0, the asset price follows

$$\begin{aligned}
dS_t &= S_t(-\lambda_t E_Y[(Y-1)e^{C_0(Y-1)}]dt + \sigma_t S_t d\tilde{Z}_t + S_t(Y-1)dN_t^* \\
&= S_t(-\lambda_t^* E_Y^*[(Y-1)]dt + \sigma_t S_t d\tilde{Z}_t + S_t(Y-1)dN_t^*
\end{aligned}$$

Since option value is an martingale under Q_0, the equation for v follows from the jump-diffusion version of Ito's lemma □

We remark here that in a recent paper Miyahara (2001) obtained a change of measure from an objective geometric Lévy process to its MEMM process. The jump-diffusion process can be regarded as a special case of the geometric Lévy process. The analysis in this article takes advantage of the HJB-type equation.

Although theoretically beautiful, the utility approach does not generate meaningful hedge ratio for risk-neutral valuation. It can be easily seen that, if the expected rate of return of S_t is positive, then utility maximization will result in an infinite long delta position. On the other hand, if the expected rate of return of S_t is negative, it will lead to the infinite short delta position. This is of course impractical. For risk neutral valuation, we need additional criteria to define the hedge ratio. Intuitively, the minimal variance hedge ratio is a rather natural choice.

The idea of minimal variance hedge is to choose the delta such that the variance of the delta-hedged option portfolio is minimized. Consider a hedged portfio

$$\Pi_t = -V(S,t) + \Delta_t S.$$

We want to choose a Δ_t to minimize the variance of $d\Pi_t$, and the answer is

$$\Delta_t^* = \frac{Cov^P(dV, dS)}{Var^P(dS)}.$$

For the jump-diffusion process (7) we can derive, by using Ito's lemma,

$$\Delta_t^* = \frac{\lambda_t E_t^P[(Y-1)(V(SY,t)-V(S,t))] + \sigma^2 S \frac{\partial V}{\partial S}}{\lambda_t S E_t^P[(Y-1)^2] + \sigma^2 S}. \tag{24}$$

This is obtained by Wilmott (1998). By assuming that the hedged portfolio has an expected return equal to the risk-free rate, identical to the assumption by Merton (1976), Wilmott created a partial differential equation for the option value. Under minimal variance hedge the hedged porfolio is uncorrelated with the underlying. Hence, when the underlying is the market portfolio, the hedged portfolio is uncorrelated with the market, and the assumption is justified by CAPM. The discrete version of (24) will be presented in the next section.

For standard call and put options, initial-value problem (21,22) with time dependent intensity can be solved in closed form. This will be a generalization to the Merton's formula (Merton, 1976). Here we would like to highlight the importance of time-dependent intensity. It has been a pattern that implied volatility soars before price sensitive events and recedes afterwards. These events include the announcement of rate-set meeting, major economic data, corporate earnings and etc. The market practitioners are used to increase or decrease the Black's volatilities to accommodate the price appreciation or depreciation of options. While it may work for describing the trend of option price variation, it does not help too much, and maybe even misleading, for option hedging. We believe that the substantial variation of options' implied volatility is mainly attributed to variation of market's anticipation of jump intensity. We argue that very often the nature of the risk brought forward by a price sensitive event is jump, instead of diffusion. Effective hedging strategy should be built upon correct characterization of risk.

Another important reason for considering time-dependent jump intensity is to capture simultaneously the steep short-term and mild long term skews. It seems that markets tend to ignore the jump risk in the far future. We may need a very big jump intensity to price the risk of an imminent jump. Yet if we use such big intensity across time, we will likely overprice long-term options. The use of time-dependent intensity will allow us to appropriately price the risk and establish a more effective hedge.

Parallel to Merton's arguments (1976), we can derive the following closed-form solution to problem (21,22) for a call option:

$$V(S,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\bar{\lambda}^*(\tau)\tau}[\bar{\lambda}^*(\tau)\tau]^n}{n!} E_Y^{Q_0}[BS(SX_n e^{-k^*\bar{\lambda}^*(\tau)\tau}, \tau; X, \sigma^2, 0)],$$

where $\tau = T - t$, $\bar{\lambda}^*(\tau)$ is the averaged jump intensity over time period (t, T):

$$\bar{\lambda}^*(\tau) = \frac{1}{\tau} \int_t^T \lambda^*(s) ds,$$

k^* is the mean jump size

$$k^* = E_Y^{Q_0}[Y - 1],$$

and $BS(\cdot)$ is the Black-Scholes formula:

$$BS(S, \tau; X, \sigma^2, r) = SN(d_1) - e^{-rT} XN(d_2),$$

with

$$d_1 = \frac{\ln \frac{S}{X} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = d_1 - \sigma\sqrt{\tau}.$$

We remark that if the objective distribution of the jump size Y is normal, then its risk-neutralized distribution (corresponding to *) is lognormal. In such case the expectations in Merton's formula can be worked out. For other distributions, we in general will have to evaluate the expectations numerically. In fact, by considering utility indifference pricing in discrete setting, which will be discussed in the next section, we will be able to develop efficient numerical methods for (21,22).

Finally in this section we briefly introduce the popular distributions for the jump size Y, which include lognormal jumps, bivariate jumps, bankruptcy inducing jumps, and double-exponential jumps. As we will see later that numerical implementation with any jump-size distribution is similar.

3.1. **Bivariate jumps.** The bivariate jump distribution is defined as

$$\ln(Y) = \begin{cases} \beta & \text{with probability } P_b \\ -\beta & \text{with probability } 1 - P_b \end{cases} \tag{25}$$

for some constant $P_b \in (0, 1)$. The corresponding mean percentage jump is

$$k = E_Y[Y - 1] = P_b e^{\beta} + (1 - P_b)e^{-\beta} - 1. \tag{26}$$

The β corresponds to, say for example, a 10% jump is

$$\beta = \ln(1 + 10\%) = 0.1.$$

The bivariate jump process will be good candidate to present our numerical method.

3.2. **Bankruptcy induced jumps.** This is the extreme case of bivariate jump distribution. Upon jump the stock price becomes zero (Samuelson, 1973), corresponding to a nonstochastic jump variable $Y = 0$. Under risk-neutral valuation we know that the price of an European call option is given by the Black-Scholes formula with the risk-free interest rate 0 replaced by λ^* (Recall that we use the forward price).

3.3. **Lognormally distributed jumps.** Let $\ln Y$ follow the normal process with mean μ_J and variance σ_J (Merton, 1979). In this case, $k = E_Y[Y-1] = \exp(\mu + \sigma^2/2)$-1. This is the most popularly used jump distribution as it gives rise to the closed-form formula for the option price under Merton's (1976) model.

3.4. **Double exponential jump process.** In the double exponential jump process (Kou, 1999) $\ln Y$ has density function

$$f_Y(y) = \frac{1}{2\eta} e^{-|y-\kappa|/\eta}, \qquad 0 < \eta < 1,$$

which, in simple words, means

$$Y = \begin{cases} \kappa + \xi, & \text{with probability } 1/2, \\ \kappa - \xi, & \text{with probability } 1/2, \end{cases}$$

where ξ is an exponential random variable with density function

$$g(x) = \begin{cases} e^{-x/\eta}/\eta, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

The mean and variance of ξ are η and η^2, respectively. Clearly for the double exponential process we have

$$k = \kappa - 1.$$

The double-exponential distribution has interesting psychological background. It reflects the pattern of investors' overreaction to price sensitive events.

4. Pricing in discrete time — multi-nomial random walk

We can discretize (8,9) directly and solve with numerical methods. However, very often such approach for obtaining solutions is neither efficient nor intuitive. In practice, hedging takes place in discrete time. Hence, it is interesting for its own sake to study utility indifference pricing in discrete setting. By pressing into this direction, as we shall see in the subsequent discussions, we will be able to develop intuitive and robust numerical method for the valuation.

Parallel to the continuous case, we now consider the writing and subsequent hedging of an option in the discrete setting. Suppose that a seller writes an option of payoff V_T for premium v and follows a hedging strategy $\{\Delta_i\}$ at moments $\{t_i\}$, then at the option's maturity he will end up with a profit or loss in the amount

$$W_T = v - V(S_N) + \sum_{i=0}^{N-1} \Delta_i (S_{i+1} - S_i).$$

Here, we have yet again used the forward prices for all securities. The seller with exponential risk aversion will seek for a strategy to maximize the utility:

$$\begin{aligned}
\max_{\{\Delta_i\}} E^P[U(W_T)] &= \max_{\{\Delta_i\}} E^P \left[-exp\left(-\alpha \left(v + \sum_{j=0}^{N-1} \Delta_j (S_{j+1} - S_j) - V(S_N) \right) \right) \right] \\
&= e^{-\alpha v} \max_{\{\tilde{\Delta}_i\}} E^P \left[-exp\left(\sum_{j=0}^{N-1} \tilde{\Delta}_j (S_{j+1} - S_j) + \alpha V(S_N) \right) \right].
\end{aligned}$$

Define

$$\begin{aligned}
J^{(0)}(S_i, t_i) &= \min_{\{\tilde{\Delta}_j\}} E^P_{t_i} \left[exp\left(\sum_{j=i}^{N-1} \tilde{\Delta}_j (S_{j+1} - S_j) \right) \right], \\
J^{(1)}(S_i, t_i) &= \min_{\{\tilde{\Delta}_j\}} E^P_{t_i} \left[exp\left(\sum_{j=i}^{N-1} \tilde{\Delta}_j (S_{j+1} - S_j) + \alpha V(S_N) \right) \right].
\end{aligned}$$

Then the utility indifference option price is defined by

$$v = \frac{1}{\alpha} \left(\ln J^{(1)}(S_0, 0) - \ln J^{(0)}(S_0, 0) \right).$$

The valuation of $J^{(l)}(S_0, t_0), l = 0, 1$ can be achieved with a dynamical programming procedure, which is stated in the following Proposition.

Proposition 4.1. *Function $J^{(l)}(S_i, t_i), l = 0, 1$ satisfy*

$$J^{(l)}(S_i, t_i) = \min_{\tilde{\Delta}_i} E^P_{t_i} [exp(\tilde{\Delta}_i (S_{i+1} - S_i)) J^{(l)}(S_{i+1}, t_{i+1})], \tag{27}$$

with

$$J^{(0)}(S_N, t_N) = 1, \quad \textit{and} \quad J^{(1)}(S_N, t_N) = exp(\alpha V(S_N)). \tag{28}$$

PROOF: We only need to prove the proposition for $l = 1$ as $l = 0$ is just a special case such that $V_T = 0$. Using the statistical independence we have

$$\begin{aligned} J^{(1)}(S_i, t_i) &= \min_{\{\tilde{\Delta}_j\}} E^P_{t_i}\left[exp\left(\tilde{\Delta}_i(S_{i+1} - S_i)\right) E^P_{t_{i+1}}\left[exp\left(\sum_{j=i+1}^{N-1} \tilde{\Delta}_j(S_{j+1} - S_j) + \alpha V(S_N)\right)\right]\right] \\ &= \min_{\tilde{\Delta}_i} E^P_{t_i}\left[exp\left(\tilde{\Delta}_i(S_{i+1} - S_i)\right) \min_{\{\tilde{\Delta}_j\}} E^P_{t_{i+1}}\left[exp\left(\sum_{j=i+1}^{N-1} \tilde{\Delta}_j(S_{j+1} - S_j) + \alpha V(S_N)\right)\right]\right] \\ &= \min_{\tilde{\Delta}_i} E^P_{t_i}\left[exp\left(\tilde{\Delta}_i(S_{i+1} - S_i)\right) J^{(1)}(S_{i+1}, t_{i+1})\right]. \end{aligned}$$

The proposition is thus proved $\square$

Under the backward dynamics of stochastic control, scheme (27) has a single control variable and is easily implementable. Differentiating with respect to $\tilde{\Delta}_i$ we obtain the first-order condition

$$0 = E^P_{t_i}[(S_{i+1} - S_i)exp(\tilde{\Delta}_i(S_{i+1} - S_i))J^{(l)}(S_{i+1}, t_{i+1})]. \tag{29}$$

The above equation implicitly defines $\tilde{\Delta}_i$, which, for known function $J(S_{i+1}, t_{i+1})$, can be solved numerically. In fact the right hand side of (29) is a monotonic function of $\tilde{\Delta}_i$ and the solution is thus unique. To see this we denote the right-hand side by

$$f(\tilde{\Delta}_i) = E^P_{t_i}[(S_{i+1} - S_i)exp(\tilde{\Delta}_i(S_{i+1} - S_i))J^{(l)}(S_{i+1}, t_{i+1})].$$

Apparently we have

$$\frac{df(\tilde{\Delta}_i)}{d\tilde{\Delta}_i} = E^P_{t_i}[(S_{i+1} - S_i)^2 exp\{\tilde{\Delta}_i(S_{i+1} - S_i)\}J^{(l)}(S_{i+1}, t_{i+1})] > 0$$

for $J^{(l)}(S_{i+1}, t_{i+1}) > 0$. Using the Newton-Raphson method we can calculate $\tilde{\Delta}_i$ in just a few steps of iterations. Proceeding backwardly with (27) we will eventually obtain $J^{(l)}(S_0, 0), l = 0, 1$. For later reference we denote the optimal control for $J^{(0)}$ and $J^{(1)}$ by $\{\tilde{\Delta}^*_i\}$ and $\{\tilde{\Delta}^{*,1}_i\}$, respectively.

Similar to the continuous case, the valuation of $J^{(0)}(S_i, t_i)$ can be significantly simplified. With induction we can prove that $J^{(0)}(S_i, t_i)$ is independent of S_i. In such case equation (29) reduces to

$$0 = E^P_{t_i}\left[\left(\frac{S_{i+1}}{S_i} - 1\right) exp\left(\tilde{\Delta}_i S_i(\frac{S_{i+1}}{S_i} - 1)\right)\right].$$

Since the distribution of $\frac{S_{i+1}}{S_i}$ is independent of S_i, the solution to the above equation is of the form

$$\tilde{\Delta}_i S_i = c_0 = const.$$

Consequently, we have

$$
\begin{aligned}
J^{(0)}(S_i,t_i) &= E^P[exp(c_0(\frac{S_{i+1}}{S_i}-1))]J^{(0)}(S_{i+1},t_{i+1}) \\
&= \left(E^P[exp(c_0(\frac{S_{i+1}}{S_i}-1))]\right)^2 J^{(0)}(S_{i+2},t_{i+2}) \\
&= \dots \\
&= \left(E^P[exp(c_0(\frac{S_{i+1}}{S_i}-1))]\right)^{N-i}.
\end{aligned}
$$

For jump-diffusion underlying process, scheme (27,28) can be implemented with a multi-nomial tree. Without loss of generality, let us focus on the jump-diffusion process with bivariate jump-size distribution (25,26). Over a small time interval δt, the jump-diffusion process can be approximated by a one-period quadri-nomial tree, as is shown in Figure 1, where $J > 1$ is a positive integer (to be determined).

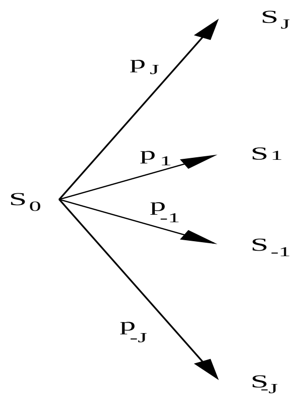

FIGURE 1 The quadrinomial tree.

The ending nodes relate to the root S_0 by

$$S_i = S_0 e^{i\sigma\sqrt{\delta t}}, \quad i = -J, -1, 1 \text{ and } J.$$

Intuitively, the two branches in the middle correspond to diffusion, while the other two branches correspond to jump. The objective probabilities to reach those nodes, matching to the subindices, are

$$p_{-J} = \lambda\delta t(1-P_b), p_{-1} = (1-\lambda\delta t)(1-p_d), p_1 = (1-\lambda\delta t)p_d \quad \text{and} \quad p_J = \lambda\delta t P_b,$$

where, similar to Amin (1994),

$$
\begin{aligned}
P_b &= \frac{(1+k)-d^J}{u^J-d^J}, \\
p_d &= \frac{R'-d}{u-d}, \qquad R' = \frac{e^{\mu\delta t}-\lambda\delta t(1+k)}{1-\lambda\delta t},
\end{aligned}
$$

and

$$u = 1/d = e^{\sigma\sqrt{\delta t}},$$
$$k = E_Y[Y-1] = P_b e^{\beta} + (1-P_b)e^{-\beta} - 1.$$

The jump step J is given by

$$J = \left[\frac{\beta}{u-1}\right] = \left[\frac{\beta}{e^{\sigma\sqrt{\delta t}} - 1}\right],$$

where $[x]$ means the largest integer smaller or equal to x.

For longer time horizon, the jump-diffusion process can be approximated by a multi-period quadri-nomial tree, as is shown in Figure 2, where each node is indexed by a pair of integers $\{i, j\}$.

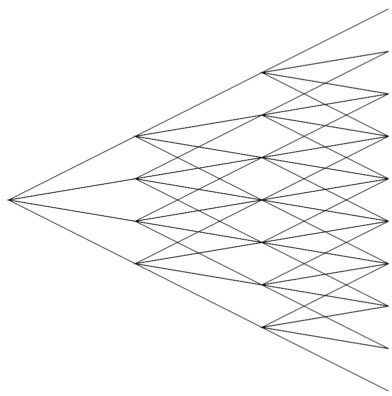

FIGURE 2 The quadrinomial tree.

The implementation of scheme on the multi-period tree is rather straight forward. Below is the algorithm.

`/* Algorithm for utility valuation */`

`/* Compute terminal values */`

`For` $i = -3N-1, \dots, 3N+1$ `compute`

$J^{(1)}_{i,N} = exp(\alpha V_T(S_{i,N}))$

`end`

`/* Valuate` $J^{(1)}_{00}$ `*/`

`For` $j = N-1, N-2, \dots, 0$

`For` $i = -3j-1, \dots, 3j+1$

`Solve` $0 = E_{t_i}[(S_{\cdot,j+1} - S_{ij})exp(\tilde{\Delta}_i(S_{\cdot,j+1} - S_{ij}))J^{(1)}_{\cdot,j+1}]$ `for` $\tilde{\Delta}^{*,1}_i$;

`Evaluate` $J^{(1)}_{ij} = E_{t_i}[exp(\tilde{\Delta}^{*,1}_i(S_{\cdot,j+1} - S_{ij}))J^{(1)}_{\cdot,j+1}]$,

```
        end
    end
/* Valuate J_{00}^{(0)} */
    Solve c_0 from 0 = E[(S_{.,1}/S_{00} - 1)exp(c_0(S_{.,1}/S_{00} - 1))],
    Compute J_{00}^{(0)} = (E[exp(c_0(S_{.,1}/S_{00} - 1))])^N.
/* Compute the option value */
    v = (1/α)(ln J_{00}^{(1)} - ln J_{00}^{(0)}).
/* The end of the algorithm */
```

Although simple, the above algorithm has two serious disadvantages. First, the number of nodes increase too fast when we reduce the time-step size δt, causing the implementation a computational burden. Second, the arithmetic with exponential function may suffer from numerical underflows or overflows. Fortunately, both problems can be fixed.

To avoid large node number, we only need to implement the algorithm over a trimmed tree, as is shown in Figure 3. The use of the trimmed tree is based on the observation that, in terms of forward prices, the market values of both deeply in-the-money and deeply

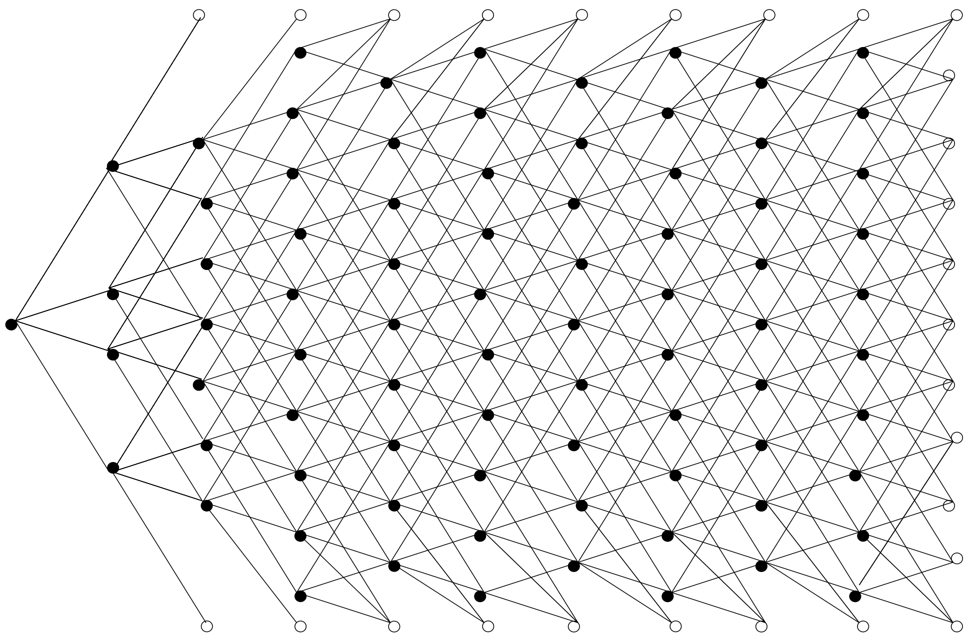

FIGURE 3 Stock price tree. The dots correspond to unknowns and the circles correspond to boundary condition.

out-of-the-money options are very close their intrinsic values. Hence, we can apply the intrinsic values as boundary conditions along the line where the tree is trimmed.

When working with the trimmed tree we must make sure that relevant index does not go beyond range. Suppose stock price index ranges from $-M$ to M, then in the algorithm above we only need to impose

$$\min\{-M+1, -3j-1\} \leq i \leq \min\{M-1, 3j+1\}.$$

Accordingly, the jump sizes are chosen such that

$$J_i^U = \min\{J, M-i\}$$

for upward jumps and

$$J_i^D = \min\{J, i-M\}$$

for downward jumps.

For the choice of M, we may simply take $M = N$, the number of time steps. In such case the collection of discrete asset prices are identical to those of a binomial tree of time step N. For deeply in- or out-of-the money options, it may be helpful to take M bigger then N.

To alleviate the problem of overflows or underflows we add two additional treatments to the algorithm. First we scale down all asset prices by dividing them by the spot asset price S_0. Consequently, most arithmetic in the algorithm will be performed with numbers around 1. Second, we can instead work the "log" of the original function

$$V(S_i, t_i) = \frac{1}{\alpha} \ln J^{(1)}(S_i, t_i).$$

Under $V(S_i, t_i)$ scheme (27) becomes

$$V(S_i, t_i) = \frac{1}{\alpha} \ln \left\{ E_{t_i}^P [exp\{\tilde{\Delta}_i^{*,1}(S_{i+1} - S_i) + \alpha V(S_{i+1}, t_{i+1}))\}] \right\}.$$

By working with V instead of $J^{(1)}$ we can reduce the propagation of roundoff errors.

It is time to focus on two special cases of utility indifference pricing, namely, risk-neutral valuation and absolute risk-averse valuation, corresponding to $\alpha \to 0$ and $\alpha \to +\infty$, respectively. As was commented earlier, risk aversion can translate into a price spread over the risk-neutral price. Risk-neutral valuation hence plays a central role for option valuation and deserves more attention.

4.1. **Risk neutral valuation.** The algorithm described earlier is not valid for risk neutral valuation when $\alpha \to 0$. Using the duality, we can show that the option value is instead given by

$$v = E^{Q_0}[V_T], \tag{30}$$

where, in the discrete setting, Q_0 is the minimal entropy martingale measure (MEMM) defined by

$$\begin{aligned} \frac{dQ_0}{dP} =& \frac{exp\{\sum_{i=0}^{N-1} \tilde{\Delta}_i^*(S_{i+1} - S_i)\}}{E^P[exp\{\sum_{i=0}^{N-1} \tilde{\Delta}_i^*(S_{i+1} - S_i)\}]} \\ =& \frac{exp\{N \cdot c_0(\frac{S_1}{S_0} - 1)\}}{E^P[exp\{N \cdot c_0(\frac{S_1}{S_0} - 1)\}]} \\ =& \frac{exp\{N \cdot c_0 \frac{S_1}{S_0}\}}{E^P[exp\{N \cdot c_0 \frac{S_1}{S_0}\}]}. \end{aligned} \tag{31}$$

The right hand side can be interpreted as the Radon-Nikodym derivative for path probabilities. Expression (31) is useful for Monte Carlo simulation. Its one-period version, meanwhile, is the key for valuation with a multi-nomial tree. Take the quadri-nomial tree for example, the minimal entropy martingale probabilities for branching out are given by

$$q_i = p_i \frac{exp\{c_0 \frac{S_{i,1}}{S_{00}}\}}{E^P[exp\{c_0 \frac{S_{\cdot,1}}{S_{00}}\}]}, \quad i = -J, -1, 1, J.$$

The risk-neutral value can be calculated by backward induction from maturity. The algorithm is much simpler than the previous one.

`/* Algorithm for risk-neutral valuation */`

`/* Compute the MEM probabilities */`

  `Solve` c_0 `from` $0 = E[(\frac{S_{\cdot,1}}{S_{00}} - 1)exp(c_0(\frac{S_{\cdot,1}}{S_{00}} - 1))]$,

  `For` $i = -J, -1, 1$ `and` J `compute` $q_i = p_i exp\{c_0 \frac{S_{i,1}}{S_{00}}\}/E^P[exp\{c_0 \frac{S_{\cdot,1}}{S_{00}}\}]$.

`/* Backward induction */`

  `For` $j = N-1, N-2, \ldots, 0$

    `For` $i = -3j-1, \ldots, 3j+1$

      $v_{ij} = q_{-J} v_{i-J,j} + q_{-1} v_{i-1,j} + q_1 v_{i+1,j} + q_J v_{i+J,j}$;

    `end`

  `end`

  `Take` v_{00} `as the option value.`

```
/* The end of the algorithm */
```

Note that the induction scheme reduces to Cox-Ross-Rubinstein (1976) binomial scheme if $\lambda = 0$.

Risk-neutral valuation does not render the hedge ratio. Similar to what we did in continuous-time case, we can consider minimal variance hedge. The hedge ratio is given by

$$\begin{aligned}\Delta_{00} &= \frac{Cov_P(S_{\cdot,1}, v_{\cdot,1})}{Var_P(S_{\cdot,1})}\\ &= \frac{\sum p_i(v_{i,1} - \bar{v}_1)(S_{i,1} - \bar{S}_1)}{\sum p_i(S_{i,1} - \bar{S}_1)^2},\end{aligned}$$

where the summation is made for $i = -J, -1, 1$ and J, and the bars indicate mean values under the objective measure:

$$\bar{S}_1 = \sum p_i S_{i,1}, \quad \text{and} \quad \bar{v}_1 = \sum p_i v_{i,1}.$$

The computation is very simple.

4.2. **Absolutely risk-averse valuation.** For $\alpha \to +\infty$ the seller will tolerate no risk of loss at all. In this limit the valuation will take a very different form, and it is given in the following proposition.

Proposition 4.2. *For $\alpha \to +\infty$, v_α tends to the limit*

$$v = \inf_{\{\Delta_i\}} \sup_{P(S)} \left\{ V(S_N) - \sum_{i=0}^{N-1} \Delta_i(S_{i+1} - S_i) \right\}$$

Step-wise, the valuation takes the form

$$V(S_i, t_i) = \inf_{\Delta_i} \sup_{S_{i+1}} \left\{ V(S_{i+1}, t_{i+1}) - \Delta_i(S_{i+1} - S_i) \right\}, \tag{32}$$

and $v = V(S_0, 0)$.

Proof: From the definition of utility indifference price we know

$$e^{v_\alpha} = \left(J^{(1)}(S_0, 0)\right)^{\frac{1}{\alpha}} / \left(J^{(0)}(S_0, 0)\right)^{\frac{1}{\alpha}}, \tag{33}$$

with

$$\begin{aligned}J^{(1)}(S_i, t_i) &= \min_{\{\Delta_i\}} E^P_{t_i}\left[exp\left(\alpha(-\sum_{j=i}^{N-1} \Delta_j(S_{j+1} - S_j) + V(S_N)) \right) \right],\\ J^{(0)}(S_i, t_i) &= \min_{\{\tilde{\Delta}_i\}} E^P_{t_i}\left[exp\left(\sum_{j=i}^{N-1} \tilde{\Delta}_j(S_{j+1} - S_j)) \right) \right].\end{aligned}$$

Apparently that in the limit $\alpha \to +\infty$, the second term in (33) tends to 1. For the first term, we quote the well-known result: for a continuous function $g(x)$ defined on $\Omega \subset R^n$,

$$\lim_{\alpha\to\infty}\left(\int_\Omega |g(x)|^\alpha d\mu(x)\right)^{\frac{1}{\alpha}} = \|g\|_\infty = \sup_{x\in\Omega}|g(x)|,$$

where $\mu(x)$ is a measure over Ω. Hence we obtain

$$\left(J^{(1)}(S_0, 0)\right)^{\frac{1}{\alpha}} \to \inf_{\{\Delta_i\}}\sup_{\{S_i\}}\left\{V(S_N) - \sum_{i=0}^{N-1}\Delta_i(S_{i+1} - S_i)\right\}.$$

The step-wise scheme follows obviously. The proof is then completed $\square$

Apparently scheme (32) produces the premium for the minimal super-replication. Such scheme was first introduced by Britton-Jones and Neuberger (1997). Note that Britton-Jones and Neuberger used the so-called "variance-adjusted time" such that the variation along all paths equals to a constant. The "time" we use in (32) is the usual calendar time and the scheme carries flexibility for more volatility and jump-intensity structures. Scheme (28) poses a linear programming problem for the "convex hull" and its implementation is easy.

5. Numerics — smile, skew, spread and risk aversion

We devote this section to the pricing experiments of the utility indifference option valuation model developed in the previous sections. We will examine the model on several aspects. First, we want to see how the option price changes in response to the changes in major input parameters, including rate of return, risk aversion, jump intensity and jump size. In particular, we want to see how the shape of volatility smile or skew varies in response to the change in the mean jump size, and compare with what was observed in the market place. For $\alpha = 1$, a particular level of risk aversion, we will compare the optimal trading strategy suggested by the new model with the Black-Scholes delta calculated with implied volatilities. Through the deviation of the risk averse prices from the risk neutral price, we will acquire a new understanding of the relation between the spread and the risk aversion: the former is a monotonic function of the latter. It may be an interesting question to establish an explicit functional relation between the two quantities.

Without loss of generality, we consider options on a underlying asset which follows the jump-diffusion process with the bivariate jumps. The fixed parameters of the jump-diffusion process are

- spot asset price $S_0 = 1$;
- annualized return in the absence of jump $\mu = 10\%$;

- annualized volatility for diffusion $\sigma = 25\%$;
- jump intensity $\lambda = 12$.

We will take various jump sizes and mean jump sizes. Note that $\lambda = 12$ means on average one jump per month. It is very big compared to the jump intensities used in other studies (see for example, Andersen and Andreasen, 2000). We consider call options across a range of strikes yet with a fixed maturity. In specific, they are

- maturity $T = 1$ months or $1/12$ year;
- strikes X: from 85% to 115% of S_0.

In our numerical scheme we take time-step size $\delta t = 1/365$. We will divide the results into five examples. Please note that, to abide to market convention, we quote option prices in terms of their Black's implied volatilities.

Example 1: Rate of return and option value. In the complete market driven by Brownian diffusion, the drift term μ_t does not enter pricing. Conceivably, it is no longer the case in an incomplete market. The question then is: how important is the drift term? Or how sensitive is an option price to the drift term? Figure 4 offers some clues to the answer. It appears that option values are insensitive to but not independent of the change in the rate of return. A more careful analysis may be useful. In this calculation the mean jump size is $k = -0.05$, which is responsible for the skew.

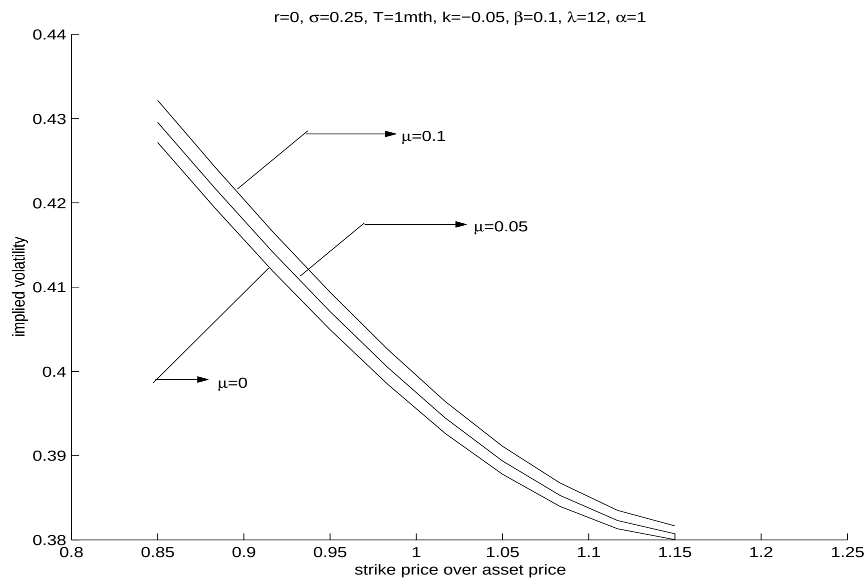

FIGURE 4. Price sensitivity to the rate of return

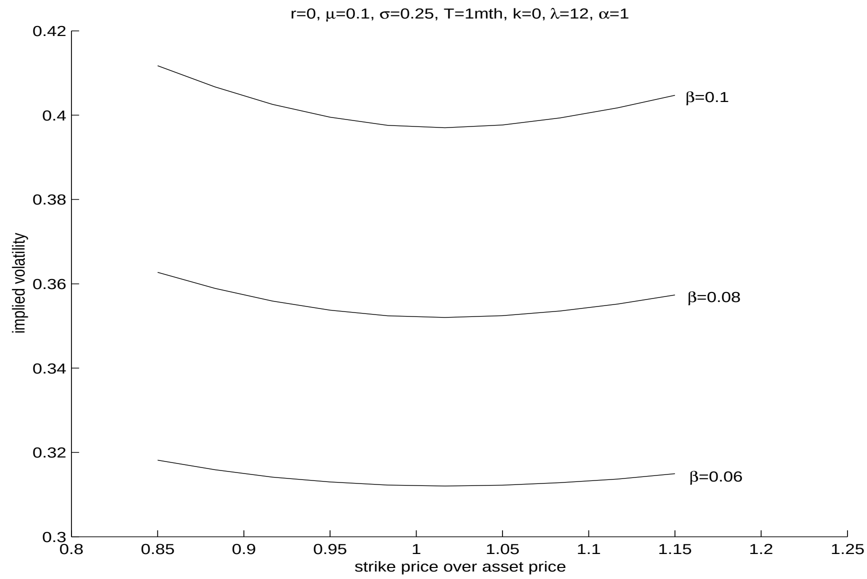

Figure 5. Implied vol. v.s. jump size

Example 2: Jump size and option value. In this example we examine the Black's volatilities for increasing jump sizes: $\beta = 6\%, 8\%$ and 10%, while taking mean jump size $k = 0$. The results are displayed in Figure 5. Figure 5 shows that the model has the property that the option value is proportional to the anticipated jump size. For zero mean jump sizes, the implied volatility curves appear like smiles. This is consistent with practical observations.

Example 3: Volatility skews and "crash-o-phobia". We would like to use this example to explain the so-called "crash-o-phobia": as the downward jump risk mounts, the in-the-money calls or out-of-the money puts will become more valuable (in terms of the implied volatility). For different mean jump size k, we calculate the option values and then translate to the Black's volatilities. The results are shown in Figure 6. It shows that, as the mean jump size of downward jump increases, the volatility skew becomes steeper. Such pattern of variation is in good agreement what is observed in the market place.

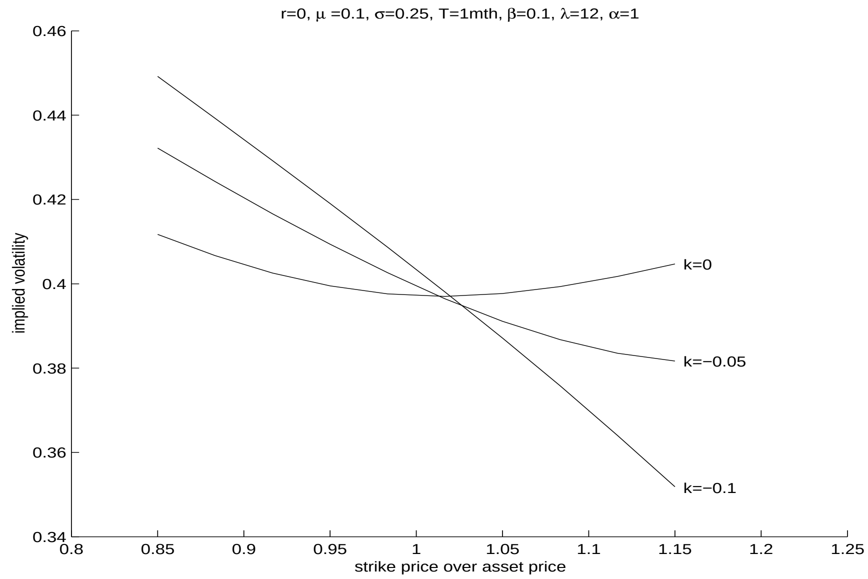

FIGURE 6. Expected jump size and smile/skew

Example 4: Optimal hedging v.s. the Black's delta hedging. In financial engineering it is very important to establish a "good" hedge to a risky position. Hence, a comparison between the optimal hedging ratio recommenced by the new model and the classic Black's delta is meaningful and may shed some light on the search of better hedging strategy. Figure 7 actually contains four curves, as the two Black's delta curves almost coincide. The Black's delta is calculated from the implied Black's volatility. The risk aversion parameter for utility indifference pricing is $\alpha = 1$. The results suggests that if the return of the stock is higher, the hedger should take bigger hedging ratio. This is meaningful since under the new model, the delta is for optimal investment instead of hedging. The Black's delta, meanwhile, is very insensitive to the rate of asset return. Also, when there is no excess of return over the risk-free rate, the delta suggested by the model is very close to the Black's delta. This is not surprising since Black's delta aims at eliminating any excess of return.

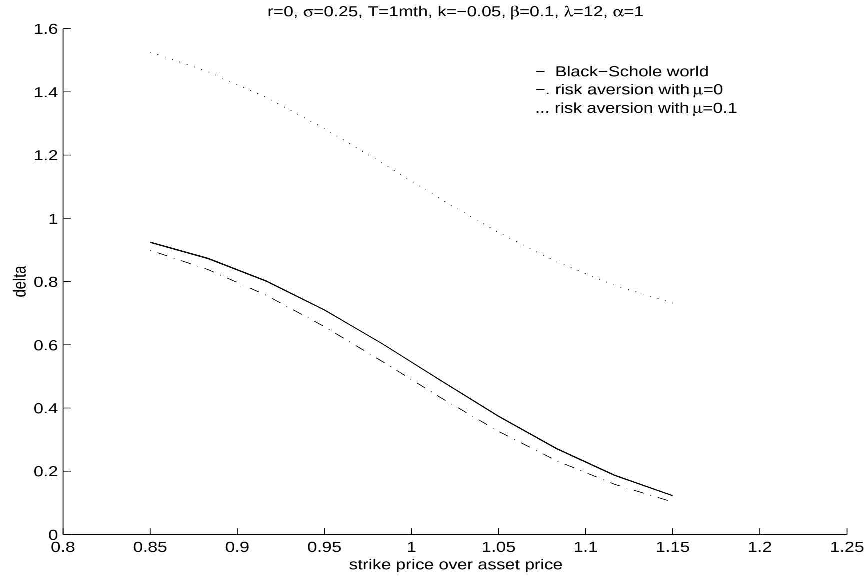

FIGURE 7. Comparison of hedging ratios

Example 5: Spread over risk-neutral price and risk aversion. In this example we examine how option prices change in response to the change of risk aversion. It is obvious that the risk neutral value is the lower bound for the risk-averse price. When $\alpha \to 0$, the risk averse price converges to the risk neutral price. When α increases, the gap between the risk-averse price and risk neutral price widens. Since the absolutely risk-averse price is finite, there is a finite asymptotic limit to the implied volatilities for $\alpha \to 0$. The interval bounded by the implied volatilities of risk-neutral price and absolute risk-averse price defines the range of non-arbitrageable prices. From this figure, we may think that the risk aversion as a generator of spread over the the risk-neutal price in terms of the implied volatility.

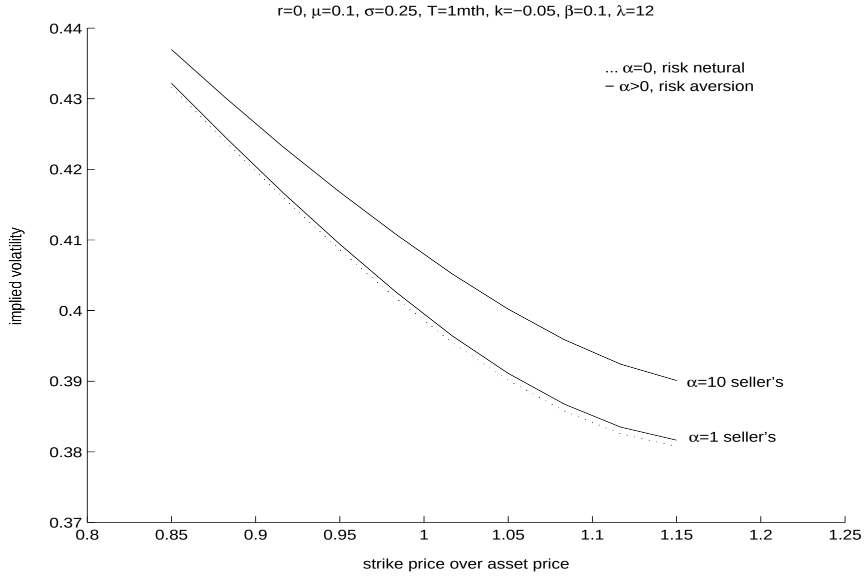

FIGURE 8. Relation between risk aversion and prices

6. American options

We briefly discuss the pricing of American options. With an American option a holder can choose to exercise early so as to maximize his profit, which is exactly against the interest of the writer. Let S_t remain to be the forward price of the underlying for delivery at maturity T, θ be the stopping time (i.e., exercise time), and $V(S_\theta)$ be the time-θ forward price of the intrinsic value of an American option, then the utility indifference price[1] to the writer is defined by

$$\sup_{\{\Delta_\tau\}} \inf_{\theta} E^P\left[-\exp\left(-\alpha[v+\int_0^T \Delta_\tau dS_\tau - V(S_\theta)]\right)\right]$$
$$= \sup_{\{\Delta_\tau\}} E^P\left[-\exp\left(-\alpha\int_0^T \Delta_\tau dS_\tau\right)\right], \quad (34)$$

[1]We try to keep all notations for European options

where v is the forward price of the option seen at time $t = 0$. From the above definition, we can solve and obtain the forward price of the option as

$$
\begin{aligned}
v = & \frac{1}{\alpha}\left(\ln \inf_{\{\Delta_\tau\}} \sup_\theta E^P\left[\exp\left(-\alpha[\int_0^T \Delta_\tau dS_\tau - V(S_\theta)]\right)\right]\right. \\
& \left. - \ln \inf_{\{\Delta_\tau\}} E^P\left[\exp\left(-\alpha\int_0^T \Delta_\tau dS_\tau\right)\right]\right). \\
= & \frac{1}{\alpha}\left(\ln \inf_{\{\tilde{\Delta}_\tau\}} \sup_\theta E^P\left[\exp\left(\int_0^T \tilde{\Delta}_\tau dS_\tau + \alpha V(S_\theta)\right)\right]\right. \\
& \left. - \ln \inf_{\{\tilde{\Delta}_\tau\}} E^P\left[\exp\left(\int_0^T \tilde{\Delta}_\tau dS_\tau\right)\right]\right). \\
\equiv & \frac{1}{\alpha}\left(\ln J^{(1)}(S_0, 0) - \ln J^{(0)}(S_0, 0)\right),
\end{aligned}
\tag{35}
$$

where

$$
J^{(1)}(S_t, t) = \inf_{\{\tilde{\Delta}_\tau\}} \sup_\theta E^P\left[\exp\left(\int_t^T \tilde{\Delta}_\tau dS_\tau + \alpha V(S_\theta)\right)\right],
$$

and $J^{(0)}(S_t, t)$ is the same function as that for a European option. The above expression can be evaluated numerically. We now take specifically an American put option for example (It is well-known that it is never optimal to exercise an American call option early in the absence of dividend, disregard the underlying process). Let X be the strike price, then the forward price of the intrinsic value (or payoff) of the option seen at time θ is

$$
V(S_\theta) = \max(e^{r(T-\theta)}X - S_\theta, 0),
$$

where the risk-free rate r enters the pricing procedure explicitly. Let us consider discrete exercise at times with gap δt. Using arguments similar to those for European option pricing, we can derive

$$
J^{(1)}(S_t, t) = \max\left(J^{(0)}(S_t, t) \cdot e^{\alpha V(S_t)},\ \min_{\tilde{\Delta}_t} E_t\left[exp(\tilde{\Delta}_t(S_{t+\delta t} - S_t)J^{(1)}(S_{t+\delta t}, t + \delta t)\right]\right). \tag{36}
$$

If the first term dominates, then we end up with the intrinsic value for the American option, indicating an early exercise at time t for price S_t. In fact, equation (36) ensures the no-arbitrage condition such that the values of American options are always bigger or equal to their intrinsic values.

Figure 9 offers a comparison between the American and European put options. The data used are indicated in the figure where $r = 0.05$. One can see that the value curve of American option stays above the intrinsic value, and pastes smoothly to the latter as asset price decreases. Figure 10 displays the early exercise boundary as the function of time-to-maturity. Due to the possibility of jumps, the limit of the exercise boundary as time to maturity approaches 0 is strictly less than the strike price. This numerical result is in consistence with that of Amin (1993).

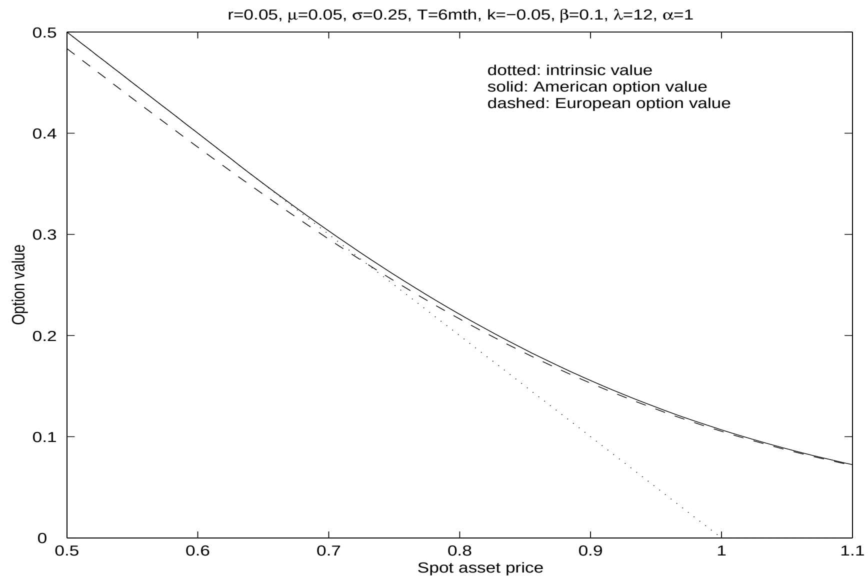

FIGURE 9. American put option value v.s. European put option value

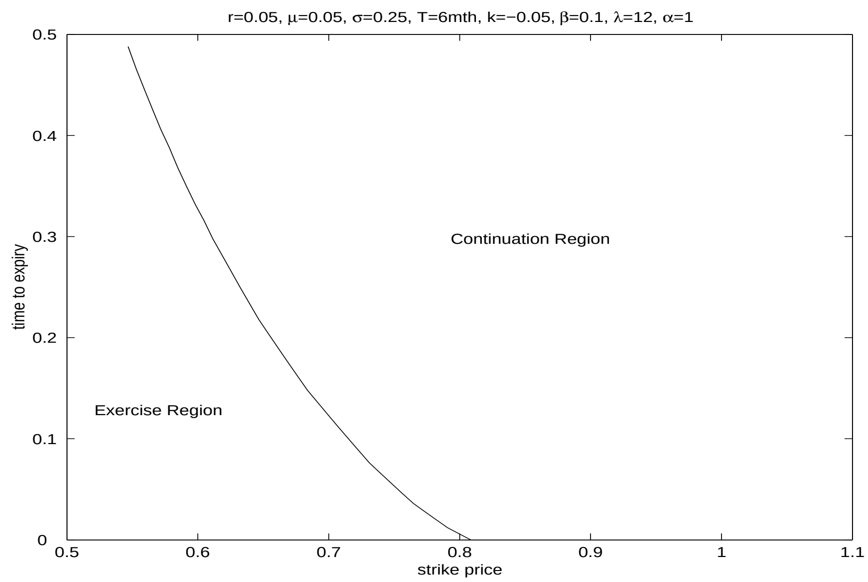

FIGURE 10. Early exercise boundary

7. Conclusion

In this paper we have studied option pricing-hedging with a jump-diffusion process along the approach of utility maximization. Option prices are defined according to the principle of utility indifference , while hedging is made through taking the optimal investment strategy under risk aversion. We have made a complete characterization of the *risk neutralized jump-diffusion process* which plays the central role in option pricing in the incomplete market represented by the jump-diffusion processes. Also, we have extended the Merton's option formula to time-dependent jump intensity. Parallel study was made on the discrete setting where hedging takes place in discrete moments. A powerful and robust scheme was proposed to valuate options with a multi-nomial tree, which can approximate a jump-diffusion process. Numerical examples show that utility indifference prices react reasonably to the changes in input parameters. The quality of computational output is excellent and the schemes show remarkable robustness and efficiency.

The implementation with multi-nomial tree can be extended to Monte Carlo simulation. For risk-neutral valuation, in particular, we will reproduce the weighted Monte Carlo method by Avellaneda *et. al.* (1998), where diffusion is taken as the underlying driving dynamics. It will be very helpful to find an efficient way to calculate the minimal variance hedge ratio with the Monte Carlo method.

The idea of this paper can be extended to price and hedging a portfolio of options on multiple assets. For that purpose we will have to replace the exponential utility function by some other utility functions which 1) reflect constant risk aversion and 2) can take negative arguments. A piece-wise quadratic-log function, for example, may be a candidate for such utility function. Föllmer (1999) had considered maximizing the log utility for the shortfall of a hedged portfolio. Still, there are a lot of work remain to be done in this area.

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