JOURNAL OF INDUSTRIAL AND
MANAGEMENT OPTIMIZATION
Volume **2**, Number **2**, May **2006**

Website: http://AIMsciences.org

pp. **199–227**

LIBOR MARKET MODEL WITH STOCHASTIC VOLATILITY

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Abstract. In this paper we extend the standard LIBOR market model to accommodate the pronounced phenomenon of implied volatility smiles/skews. We adopt a multiplicative stochastic factor to the volatility functions of all relevant forward rates. The stochastic factor follows a square-root diffusion process, and it can be correlated to the forward rates. For any swap rate, we derive an approximate process under its corresponding forward swap measure. By utilizing the analytical tractability of the approximate process, we develop a closed-form formula for swaptions in term of Fourier transforms. Extensive numerical tests are carried out to support the swaptions formula. The extended model captures the downward volatility skews by taking negative correlations between forward rates and their volatilities, which is consistent with empirical findings.

1. **Introduction.**

In this paper we extend the standard LIBOR market model (Brace, Gatarek and Musiela, 1997; Jamshidian, 1997; and Miltersen, Sandmann and Sondermann, 1997) by adopting stochastic volatilities. Over the past decade the market model, which is based on lognormal assumption for forward rates, has established itself as the benchmark model for interest-rate derivatives. One of many virtues of the market model is that it justifies the use of Black's formula for caplet and swaption prices, which has long been a standard market practice. The Black's formula establishes a relationship between option prices and local volatilities of the forward rates, and such a relationship has enabled fast calibration of the standard model (Wu, 2003). Nevertheless, the standard market model is also known for its limitation: it only generates flat implied volatility curves, whereas the implied volatility curves observed in LIBOR markets often have the shape of a smile or skew. The implication is that, after being calibrated to at-the-money options, the model underprices off-the-money options. Because the benchmark role of caps and swaptions in the fixed-income derivatives markets, there have been great interests in

2000 *Mathematics Subject Classification.* 60J60, 90C47.

Key words and phrases. LIBOR model, stochastic volatility, square-root process, swaptions, Fast Fourier transform (FFT).

This research is partially supported by RGC Grant HKUST6145/01P. Early versions of this paper were presented in Ecole Polytechnique (June 2002), Stanford University (September 2002), University of Texas at Austin (November 2002), Quantitative Methods in Finance 2002 (December), BNP Paribas (February 2003, New York), The Annual Meeting of Canadian Mathematical Society of 2003, and other occasions. We would like to thank Alan Brace, Rama Cont, Paul Glasserman, George Papanicolaou, Thaleia Zariphopoulou for helpful comments, and Mr. Kalok Chau of HSBC for supplying data. We are responsible for any remaining errors.

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extending the standard model so as to fix the problem of underpricing, or, speaking in terms of implied volatilities, capture the smiles and skews.

There have been mainly three strands of extensions to the standard market model, and each of them is based on a stochastic process more general than the lognormal process for forward rates (the state variables). Andersen and Andreasen (2000a) adopt Constant Elasticity Variance (CEV) processes, which can generate either monotonically decreasing or increasing implied volatility skews. Paralleling to the standard model, the CEV model renders closed-form pricing for options, in terms of χ^2 functions, yielding thus a price - local volatility relationship similar to that under the standard market model. Owning to such similarity, calibration methodology for the standard model is essentially applicable to the CEV model (Wu, 2003). Joshi and Rebonato (2002), instead, take displaced diffusion (DD) processes, which also generate monotonic volatility skews. Closed-form pricing under such a model is, however, limited to caplets. A major problem with both CEV and DD models is the monotonicity of the implied volatility skews, which is often not the case in reality. The remedy to such a problem has been to adopt stochastic volatilities (Joshi and Rebonato, 2002; Andersen and Brotherton-Ractliffe, 2002), which effectively produce additional curvature to the implied volatility curves. The use of stochastic volatilities increases the capacity of the models, making them capable of fitting even “hockey stick” shaped volatility curves, which had appeared in the aftermath of Rubble crisis in 1998. The third strand of extension is represented by the jump-diffusion model of Glasserman and Kou (2000) and Glasserman and Merener (2001). With such a model one can manipulate the slope and the curvature of a skewed smile through changing jump intensity and jump sizes. One comparative advantage of the jump-diffusion model is its ability to generate sharp short-term skews for caplets and swaptions, which is not rare in the markets. A recent generalization along this line is made with Lévy processes (Eberlein and Özkan, 2004). Other interesting extensions include a model based on mixed lognormal densities for LIBOR (Brigo and Mercurio, 2003).

In financial literature, an implied volatility smile/skew is directly linked to the leptokurtic feature[1] in the return distribution of a state variable. Such a feature has been commonly seen in many financial time series data, including interest rates and equities. With prices of many financial instruments, empirical studies have identified stochastic volatilities and jumps as the two primary factors responsible to such a feature. In the LIBOR derivatives markets, however, the dynamics of stochastic volatilities is believed to dominate that of jumps (Chen and Scott, 2001), yet such dominance is not reflected in any of the models mentions above. In this paper, we will fill this gap and develop a genuine stochastic volatility model for the LIBOR derivatives. Specifically, we will make an extension to the standard market model in the spirit of the Heston’s model (1993) for equity options. The latter is known for its sound empirical properties as well as analytical tractability[2].

Our LIBOR version of the Heston’s option model takes the following form: a single multiplicative stochastic factor is adopted for forward-rate volatilities, and the factor follows a square-root process. Similar multiplicative stochastic factors have been considered in Brace (2000) and Chen and Scott (2001), and, especially, in the aforementioned work of Andersen and Brotherton-Ratcliffe (2001). In formalism, the model we take appears like a special case (corresponding to elasticity constant

[1]Higher peak and fatter tails than those of normal distribution.

[2]See Lewis (2000) for an excellent survey of the models.

one) of the CEV LIBOR model with stochastic volatilities , but our handling of stochastic volatilities renders the model novel in the context of LIBOR: we allow correlations between the forward rates and the stochastic factor (hereafter called "rate - factor correlation" for short). Under our model, the "volatility of volatility" determines the curvature of an implied volatility smile/skew, while the rate - factor correlation is responsible for its sloping. Specifically, a downward volatility skew is associated to a negative rate - factor correlation. Such a mechanism for the implied volatility smiles/skews is much more plausible to practitioners.

Other than the appealing mechanism for the volatility smiles/skews, a major technical contribution of this paper is the closed-form pricing for caplets and swaptions. It is well known that the key to the analytical tractability of the Heston's equity model lies in the existence of analytical moment generating function. For the LIBOR version of the Heston's model, however, analytical moment generating function does not exist. Subtle treatments are thus made to approximate forward rate and forward swap-rate processes, after changes of measure, by Heston's type processes. Extensive numerical studies and comparisons are subsequently carried out to justify the approximations. For numerical swaption pricing, we have adopted the technique of Carr and Madam (1998) and utilized fast Fourier transforms (FFT), which then enables marking-to-market cap and swaption prices in real-time.

The model presented in this paper can serve as a stepping stone for more comprehensive models. It is parsimonious and has the property of time-stationarity. Sometimes, there may be needs to account for additional sources of risks so as to enhance the empirical properties of the model. For example, it helps to include a jump component in order to generate a steep downward volatility skew, as otherwise we may occasionally need to use an unrealistically large negative correlation. A possible extension is a model based on time-changed Lévy processes, which effectively takes both stochastic volatilities and jumps into account.

The remaining part of the paper will be organized as follows. In section 2 we lay out the LIBOR market model with stochastic volatility, and derive approximate formula for caplet prices. In section 3 we address swaption pricing, where we introduce the necessary approximations to retain analytical tractability. In section 4 we discuss the analytical solution of the moment generating function under the assumption of piecewise constant rate - factor correlations. In section 5 we introduce the FFT method for numerical option valuation. In section 6 we present pricing comparisons between our closed-form swaption formula and Monte Carlo method, and study the relation between volatility smiles/skews and the rate - factor correlations. Finally in section 7 we conclude. Most technical details are put in the appendix.

2. **The Market Model with Stochastic Volatility.**

The derivation of market model can start from the price process of zero-coupon Treasury bonds. Let $P(t,T)$ be the zero-coupon Treasury bond maturing at $T > t$ with par value \$1. Under the risk-neutral measure, denoted by $\mathbb{Q}$, $P(t,T)$ is assumed to follow the lognormal process

$$dP(t,T) = P(t,T)\left(r_t dt + \sigma(t,T)\cdot d\mathbf{Z}_t\right), \tag{2.1}$$

where r_t is the risk-free rate, $\sigma(t,T)$ is the volatility vector of $P(t,T)$, and $\mathbf{Z}_t$ is vector of independent Wiener processes under the risk-neutral measure, and "$\cdot$" means scalar product.

Using Ito's lemma we can derive the equation for $\ln P(t,T)$:

$$d\ln P(t,T) = \left(r_t - \frac{\sigma(t,T)\cdot\sigma(t,T)}{2}\right)dt + \sigma(t,T)\cdot d\mathbf{Z}_t.$$

Differentiating the above equation (assuming adequate regularity) with respect to maturity T, we obtain

$$df(t,T) = \sigma_T \cdot \sigma dt - \sigma_T \cdot d\mathbf{Z}_t, \tag{2.2}$$

where

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}$$

is the instantaneous forward rate maturing at T, and

$$\sigma_T(t,T) = \frac{\partial\sigma(t,T)}{\partial T} \quad \text{or} \quad \sigma(t,T) = \int_t^T \sigma_T(t,s)ds. \tag{2.3}$$

Equation (2.2) is the well-known Heath-Jarrow-Morton equation (Heath *et al.*, 1992), which states that, under the risk neutral measure, the drift term of the forward rate is a function of its volatility. The HJM model is also regarded as a framework for arbitrage-free models. By choosing specific volatility function for the forward rate, one can deduce various interest-rate models, including Ho-Lee (1977), Cox-Ross-Ingersoll (1985), Hull-White (1990) and other popular ones.

Although theoretically appealing, the HJM model is not convenient for applications. There are mainly two reasons for that. First, it takes the instantaneous forward rate, a non-observable quantity, as state variable. This increases the difficulty in yield curve construction and model calibration. Second, under the HJM model the pricing of most derivative securities, including the benchmark securities (caps, floors and swaptions), has to resort to Monte Carlo simulations. In market places, meanwhile, the pricing of those benchmark securities has instead been done with the Black's formula (Black, 1976). For some time such market practice had been considered inconsistent with the HJM theory.

The works by Brace-Gatarek-Musiela (1997), Jamshidian (1997), and Miltersen, Sandmann & Sondermann (1997) reconciled practice with theory and shifted the paradigm of the interest-rate researches. These researchers chose to model the forward term rates directly, while observing the no-arbitrage principle of the HJM framework. The state variables under the new paradigm are the arbitrage-free forward lending rates seen at time t for the period (T_j, T_{j+1}), denoted by $f_j(t) = f(t; T_j, T_{j+1}), \forall j$, which are observable and tradable quantities in the interest-rate markets (through e. g. *forward-rate agreements*). The forward rate, $f_j(t)$, relates to the zero-coupon bonds by

$$f_j(t) = \frac{1}{T_{j+1}-T_j}\left(\frac{P(t,T_j)}{P(t,T_{j+1})} - 1\right).$$

As a function of two zero-coupon bonds, the dynamics of $f_j(t)$ is determined by those of the zero-coupon bonds. By Ito's lemma we can derive

$$df_j(t) = f_j(t)\gamma_j(t)\cdot[d\mathbf{Z}_t - \sigma(t,T_{j+1})dt], \qquad 1 \leqslant j \leqslant N, \tag{2.4}$$

where $\gamma_j(t)$ is a function of zero-coupon bond volatilities:

$$\gamma_j(t) = \frac{1+\Delta T_j f_j(t)}{\Delta T_j f_j(t)}[\sigma(t,T_j) - \sigma(t,T_{j+1})], \tag{2.5}$$

and it is regarded as the volatility vector for $f_j(t)$. Under the risk neutral measure, therefore, the volatilities of the forward term rates are determined by those of the zero-coupon bonds. Such a relation can be viewed conversely. In fact, under the new paradigm we begin with prescribing the volatilities of the forward rates. The volatilities of the zero-coupon bonds are then obtained by inverting (2.5):

$$\sigma(t,T_{j+1}) = -\sum_{k=1}^{j} \frac{\Delta T_k f_k(t)}{1+\Delta T_k f_k(t)} \gamma_k(t) + \sigma(t,T_1), \tag{2.6}$$

where $\Delta T_j = T_{j+1} - T_j$. Under usual regularity conditions for $\gamma_j(t)$, Brace *et al.* (1997) proved that $f_j(t)$ does not blow up, and justified that one can set $\sigma(t,T_1) = 0$ for $t \leqslant T_1$. Equations (2.4,2.6) constitutes the so-called *market model* of interest rates. Roughly speaking, the stochastic evolution of the N forward rates is governed by covariance set

$$Cov^i_{jk} = \int_{T_{i-1}}^{T_i} \gamma_j(t)\cdot\gamma_k(t)dt, \qquad i \leqslant j,k \leqslant N, \quad 1 \leqslant i \leqslant N. \tag{2.7}$$

Note that $Cov^i_{jk} = 0$ for either $j < i$ or $k < i$ since either f_j or f_k has been reset by the time T_i (corresponding to either $\gamma_j = 0$ or $\gamma_k = 0$). The market model has the capacity to build in desirable correlation structure between the forward rates (Wu, 2003).

Under the market model, the use of the Black's formula for caplets is nicely justified. Swaptions, meanwhile, can also be priced with an approximate formula (see for instance Andersen and Andreasen (1998) and Sidenius (1999)) which is accurate within the bid-ask spread. However, this model typically cannot reproduce the market prices of caplets and swaptions, or in other words, cannot be calibrated to the implied volatility smiles or skews of those benchmark derivatives, exemplified in Figures 1 to 7 using the US dollar data of July 05, 2002 [3]. The capacity of the model is actually restricted by the specification that $\{\gamma_j(t)\}$ are deterministic functions. To accommodate the implied volatility smiles or skews, we need to relax such a specification.

In this paper, we consider taking stochastic volatilities. In specifics, we adopt a stochastic factor for forward-rate volatilities, and let this factor follow a square-root process under the risk neutral measure:

$$\begin{aligned} df_j(t) &= f_j(t)\sqrt{V(t)}\gamma_j(t)\cdot[d\mathbf{Z}_t - \sqrt{V(t)}\sigma_{j+1}(t)dt], \\ dV(t) &= \kappa(\theta - V(t))dt + \epsilon\sqrt{V(t)}dW_t, \end{aligned} \tag{2.8}$$

where κ, θ and ϵ are state-independent variables (ϵ is not necessarily a small number), W_t is an additional one-dimensional Brownian motion. Note that for the extended model (2.8), equation (2.5) remains the arbitrage-free condition. Unlike the extended CEV model of Andersen and Brotherton-Ratcliffe (2000), we allow correlation between the forward rates and the stochastic factor:

$$E^Q\left[\left(\frac{\gamma_j(t)}{\|\gamma_j(t)\|}\cdot d\mathbf{Z}_t\right)\cdot dW_t\right] = \rho_j(t)dt, \quad \text{with} \quad |\rho_j(t)| \leqslant 1. \tag{2.9}$$

Here, $(\gamma_j(t)/\|\gamma_j(t)\|\cdot d\mathbf{Z}_t)$ can be regarded as the differential of a Brownian motion that drives $f_j(t)$. The distributional properties of $V(t)$ are well understood

[3]The supply of data by Dr. Kalok Chau of HSBC (Hong Kong) is gracefully acknowledged

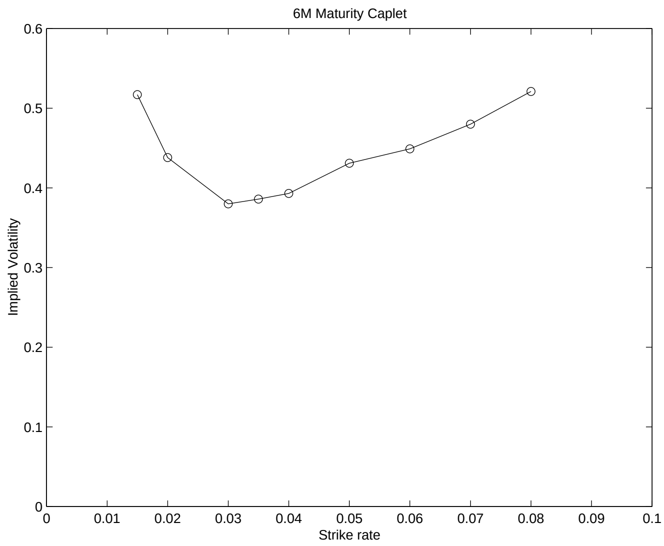

FIGURE 1. Implied volatility of 6M maturity caplet

(e.g., Avellaneda and Laurence, 2000). When $2\kappa\theta > \epsilon^2$, in particular, $V(t)$ has a stationary distribution and stays strictly positive.

In appearance, our extended market model (2.8) may look like a special case of the CEV LIBOR model (with elasticity one), yet appearance is the only similarity between the two models. To see the differences between the models, we compare their mechanisms for smile/skew generation and their capacity for swaption pricing. With the extended CEV model, volatility skews are generated by taking non-unitary elasticity constant. The use of the volatility factor, which is uncorrelated to the forward rates, is for generating additional curvature to the otherwise monotonic skews. With (2.8), as we shall see, the sloping of the smiles/skews are determined by the rate - factor correlations. Specifically, downward skews and upward skews are generated simply by taking zero, negative and positive correlation respectively[4]. On swaption pricing, the extended CEV model requires a uniform elasticity parameter for all forward-rate processes. Such a requirement is often not met by a CEV model determined by calibration. The extended LIBOR model, (2.8), meanwhile, prices caplets and swaptions in the same framework without requiring a uniform rate - factor correlation. In addition, the numerical option valuation procedures for the two models are also very different. The extended CEV model relies on asymptotic expansion, whenever feasible, while the extended LIBOR model can take advantage of fast Fourier transformations.

We now consider the pricing of a caplet on $f_j(t)$ under the extended LIBOR model (2.8). A caplet is a call option on the forward rate. Assume a \$1 notional value for the caplet, then its payoff is

$$\Delta T_j (f_j(T_j) - K)^+ \stackrel{\triangle}{=} \Delta T_j \max\{f_j(T_j) - K, 0\},$$

[4]The empirical results of Chen and Scott (2002) suggest the independence between the near-term futures rate and its implied volatility. The particular smile shown in Figure 1 for short-maturity caplets is supportive of such a suggestion.

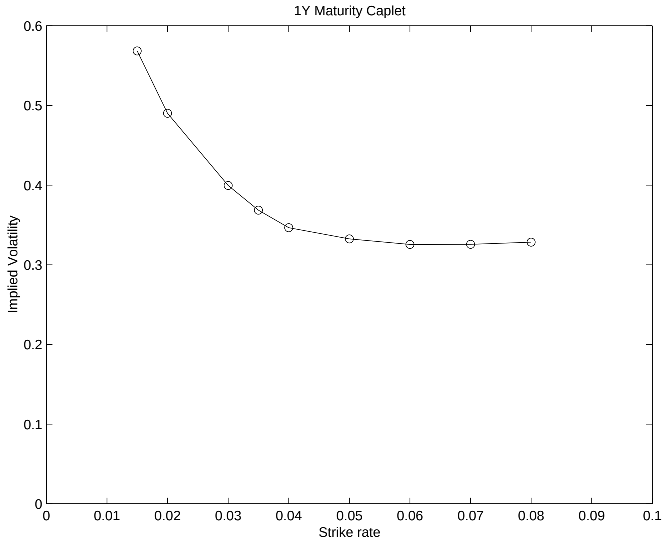

FIGURE 2. Implied volatility of 1Y maturity caplet

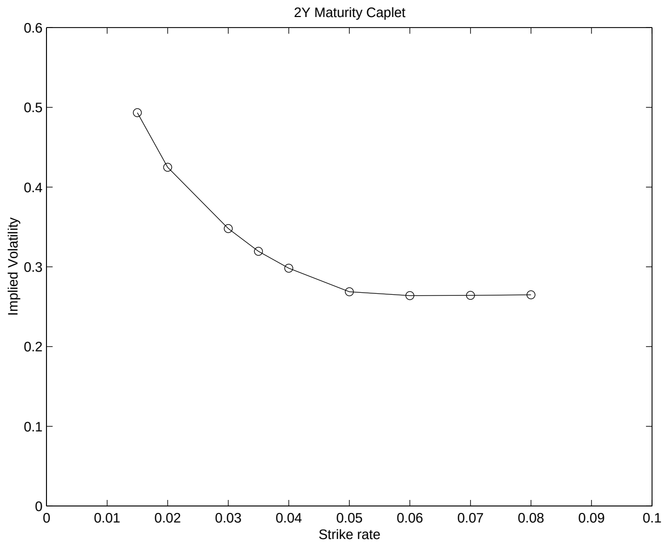

FIGURE 3. Implied volatility of 2Y maturity caplet

where ΔT_j is the corresponding forward period for $f_j(t)$. To price the caplet we choose, in particular, $P(t, T_{j+1})$ as a new numeraire, and let $\mathbb{Q}^{j+1}$ denote the corresponding forward measure. The next proposition establishes the relation between the Brownian motions under the risk-neutral measure and under the forward measure. The proof is put in the appendix.

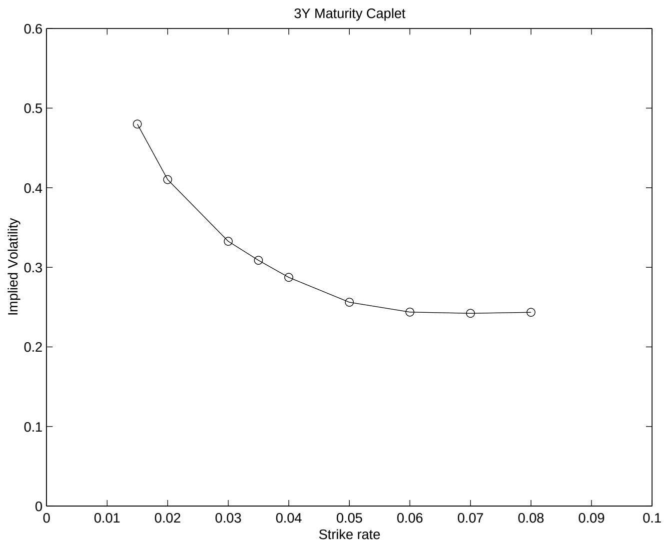

FIGURE 4. Implied volatility of 3Y maturity caplet

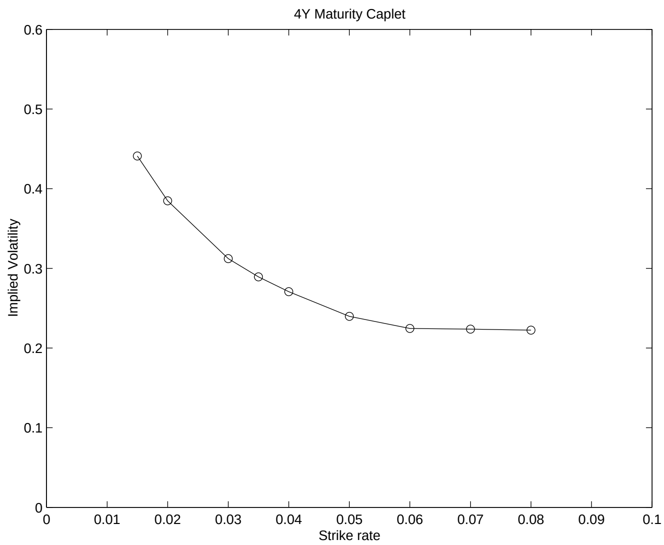

FIGURE 5. Implied volatility of 4Y maturity caplet

Proposition 2.1. *Let* $\mathbf{Z}_t$ *and* W_t *be Brownian motions under* $\mathbb{Q}$*. Define* $\mathbf{Z}_t^{j+1}$ *and* W_t^{j+1} *through*

$$\begin{aligned} d\mathbf{Z}_t^{j+1} &= d\mathbf{Z}_t - \sqrt{V(t)}\sigma_{j+1}(t)dt, \\ dW_t^{j+1} &= dW_t + \xi_j(t)\sqrt{V(t)}dt, \end{aligned} \tag{2.10}$$

where

$$\xi_j(t) = \sum_{k=1}^{j} \frac{\Delta T_k f_k(t)\rho_k(t)\|\gamma_k(t)\|}{1+\Delta T_k f_k(t)},$$

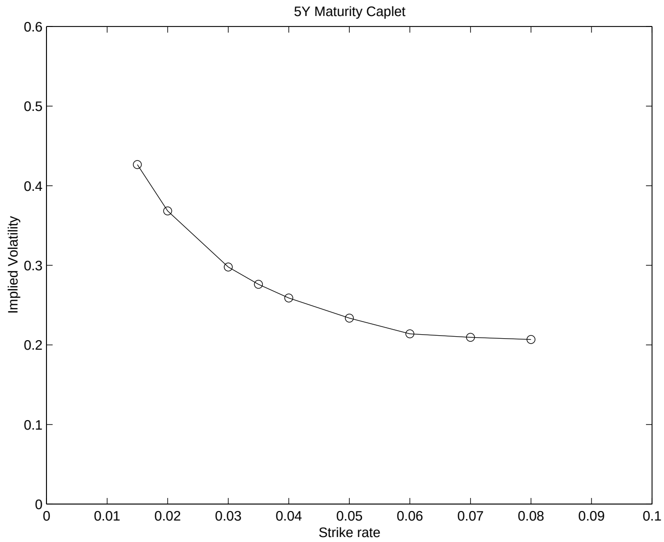

FIGURE 6. Implied volatility of 5Y maturity caplet

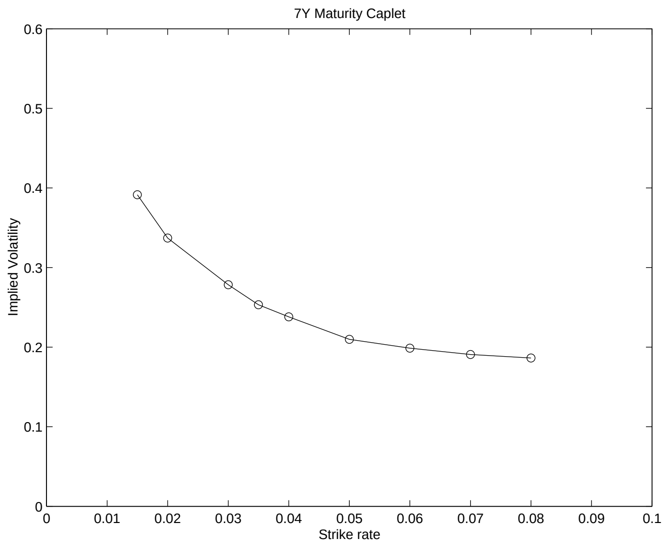

FIGURE 7. Implied volatility of 7Y maturity caplet

then $\mathbf{Z}_t^{j+1}$ *and* W_t^{j+1} *are Brownian motions under* $\mathbb{Q}^{j+1}$.

In terms of $\mathbf{Z}_t^{j+1}$ and W_t^{j+1}, the extended market model (2.8) becomes

$$\begin{aligned} df_j(t) &= f_j(t)\sqrt{V(t)}\gamma_j(t)\cdot d\mathbf{Z}_t^{j+1}, \\ dV(t) &= [\kappa\theta - (\kappa + \epsilon\xi_j(t))V(t)]dt + \epsilon\sqrt{V(t)}dW_t^{j+1}. \end{aligned}$$

Note that $\xi_j(t)$ depends on $f_k, k \leqslant j$, which spoils the analytical tractability of a Heston's type model. To retain the analytical tractability, we propose an approximation of $\xi_j(t)$ through "freezing coefficients":

$$\xi_j(t) \approx \sum_{k=1}^{j} \frac{\Delta T_k f_k(0) \rho_k(t) \|\gamma_k(t)\|}{1 + \Delta T_k f_k(0)}. \tag{2.11}$$

In light of the martingale property of $f_j(t)$, $E_0^{Q_{j+1}}[f_j(t)] \triangleq E^{Q_{j+1}}[f_j(t)|\mathcal{F}_0] = f_j(0)$, we can to some extent regard the right-hand side of (2.11) as the leading non-stochastic term of $\xi_j(t)$. Denoting

$$\tilde{\xi}_j(t) = 1 + \frac{\epsilon}{\kappa} \xi_j(t),$$

we then regain a neat equation for $V(t)$:

$$dV(t) = \kappa[\theta - \tilde{\xi}_j(t) V(t)]dt + \epsilon \sqrt{V(t)} dW_t^{j+1}.$$

Under the forward measure $\mathbb{Q}^{j+1}$, the forward price of the caplet is a martingale and we thus have the following expression for its price

$$\begin{aligned} C_{let}(0) &= P(0, T_{j+1}) \Delta T_j E_0^{Q^{j+1}} \left[(f_j(T_j) - K)^+ \right] \\ &= P(0, T_{j+1}) \Delta T_j f_j(0)\ G(0, f_j(0), V(0), K), \end{aligned}$$

where

$$\begin{aligned} &G(0, f_j(0), V(0), K) \triangleq E_0^{Q^{j+1}} \left[(f_j(T_j)/f_j(0) - K/f_j(0))^+ \right] \\ =&E_0^{Q^{j+1}} \left[e^{\ln(f_j(T_j)/f_j(0))} \mathbf{1}_{f_j(T_j) > K} \right] - \frac{K}{f_j(0)} E_0^{Q^{j+1}} \left[\mathbf{1}_{f_j(T_j) > K} \right]. \end{aligned} \tag{2.12}$$

The two expectations above can be valuated using moment generating function of the forward rate, defined by

$$\phi(X(t), V(t), t; z) \triangleq E \left[e^{zX(T_j)} | \mathcal{F}_t \right], \quad z \in C,$$

where $X(t) = \ln(f_j(t)/f_j(0))$. Let $\phi_T(z) \triangleq \phi(0, V(0), 0; z)$ for simplicity. When z is an imaginary number, $\phi_T(z)$ reduces to the characteristic function of $X(T_j)$. From the definition of a characteristic function, one can derive that (Kendall (1994) or more recently Duffie, Pan and Singleton (2000))

$$\begin{aligned} E_0^{Q^{j+1}} \left[\mathbf{1}_{f_j(T_j) > K} \right] &= \frac{\phi_T(0)}{2} + \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Im}\{e^{-iu \ln(K/f_j(0))} \phi_T(iu)\}}{u} du, \\ E_0^{Q^{j+1}} \left[e^{X(T_j)} \mathbf{1}_{f_j(T_j) > K} \right] &= \frac{\phi_T(1)}{2} + \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Im}\{e^{-iu \ln(K/f_j(0))} \phi_T(1 + iu)\}}{u} du. \end{aligned} \tag{2.13}$$

Literally, what we need to do then is to derive $\phi_T(z)$ before performing numerical integrations, as were done in Heston (1993).

When the Brownian motions $\mathbf{Z}_t^{j+1}$ and W_t^{j+1} are independent, we can work out the moment generating function directly. In general, $\phi(x, V, t; z)$ satisfies the Kolmogrorov backward equation corresponding to the joint process of forward rate and stochastic factor:

$$\frac{\partial\phi}{\partial t} + \kappa(\theta - \tilde{\xi}_j V)\frac{\partial\phi}{\partial V} - \frac{1}{2}\|\gamma_j(t)\|^2 V\frac{\partial\phi}{\partial x} + \frac{1}{2}\epsilon^2 V\frac{\partial^2\phi}{\partial V^2} + \epsilon\rho_j V\|\gamma_j(t)\|\frac{\partial^2\phi}{\partial V\partial x} + \frac{1}{2}\|\gamma_j(t)\|^2 V\frac{\partial^2\phi}{\partial x^2} = 0, \quad (2.14)$$

subject to terminal condition

$$\phi(x, V, T_j; z) = e^{zx}. \quad (2.15)$$

As we shall show in section 4, the above equation admits an analytical solution if the coefficients are piecewise constants of t. Otherwise we can at least solve for $\phi(x, V, t; z)$ numerically.

3. **Swaption Pricing.**

The equilibrium swap rate for the period (T_m, T_n) is defined by

$$R_{m,n}(t) = \frac{P(t,T_m) - P(t,T_n)}{B^S(t)},$$

where

$$B^S(t) = \sum_{j=m}^{n-1} \Delta T_j P(t, T_{j+1})$$

is an annuity. The payoff of a swaption at T_m can be expressed as

$$B^S(T_m) \cdot \max(R_{m,n}(T_m) - K, 0),$$

where K is the strike rate.

The swap rate can be regarded as the price of a traded portfolio (consists of being long one T_m-maturity and being short one T_n-maturity zero-coupon bonds) measured by the annuity $B^S(t)$. According to the arbitrage pricing theory (Harrison and Pliska, 1981), the swap rate must be a martingale under the measure corresponding to the numeraire $B^S(t)$. This measure, denoted by $\mathbb{Q}^S$, is called the *forward swap measure* (Jamshidian, 1997). In the next proposition, we characterize the Brownian motions under $\mathbb{Q}^S$, parallel to Proposition 2.1.

Proposition 3.1. *Let $\mathbf{Z}_t$ and W_t be Brownian motions under $\mathbb{Q}$. Define $\mathbf{Z}_t^S$ and W_t^S through*

$$\begin{aligned} d\mathbf{Z}_t^S &= d\mathbf{Z}_t - \sqrt{V(t)}\sigma^S(t)dt, \\ dW_t^S &= dW_t + \sqrt{V(t)}\xi^S(t)dt, \end{aligned} \quad (3.1)$$

where

$$\sigma^S(t) = \sum_{j=m}^{n-1} \alpha_j(t)\sigma(t, T_{j+1}), \qquad \xi^S(t) = \sum_{j=m}^{n-1} \alpha_j(t)\xi_j(t), \quad (3.2)$$

with weights

$$\alpha_j(t) = \frac{\Delta T_j P(t, T_{j+1})}{B^S(t)},$$

then $\mathbf{Z}_t^S$ and W_t^S are Brownian motions under $\mathbb{Q}^S$.

Using Ito's lemma, we can derive the swap rate as

$$\begin{aligned} dR_{m,n}(t) &= \sqrt{V(t)}\sum_{j=m}^{n-1}\frac{\partial R_{m,n}(t)}{\partial f_j}f_j(t)\gamma_j(t)\cdot d\mathbf{Z}^S(t), \\ dV(t) &= \kappa[\theta-\tilde{\xi}^S(t)V(t)]dt+\epsilon\sqrt{V(t)}dW^S(t), \end{aligned} \tag{3.3}$$

where

$$\tilde{\xi}^S(t)=1+\frac{\epsilon}{\kappa}\xi^S(t).$$

The partial derivatives in (3.3) can be evaluated using an alternative definition of the swap rate:

$$R_{m,n}(t)=\sum_{k=m}^{n-1}\alpha_k(t)f_k(t).$$

The results are stated below.

Proposition 3.2. *We have*

$$\frac{\partial R_{m,n}(t)}{\partial f_j}=\alpha_j(t)+\frac{\Delta T_j}{1+\Delta T_j f_j(t)}\left[\sum_{l=m}^{j-1}\alpha_l(t)(f_l(t)-R_{m,n}(t))\right], \quad m\leqslant j\leqslant n-1$$

Under the forward swap measure, we have the following expression for the swaption price

$$\begin{aligned} PS(0) &= B^S(0)E_0^S\left[(R_{m,n}(T_m)-K)^+\right] \\ &= B^S(0)\left(E_0^S\left[R_{m,n}(T_m)\mathbf{1}_{R_{m,n}(T_m)>K}\right]-KE_0^S\left[\mathbf{1}_{R_{m,n}(T_m)>K}\right]\right) \\ &= B^S(0)R_{m,n}(0)\left(E_0^S\left[e^{\ln(R_{m,n}(T_m)/R_{m,n}(0))}\mathbf{1}_{R_{m,n}(T_m)>K}\right]\right. \\ &\quad \left.-\frac{K}{R_{m,n}(0)}E_0^S\left[\mathbf{1}_{R_{m,n}(T_m)>K}\right]\right), \end{aligned} \tag{3.4}$$

where $E_0^S[\cdot]$ stands for the expectation under the forward swap measure conditioned to the filtration at time $t=0$. In view of the complexity of (3.3), we would not think that an exact valuation of (3.4) is possible. Hence, we make the following lognormal approximations to the swap-rate process:

$$\begin{aligned} dR_{m,n}(t) &= R_{m,n}(t)\sqrt{V(t)}\sum_{j=m}^{n-1}w_j(0)\gamma_j(t)\cdot d\mathbf{Z}^S(t), \qquad 0\leqslant t<T_m, \\ dV(t) &= \kappa[\theta-\tilde{\xi}_0^S(t)V(t)]dt+\epsilon\sqrt{V(t)}dW^S(t), \end{aligned} \tag{3.5}$$

where

$$w_j(0)=\frac{\partial R_{m,n}(0)}{\partial f_j(0)}\frac{f_j(0)}{R_{m,n}(0)},$$

$$\tilde{\xi}_0^S(t)=1+\xi_0^S(t), \quad \text{and} \quad \xi_0^S(t)\triangleq\sum_{j=m}^{n-1}\alpha_j(0)\xi_j(t).$$

Yet again, the approximations taken in (3.5) are aimed at a Heston's type model, for which the moment generating function can be derived analytically and then used for valuing (3.4). Note that the technique of "frozen coefficient" simply takes advantage of the low variability of certain functions, and the technique is familiar to practitioners. The approximations made in (3.5) are in the spirit of Anderson and Andreasen (1998), which freezes coefficients after having applied Ito's lemma.

With the approximate swap-rate process, "closed-form" swaption pricing is done analogously to that of caplets. For brevity, we omit the details.

Pricing caplets and swaptions through evaluating the integrals as in (2.13) is not fast enough for production purpose. The need for frequent calibration of the model (to around-the-money caps and at-the-money swaption of various maturities) calls for a faster valuation method, which will be introduced in section 5.

4. **Solving for the Moment Generating Functions.** The moment generating functions, $\phi(x, V, t; z)$, for both a forward rate and a forward swap rate satisfy the following Kolmogorov partial differential equation

$$\frac{\partial \phi}{\partial t}+(\kappa\theta-\kappa\xi V)\frac{\partial \phi}{\partial V}-\frac{1}{2}\lambda^2 V\frac{\partial \phi}{\partial x}+\frac{1}{2}\epsilon^2 V\frac{\partial^2 \phi}{\partial V^2}+\epsilon\rho\lambda V\frac{\partial^2 \phi}{\partial V\partial x}+\frac{1}{2}\lambda^2 V\frac{\partial^2 \phi}{\partial x^2}=0, \quad (4.1)$$

with terminal condition

$$\phi(x, V, T; z) = e^{zx}. \quad (4.2)$$

Here, ξ, λ and ρ take different functions for forward rates and for swap rates. For f_j,

$$\xi = \tilde{\xi}_j(t), \quad \lambda = \|\gamma_j(t)\|, \quad \text{and} \quad \rho = \rho_j.$$

For $R_{m,n}$,

$$\xi = \tilde{\xi}^S(t), \quad \lambda = \|\sum_{j=m}^{n-1} w_j(0)\gamma_j(t)\|, \quad \text{and} \quad \rho = \frac{1}{\lambda}\sum_{j=m}^{n-1} w_j(0)\|\gamma_j(t)\|\rho_j(t).$$

Notice that all coefficients are either zero order or first order in V.

Following Heston (1993), we consider solution of the form

$$\tilde{\phi}(x, V, \tau; z) = e^{A(\tau,z)+B(\tau,z)V+zx}(=\phi(x, V, t; z)),$$

here $\tau = T - t$ is the time to maturity. Substituting the above formal solution to (4.1,4.2), we obtain the following equations for the two undetermined functions:

$$\begin{aligned}\frac{dA}{d\tau} &= \kappa\theta B,\\ \frac{dB}{d\tau} &= \frac{1}{2}\epsilon^2 B^2 + (\rho\epsilon\lambda z - \kappa\xi)B + \frac{1}{2}\lambda^2(z^2 - z),\end{aligned} \quad (4.3)$$

subject to initial conditions

$$A(0, z) = 0, \qquad B(0, z) = 0. \quad (4.4)$$

The equation for B is a Riccatti equation, which is known to have no analytical solution for general coefficients. Yet an analytical solution exists in recursive form for piece-wise constant coefficients, which will be the case if both λ and ρ are piece-wise constant functions. For application purposes, this is by no means a restriction.

Proposition 4.1. *For piece-wise constant coefficients and for $\epsilon \neq 0$, equations (4.3,4.4) admit a unique solution of the form*

$$\begin{cases} A(\tau, z) = A(\tau_j, z) + \dfrac{\kappa\theta}{\epsilon^2}\left\{(a+d)(\tau-\tau_j) - 2\ln\left[\dfrac{1-g_j e^{d(\tau-\tau_j)}}{1-g_j}\right]\right\}, \\[2ex] B(\tau, z) = B(\tau_j, z) + \dfrac{(a+d-\epsilon^2 B(\tau_j, z))(1-e^{d(\tau-\tau_j)})}{\epsilon^2(1-g_j e^{d(\tau-\tau_j)})}, \\ \qquad\qquad \text{for} \quad \tau_j \leqslant \tau < \tau_{j+1}, \quad j = 0, 1, \ldots, m-1, \end{cases} \quad (4.5)$$

where

$$a = \kappa\xi - \rho\epsilon\lambda z, \quad d = \sqrt{a^2 - \lambda^2\epsilon^2(z^2 - z)}, \quad g_j = \frac{a + d - \epsilon^2 B(\tau_j, z)}{a - d - \epsilon^2 B(\tau_j, z)}.$$

The above analytical solution is just as good for computational purpose. Having $\phi_T(z) = \phi(0, V, 0; z) = \tilde{\phi}(0, V, T; z)$ in closed-form, we are ready to consider numerical valuation.

5. **Option pricing via Fast Fourier Transform.** Fast Fourier transform (FFT) has been applied in numerical option valuation since early nineteen nineties, yet mostly used as an auxiliary tool. It was the pioneering work of Carr and Madan (1998) that gave FFT a major role. Carr and Madan discovered that the Fourier transform of an option price can be expressed in terms of the moment generating function of the underlying state variable. As a consequence, once the moment generating function is available, the option price can be obtained via an inverse Fourier transform. This procedure is made very fast by FFT, and it has been applied to option pricing under a broad range of driving processes, including Lévy processes and affine jump-diffusion processes. In the rest of this section we present the two transformation methods of Carr and Madam (1998), along with some subtle treatments tailored to caplet and swaption pricing.

5.1. **Fourier method with dampened option value.** We will illustrate the method with the valuation of a caplet. The first step is to treat the forward price of the option, given in (2.12), as a function of strike:

$$G_T(k) \triangleq G(0, f_j(0), V(0), K) = E_0^{Q^{j+1}}\left[(f_j(T_j)/f_j(0) - K/f_j(0))^+\right],$$

where $k = \ln K/f_j(0)$. Then, let $q_T(s)$ denote the density function of $X(T) = \ln f_j(T)/f_j(0)$, we write,

$$G_T(k) = \int_k^\infty (e^s - e^k) q_T(s) ds.$$

Note that $G_T(k)$ is not square integrable over $(-\infty, \infty)$ as it tends to 1 when k tends to $-\infty$. But this problem can be removed by damping:

$$g_T(k) = e^{ak} G_T(k), \quad \text{for some constant } a > 0.$$

The Fourier transform of the dampened function exists and is given by

$$\begin{aligned}
\psi_T(u) &= \int_{-\infty}^\infty e^{iuk} g_T(k) dk = \int_{-\infty}^\infty e^{iuk} \int_k^\infty e^{ak}(e^s - e^k) q_T(s) ds dk \\
&= \int_{-\infty}^\infty q_T(s) \int_{-\infty}^s (e^{s+ak} - e^{(1+a)k}) e^{iuk} dk ds \\
&= \int_{-\infty}^\infty q_T(s) \left[\frac{e^{(a+1+iu)s}}{a + iu} - \frac{e^{(a+1+iu)s}}{a + 1 + iu}\right] ds \\
&= \frac{\phi_T(1 + a + iu)}{(a + iu)(1 + a + iu)}.
\end{aligned}$$

Given $\psi_T(u)$, the caplet price follows from an inverse Fourier transform

$$G_T(k) = exp(-ak) g_T(k) = \frac{exp(-ak)}{\pi} \int_0^\infty e^{-iuk} \psi_T(u) du. \tag{5.1}$$

Implementation via FFT will be explained in the end of this section.

5.2. **Fourier method with intrinsic value of Call/Put options.** The alternative method is to consider the *time value* of the caplet:

$$z_T(k) \triangleq G_T(k) - (1 - K/f_j(0))^+.$$

By put-call parity,

$$z_T(k) = \int_{-\infty}^{\infty} [e^k - e^s)\mathbf{1}_{s<k}\mathbf{1}_{k<0} + (e^s - e^k)\mathbf{1}_{s>k}\mathbf{1}_{k>0}]q_T(s)ds.$$

Assume that $z_T(k)$ is in $L^2(R)$ and perform a Fourier transform, we obtain

$$\begin{aligned}
\eta_T(u) &\triangleq \int_{-\infty}^{\infty} e^{iuk} z_T(k)dk \\
&= \int_{-\infty}^{0} dke^{iuk} \int_{-\infty}^{k} (e^k - e^s)q_T(s)ds + \int_{0}^{\infty} dke^{iuk} \int_{k}^{\infty} (e^k - e^s)q_T(s)ds \\
&= \int_{-\infty}^{0} dsq_T(s)\int_{s}^{0} [e^{(1+iu)k} - e^s e^{iuk}]dk + \int_{0}^{\infty} dsq_T(s)\int_{0}^{s} [e^s e^{iuk} - e^{(1+iu)k}]dk \\
&= \frac{1 - \phi_T(1+iu)}{u^2 - iu}.
\end{aligned}$$

Using the martingale property of $X(t)$, we can show that $\phi_T(1) = 1$ and $u = 0$ is a removable singularity of $\eta_T(u)$. An inverse Fourier transform then yields

$$G_T(k) = (1 - K/f_j(0))^+ + \frac{1}{\pi}\int_0^{\infty} e^{-iuk}\eta_T(u)du. \tag{5.2}$$

The inverse Fourier transforms will be evaluated numerically. For that we need to make truncation of the infinite domain. Take the inverse transform of η_T for example. By the martingale property of $X(t)$, we have

$$|\phi_T(1+iu)| = \left| E_0^{Q^{j+1}}[e^{(1+iu)X(T)}] \right| \leqslant E_0^{Q^{j+1}}[\left| e^{X(T)} e^{iuX(T)} \right|] = E_0^{Q^{j+1}}[e^{X(T)}] = 1.$$

It follows that

$$|\eta_T(u)| = \left| \frac{1 - \phi_T(1+iu)}{u^2 - iu} \right| \leqslant \left| \frac{2}{\sqrt{u^4 + u^2}} \right| < \frac{2}{u^2},$$

and

$$\left| \int_A^{\infty} e^{-iuk}\eta_T(u)du \right| \leqslant \int_A^{\infty} \frac{2}{u^2}du = \frac{2}{A}.$$

Therefore, to ensure an accuracy in the order of one basis point, we should truncate the integral at $A = 10^4$. Similar analysis shows that $A = 10^4$ also applies to truncating (5.1). Our experiences, however, suggest that such a large truncation is too conservative.

After an truncation at A, we can now talk about the numerical scheme. We consider the *composite trapezoidal rule* for the numerical integration:

$$H(k) \triangleq \frac{1}{\pi}\left(\frac{\eta_T(0)}{2} + \sum_{m=1}^{N-1} e^{-iu_m k}\eta_T(u_m) + \frac{e^{-iu_N k}\eta_T(u_N)}{2} \right)\Delta u, \tag{5.3}$$

where $u_m = m\Delta u$ and $\Delta u = A/N$. The composite trapezoidal rule has the order of accuracy of $O(\Delta u^2)$. Since we are interested mainly in the around-the-money options, we take k's around zero:

$$k_n = -b + n\Delta k, \quad \text{for some } b > 0 \text{ and } n = 0, 1, \ldots, N-1,$$

with

$$\Delta k = \frac{2b}{N}.$$

Hence, for $n = 0, 1, \ldots, N-1$,

$$H(k_n) = \frac{1}{\pi}\left(\frac{\eta_T(0)}{2} + \sum_{m=1}^{N-1} e^{-i\Delta u \Delta k mn}[e^{ibu_m}\eta_T(u_m)] + \frac{e^{-i\Delta u\Delta k N n}e^{ibu_N}\eta_T(u_N)}{2}\right)\Delta u.$$

We now choose, in particular

$$\Delta u \Delta k = \frac{2\pi}{N}, \quad \text{or} \quad b = \frac{\pi N}{A},$$

which results in

$$H(k_n) = \frac{1}{\pi}\left(\frac{\eta_T(0) + e^{ibu_N}\eta(u_N)}{2} + \sum_{m=1}^{N-1} e^{-i\frac{2\pi}{N}mn}[e^{ibu_m}\eta_T(u_m)]\right)\Delta u, \qquad n = 0, 1, \ldots, N-1.$$

The expression of $H(k_n)$ fits into the definition of discrete Fourier transform, and it is valuated via FFT (see for example, Press *et al.* (1992)). For later references, we call the Fourier option pricing method the *FFT method*.

One can consider more accurate numerical quadrature scheme. But we have found the composite trapezoidal rule accurate enough for applications.

6. **Numerical Results.** In this section, we will present the accuracy and speed of the FFT method for option pricing under our extended LIBOR model. For a given term structure of forward rates and stochastic volatilities, we compute caplet and swaption[5] prices across strikes, maturities and tenors by FFT method and Monte Carlo (MC) simulation method. For the Monte Carlo method, we have taken a small time stepping and a big number of paths, in order to get "exact solutions" for comparison. We will also demonstrate how an implied volatility curve response to the changes of rate - factor correlation.

Example: The initial term structures of forward rates, forward-rate volatilities, and the parameters for the stochastic factor process are given below.

- Forward curve: $f_j(0) = 0.04 + 0.00075j, \Delta T_j = 0.5$, for all j.
- Deterministic volatility vector:

$$\gamma_j(T_k) = (0.08 + 0.1e^{-0.05(k-j)}, 0.1 - 0.25e^{-0.1(k-j)}).$$

- Parameters for the stochastic factor: $\theta = \kappa = 1$, $\epsilon = 1.5$ and $V(0) = 1$.

The specification of volatility term structure corresponds to a two-factor model with an annualized short-rate volatility of 25%. Caplet and swaption prices, in basis points (bps), are reported in Tables 2a, 2b, 3a and 3b. In addition to the prices by both FFT and MC methods, we have also reported their corresponding implied Black's volatilities, differences between the implied volatilities, and the radius (or half of the width) of 95% confidence interval (CI) for the Monte Carlo prices. The results are presented for pairs of maturity-strike using the following format:

[5]For convenience we have taken the same ΔT for both caps and swaptions. In reality, ΔT can be different for caps and swaptions .

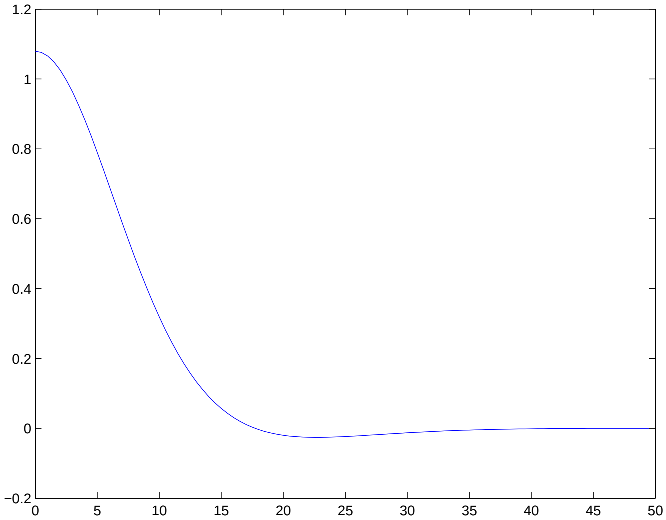

FIGURE 8. Real part of $\psi_T(u)$ for the in-1-to-1 swaption

TABLE 1. The ordering of the entries

	Strike	
Maturity	MC Price	(Implied Vol.)
	FFT Price	(Implied Vol.)
	Radius of 95% CI	(Difference of implied vol's.)

Tables 2a-2b are for the non-correlation case ($\rho = 0$), while Tables 3a-3b are for a negative correlation case ($\rho = -0.5$).

The FFT prices are calculated by the dampened-value Fourier method with dampening parameter $\alpha = 2$, truncation range $A = 50$, and number of divisions $N = 100$. This pair of A and N give a grid size $\Delta u = 0.5$ for integral discretization. This selection was made after several trials. For $\Delta u = 0.5$, an A bigger than 50 makes essentially no difference to the prices. Obviously the choice of $A = 50$ for truncating the integrals is much smaller than the priori estimation of the bound. To understand why, we plot the real and imaginary parts of $\psi_T(u)$ for in-1-to-1 swaption in Figure 8 and 9. It can be seen that beyond the interval $(-50, 50)$, both the real and imaginary parts are well under the magnitude of 10^{-4}. For swaptions with other maturities and tenors, the plots of $\psi_T(u)$ looks similar. Our experiences suggest that, for the same A and N, the dampened-value method is more accurate than the time-value method. This is probably due to the non-smoothness of time values at $k = 0$, which can causes a slower decay of Fourier transforms and thus a larger truncation error.

Next we describe the Monte Carlo simulation method for our extended market model. The MC method is implemented under the risk neutral measure. To input correlation between the forward rates and the stochastic factor, we recast the equation for the forward rates into

$$\frac{df_j(t)}{f_j(t)} = -V(t)\gamma_j(t) \cdot \sigma(t, T_{j+1})dt + \sqrt{V(t)}\left(\sqrt{1-\rho^2}\gamma_j(t) \cdot d\hat{\mathbf{Z}}_t + \rho\|\gamma_j(t)\|_2 dW_t\right), \tag{6.1}$$

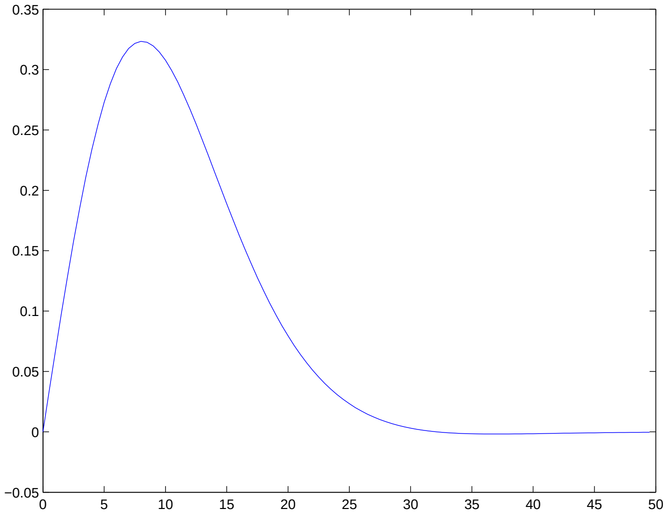

FIGURE 9. Imaginary part of $\psi_T(u)$ for the in-1-to-1 swaption

.

where $(\hat{\mathbf{Z}}_t, W_t)$ is a vector of independent Brownian motions. Treated like a lognormal process, $f_j(t)$ is advanced by the so-called *log-Euler* scheme:

$$f_j(t+\Delta t) = f_j(t)e^{-V(t)(\gamma_j(t)\cdot\sigma(t,T_{j+1})+\frac{1}{2}\|\gamma_j(t)\|^2)\Delta t+\sqrt{V(t)}\ \ \sqrt{1-\rho^2}\gamma_j(t)\cdot\Delta\hat{\mathbf{Z}}_t+\rho\|\gamma_j(t)\|_2\Delta W_t}.$$

Meanwhile, the volatility factor evolves according to a step-wisely moment-matched log-normal scheme:

$$V(t+\Delta t) = E_t^Q[V(t+\Delta t)]e^{-\frac{1}{2}\Gamma_t^2\Delta t+\Gamma_t\Delta W_t},$$

where

$$\Gamma_t^2 = \frac{1}{\Delta t}\ln\frac{E_t^Q[V^2(t+\Delta t)]}{(E_t^Q[V(t+\Delta t)])^2}, \tag{6.2}$$

with

$$\begin{aligned} E_t^Q[V(t+\Delta t)] &= \theta + (V(t)-\theta)e^{-\kappa\Delta t}, \\ E_t^Q[V^2(t+\Delta t)] &= (1+\frac{\epsilon^2}{2\kappa\theta})(E_t^Q[V(t+\Delta t)])^2 - \frac{\epsilon^2}{2\kappa\theta}e^{-2\kappa\Delta t}V^2(t). \end{aligned} \tag{6.3}$$

The derivation of (6.3) is given in the appendix. Unlike a Euler scheme (e.g. Kloeden and Platen, 1992), the moment-matched lognormal scheme for a square-root process will never break down. In our simulations, we have taken time-step size $\Delta t = 1/12$ and the number of paths to be 100,000.

It can be seen in Tables 2a-3b that, for various strikes, maturities and tenors, the price differences between the FFT and Monte Carlo method methods are generally small. In terms of implied volatilities, the differences are often within 1%, which is the typical width of bid-ask spread for swaption transactions. In term of actual prices, the differences are mostly within 1% as well, with exceptions amongst in-10-to-10 swaptions (for which price differences can reach 2%). The slow deterioration of accuracy with respect to option maturity only reflects the approximation nature of the FFT method. Overall, the accuracy of the FFT method is remarkably high.

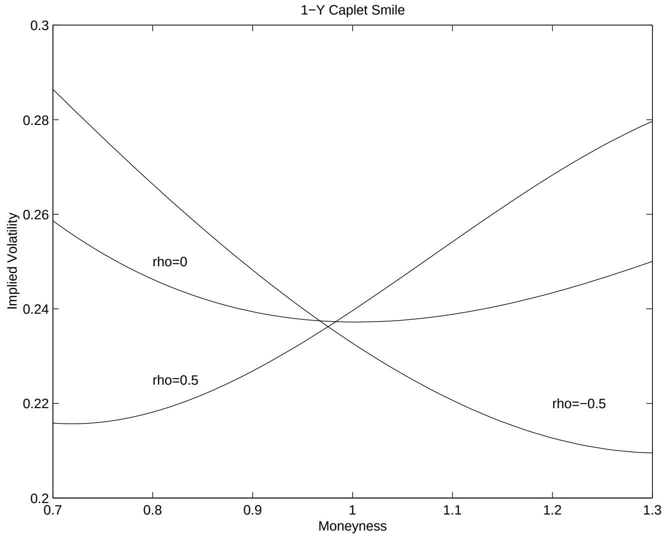

Figure 10. Volatility smile and skews for one-year caplets

Such an accuracy strongly supports the approximations taken in this paper for the swap rate processes. We have also computed the prices using Heston's method, and found them almost indistinguishable from their corresponding FFT prices (using the same resolution of discretization). We thus have omitted the Heston's prices for presentation.

To give a sense of efficiency gained by using FFT we also list, in Table 4, the CPU times for FFT, Heston's and the Monte Carlo methods. The computations are done under MATLAB-5.3 in a PC with 1.1 GHz Intel Celeron CPU. Note that each execution of FFT produces $N = 100$ prices for about 71% of the CPU time taken by the Heston's method, and the latter produces only one price. The ratio of 71% is consistent with the fact the Heston's method evaluates two integrals, while the FFT method evaluates only one.

	FFT	Heston	Monte Carlo
CPU(seconds)	0.93	1.3	39486

Table 4. CPU times of the three methods with $\rho = -0.5$.

We now highlight the most desirable feature of our extended LIBOR model: the rate - factor correlations determine the sloping of implied volatility curves of caplets and swaptions. With every other parameters identical, we calculate and plot implied volatility curve for negative, zero and positive rate - factor correlations. Figure 10 shows such curves for caplets, where the downward sloping skew corresponds to a negative correlation, $\rho = -0.5$, the upward sloping skew corresponds to a positive correlation, $\rho = 0.5$, and the nearly symmetric smile corresponds to no correlation, $\rho = 0$. Similar correspondence between the correlation and shape exists in swaptions, which is depicted in Figure 11. We emphasize here the association of downward skews to negative rate - factor correlation is consistent with the empirical finding, and it is very plausible to market practitioners.

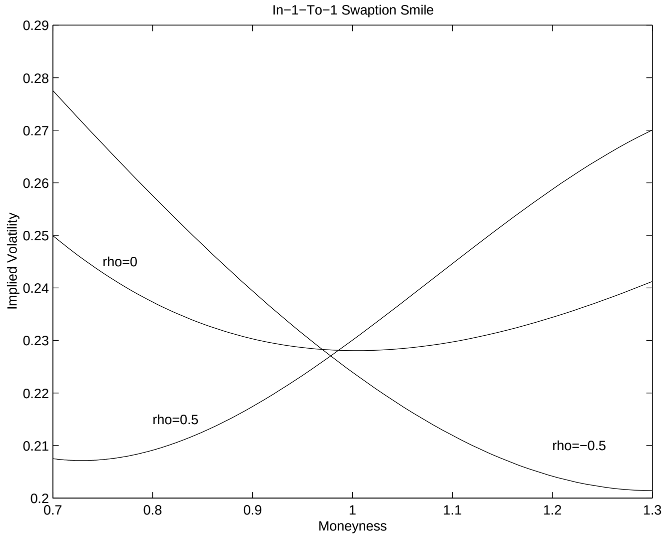

FIGURE 11. Volatility smile and skews for one-year swaptions.

Finally, we take a look at the impact of stochastic volatility on the level of implied volatility curve. Figure 12 and 12 show such curves for lognormal models with and without stochastic volatility, while the two models share the same swap-rate variance, $VaR(X(T_m))$, with $X(T) = \ln(R_{m,n}(T)/R_{m,n}(0))$. For the lognormal model without stochastic volatility, the implied volatility curve is flat. With a rate - factor correlation of $\rho = -0.5$, the implied volatility curve by the lognormal model with stochastic volatility tilts around the flat curve near the at-the-money strike. We conclude here that the adoption of a stochastic volatility generally does not change the average level of the implied volatility curve.

7. **Conclusion.** In this paper we extend the standard market model by taking a multiplicative stochastic factor for forward rate volatilities. We allow correlation between the stochastic factor and otherwise lognormal forward rates. Through appropriate approximations, we derive moment generating functions for forward rates and swap rates in closed form. Fourier transforms of caplet and swaption prices are then expressed in terms of the moment generating functions. These options are then valuated via inverse Fourier transforms, which are implemented by FFT. Typical patterns of implied volatility smiles/skews are readily generated, and the sloping of the implied volatility curve is determined by the correlation between a swap rate and the volatility factor.

Appendix A. **Details of Some Derivations.** *Proof of Proposition 2.1*:
The Radon-Nikodym derivative of $\mathbb{Q}^{j+1}$ with respect to $\mathbb{Q}$ is

$$\begin{aligned}\frac{d\mathbb{Q}^{j+1}}{d\mathbb{Q}} &= \frac{P(t,T_{j+1})/P(0,T_{j+1})}{B(t)} = e^{\int_0^t -\frac{1}{2}V(\tau)\sigma_{j+1}^2(\tau)d\tau + \sqrt{V(\tau)}\sigma_{j+1}\cdot d\mathbf{Z}_\tau} \\ &\triangleq m_{j+1}(t), \quad t \leqslant T_{j+1}.\end{aligned}$$

Clearly, we have

$$dm_{j+1}(t) = m_{j+1}(t)\sqrt{V(t)}\sigma_{j+1}(t)\cdot d\mathbf{Z}_t.$$

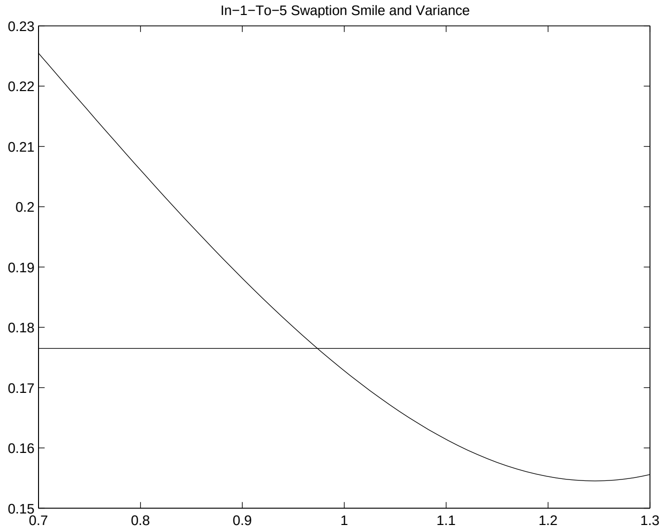

FIGURE 12. Volatility skew vs swap-rate volatility.

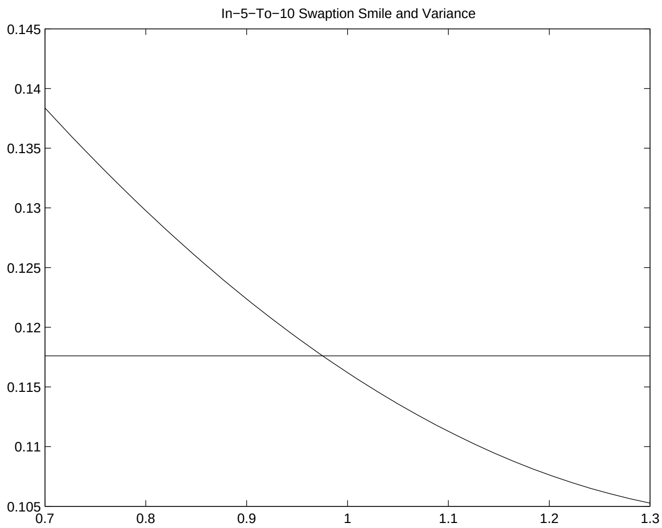

FIGURE 13. Volatility skew vs swap-rate volatility.

Let $< \cdot, \cdot >$ denote covariance. By Girsanov theorem (e.g. Hunt and Kennedy, 2000), the following equations define a new pair of Brownian motions under $\mathbb{Q}^{j+1}$:

$$\begin{aligned}
d\mathbf{Z}_t^{j+1} =& d\mathbf{Z}_t - < d\mathbf{Z}_t, dm_{j+1}(t)/m_{j+1}(t) >= d\mathbf{Z}_t - \sqrt{V(t)}\sigma_{j+1}(t)dt, \\
dW_t^{j+1} =& dW_t - < dW_t, dm_{j+1}(t)/m_{j+1}(t) >= dW_t - < dW_t, \sqrt{V(t)}\sigma_{j+1}(t) \cdot d\mathbf{Z}_t > \\
=& dW_t + \sqrt{V(t)} \sum_{k=1}^{j} \frac{\Delta T_k f_k(t) \|\gamma_k(t)\|}{1 + \Delta T_k f_k(t)} < dW_t, \frac{\gamma_k(t)}{\|\gamma_k(t)\|} \cdot d\mathbf{Z}_t > \\
=& dW_t + \sqrt{V(t)} \sum_{k=1}^{j} \frac{\Delta T_k f_k(t) \|\gamma_k(t)\|}{1 + \Delta T_k f_k(t)} \rho_k(t) dt \\
=& dW_t + \sqrt{V(t)}\xi_j(t)dt
\end{aligned}$$

Note that the Novikov condition is somewhat obvious, given the existence of moment generating function of $V(t)$ and the boundedness of $\sigma_{j+1}(t)$ and $\xi_j(t)$ $\square$.

Proof of Proposition 3.1:
The Radon-Nikodym derivative for $\mathbb{Q}^S$ is

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}} = \frac{B^S(t)/B^S(0)}{B(t)} = \frac{1}{B^S(0)} \sum_{j=m}^{n-1} \Delta T_j P(0,T_{j+1}) e^{\int_0^t -\frac{1}{2}V(\tau)\sigma_{j+1}^2(\tau)d\tau + \sqrt{V(\tau)}\sigma_{j+1}\cdot d\mathbf{Z}_\tau}$$
$$\triangleq m_S(t), \quad t \leqslant T_m.$$

We have

$$\begin{aligned}
dm_S(t) &= \frac{1}{B^S(0)} \sum_{j=m}^{n-1} \Delta T_j P(0,T_{j+1}) e^{\int_0^t -\frac{1}{2}V(\tau)\sigma_{j+1}^2(\tau)d\tau + \sqrt{V(\tau)}\sigma_{j+1}\cdot d\mathbf{Z}_t} \sqrt{V(t)}\sigma_{j+1}(t)\cdot d\mathbf{Z}_\tau \\
&= \frac{1}{B^S(0)B(t)} \sum_{j=m}^{n-1} \Delta T_j P(t,T_{j+1}) \sqrt{V(t)}\sigma_{j+1}(t)\cdot d\mathbf{Z}_t \\
&= m_S(t) \sum_{j=m}^{n-1} \alpha_j \sqrt{V(t)}\sigma_{j+1}(t)\cdot d\mathbf{Z}_t.
\end{aligned}$$

Again, the following equations define a new pair of Brownian motions under $\mathbb{Q}^S$:

$$\begin{aligned}
d\mathbf{Z}_t^S =& d\mathbf{Z}_t - < d\mathbf{Z}_t, dm_S(t)/m_S(t) >= d\mathbf{Z}_t - \sqrt{V(t)}\sum \alpha_j\sigma_{j+1}(t)dt, \\
=& d\mathbf{Z}_t - \sqrt{V(t)}\sigma^S(t)dt, \\
dW_t^S =& dW_t - < dW_t, dm_S(t)/m_S(t) >= dW_t - < dW_t, \sqrt{V(t)}\sum \alpha_j\sigma_{j+1}(t)\cdot d\mathbf{Z}_t > \\
=& dW_t + \sqrt{V(t)} \sum_{j=m}^{n-1} \alpha_j \sum_{k=1}^{j} \frac{\Delta T_k f_k(t)\|\gamma_k(t)\|}{1+\Delta T_k f_k(t)} < dW_t, \frac{\gamma_k(t)}{\|\gamma_k(t)\|}\cdot d\mathbf{Z}_t > \\
=& dW_t + \sqrt{V(t)} \sum_{j=m}^{n-1} \alpha_j \sum_{k=1}^{j} \frac{\Delta T_k f_k(t)\|\gamma_k(t)\|}{1+\Delta T_k f_k(t)} \rho_k(t)dt \\
=& dW_t + \sqrt{V(t)} \sum_{j=m}^{n-1} \alpha_j \xi_j(t)dt \\
=& dW_t + \sqrt{V(t)}\xi^S(t)dt
\end{aligned}$$

For the same reasons given in the proof of Proposition 2.1, the Novikov condition is satisfied $\square$

Proof of Proposition 3.2:
Differentiating the swap rate with respect to a forward rate, we literally have

$$\frac{\partial R_{m,n}(t)}{\partial f_j} = \alpha_j + \sum_{k=m}^{n-1} \frac{\partial \alpha_k}{\partial f_j} f_k. \tag{1.1}$$

According to price-yield relationship,

$$P(t,T_{k+1}) = \frac{P(t,T_k)}{1+\Delta T_k f_k} = \ldots = \frac{P(t,T_m)}{\Pi_{l=m}^k (1+\Delta T_l f_l)}. \tag{1.2}$$

Apparently,

$$\begin{aligned}
\frac{\partial P(t,T_{k+1})}{\partial f_j} =& \begin{cases} \frac{-\Delta T_j}{1+\Delta T_j f_j} \cdot \frac{P(t,T_m)}{\Pi_{l=m}^k (1+\Delta T_l f_l)}, & k \geqslant j, \\ 0, & k < j \end{cases} \\
=& \frac{-\Delta T_j}{1+\Delta T_j f_j} \cdot P(t,T_{k+1}) \cdot H(k-j),
\end{aligned} \tag{1.3}$$

where $H(\cdot)$ is the Heaviside function, defined such that $H(x) = 1$ for $x \geqslant 0$ and $H(x) = 0$ otherwise. It follows that

$$\begin{aligned}
\frac{\partial \alpha_k}{\partial f_j} &= \Delta T_k \cdot \quad \frac{\partial P(t, T_{k+1})}{\partial f_j} B^S(t) - P(t, T_{k+1}) \frac{\partial B^S(t)}{\partial f_j} \quad \Big/ (B^S(t))^2, \\
&= \Delta T_k \cdot \quad \frac{-\Delta T_j}{1 + \Delta T_j f_j} \cdot P(t, T_{k+1}) \cdot H(k-j) B^S(t) \\
&\qquad -P(t, T_{k+1}) \sum_{l=m}^{n-1} \Delta T_l \frac{-\Delta T_j}{1 + \Delta T_j f_j} \cdot P(t, T_{l+1}) \cdot H(l-j) \Bigg) \Big/ (B^S(t))^2 \\
&= \frac{-\Delta T_j}{1 + \Delta T_j f_j} \alpha_k \left(H(k-j) - \sum_{l=j}^{n-1} \alpha_l \right) \\
&= \frac{\Delta T_j}{1 + \Delta T_j f_j} \alpha_k \quad 1 - H(k-j) - \sum_{l=m}^{j-1} \alpha_l \Bigg).
\end{aligned} \tag{1.4}$$

Substitute the above expression to equation (1.1), we then end up with

$$\begin{aligned}
\frac{\partial R_{m,n}(t)}{\partial f_j} &= \alpha_j + \frac{\Delta T_j}{1 + \Delta T_j f_j} \sum_{k=m}^{n-1} \alpha_k \quad 1 - H(k-j) - \sum_{l=m}^{j-1} \alpha_l \Bigg) f_k \\
&= \alpha_j + \frac{\Delta T_j}{1 + \Delta T_j f_j} \left\{ \sum_{k=m}^{n-1} \alpha_k f_k [1 - H(k-j)] - (\sum_{l=m}^{j-1} \alpha_l)(\sum_{k=m}^{n-1} \alpha_k f_k) \right\} \\
&= \alpha_j + \frac{\Delta T_j}{1 + \Delta T_j f_j} \sum_{k=m}^{j-1} \alpha_k \left(f_k - R_{m,n}(t) \right)
\end{aligned} \tag{1.5}$$

Proof of Proposition 4.1:
For clarity we let

$$a = \kappa\theta, \quad b_0 = \frac{1}{2}\lambda^2(z^2 - z), \quad b_1 = (\rho\epsilon\lambda z - \kappa\xi), \quad b_2 = \frac{1}{2}\epsilon^2,$$

and consider

$$\begin{aligned}
\frac{dA}{d\tau} &= aB, \\
\frac{dB}{d\tau} &= b_2 B^2 + b_1 B + B_0,
\end{aligned} \tag{1.6}$$

subject to general initial conditions

$$A(0) = A_0, \qquad B(0) = B_0. \tag{1.7}$$

B is independent of A and thus will be solved first. In the special case when

$$b_2 B_0^2 + b_1 B_0 + b_0 = 0,$$

we have a easy solution

$$\begin{aligned}
B(\tau) &= B_0, \\
A(\tau) &= A_0 + a_0 B_0 \tau.
\end{aligned} \tag{1.8}$$

Otherwise, let Y_1 be the solution to

$$b_2 Y^2 + b_1 Y + b_0 = 0.$$

Then,

$$Y_1 = \frac{-b_1 \pm d}{2b_2}, \quad \text{with} \quad d = \sqrt{b_1^2 - 4b_0 b_2}. \tag{1.9}$$

Without making a difference to the solution of B, we take the "+" sign for Y_1. We then consider the difference between Y_1 and B:

$$Y_2 = B - Y_1.$$

Y_2 satisfies

$$\begin{aligned}\frac{dY_2}{d\tau} &= \frac{d(Y_1+Y_2)}{d\tau} = b_2(Y_1+Y_2)^2 + b_1(Y_1+Y_2) + b_0 = b_2Y_2^2 + (2b_2Y_1+b_1)Y_2 \\ &= b_2Y_2^2 + dY_2,\end{aligned} \tag{1.10}$$

with initial condition

$$Y_2(0) = B_0 - Y_1.$$

Note that in the last equality of (1.10), we have used the equation (1.9). Equation (1.10) belongs to the class of Bernoulli equations which can be solved explicitly, and the solution is

$$Y_2 = \frac{d}{b_2}\frac{ge^{d\tau}}{(1-ge^{d\tau})}, \quad \text{with} \quad g = \frac{-b_1+d-2B_0b_2}{-b_1-d-2B_0b_2}. \tag{1.11}$$

Hence we have

$$B(\tau) = Y_1 + Y_2 = \frac{-b_1+d}{2b_2} + \frac{d}{b_2}\frac{ge^{d\tau}}{(1-ge^{d\tau})} = B_0 + \frac{(-b_1+d-2b_2B_0)(1-e^{d\tau})}{2b_2(1-ge^{d\tau})}.$$

Having obtained B, we integrate the first equation of (1.6) to get A:

$$\begin{aligned}A(\tau) &= A_0 + a_0\int_0^\tau B(s)ds \\ &= A_0 + a_0B_0\tau + \frac{a_0(-b_1+d-2b_2B_0)}{2b_2}\int_0^\tau \frac{1-e^{d\tau}}{1-ge^{d\tau}}d\tau \\ &= A_0 + a_0B_0\tau + \frac{a_0(-b_1+d-2b_2B_0)}{2b_2}\quad \tau - \int_0^\tau \frac{(1-g)e^{d\tau}}{1-ge^{d\tau}}d\tau \\ &= A_0 + \frac{a_0(-b_1+d)\tau}{2b_2} - \frac{a_0(-b_1+d-2b_2B_0)}{2b_2d}\int_1^{e^{d\tau}} \frac{(1-g)}{1-gu}du \\ &= A_0 + \frac{a_0(-b_1+d)\tau}{2b_2} - \frac{a_0(-b_1+d-2b_2B_0)}{2b_2d}\frac{(g-1)}{g}\ln\quad\frac{1-ge^{d\tau}}{1-g} \\ &= A_0 + \frac{a_0}{2b_2}\quad(-b_1+d)\tau - 2\ln\quad\frac{1-ge^{d\tau}}{1-g}\quad .\end{aligned}$$

Specifying a_0, b_0, b_1, b_2 and A_0, B_0 by

$$\begin{aligned}&a_0 = \kappa\theta, \quad b_0 = \frac{1}{2}\lambda^2(z^2-z), \quad b_1 = \rho\epsilon\lambda z - \kappa\xi, \\ &b_2 = \epsilon^2/2, \quad A_0 = A(\tau_j, z), \quad B_0 = B(\tau_j, z),\end{aligned}$$

and replacing τ by $\tau - \tau_j$, we arrive at (4.5). The solution $\phi(z)$ so obtained belongs to $\mathbb{C}^1$ and hence is a weak solution to (4.1) $\square$

Table 2a. Swaption prices (bps) by MC and FFT methods, $\rho_j = 0.0$

Exp.	Strikes 0.015	0.020	0.030	0.035	0.040	0.050	0.060	0.070	0.080
	Tenor=0.5 year								
1	124.72 (0.301)	101.23 (0.274)	55.59 (0.234)	35.60 (0.221)	20.21 (0.215)	5.34 (0.222)	1.48 (0.239)	0.46 (0.254)	0.17 (0.267)
	124.63 (0.000)	101.13 (0.000)	55.44 (0.228)	35.49 (0.219)	20.20 (0.215)	5.30 (0.222)	1.41 (0.237)	0.44 (0.252)	0.16 (0.266)
	0.27 (—)	0.27 (—)	0.26 (0.006)	0.24 (0.002)	0.20 (0.000)	0.12 (0.001)	0.07 (0.002)	0.04 (0.002)	0.02 (0.001)
5	128.19 (0.211)	108.88 (0.198)	72.82 (0.182)	57.24 (0.177)	43.98 (0.175)	24.88 (0.174)	13.93 (0.176)	8.02 (0.180)	4.81 (0.184)
	128.01 (0.000)	108.70 (0.184)	72.68 (0.180)	57.14 (0.176)	43.93 (0.174)	24.95 (0.174)	14.00 (0.176)	8.04 (0.180)	4.85 (0.184)
	0.44 (—)	0.45 (0.014)	0.43 (0.002)	0.41 (0.001)	0.39 (0.000)	0.32 (-0.000)	0.26 (-0.000)	0.21 (-0.000)	0.17 (-0.000)
10	122.33 (0.175)	107.49 (0.169)	79.89 (0.161)	67.71 (0.159)	56.88 (0.157)	39.43 (0.156)	27.08 (0.156)	18.66 (0.157)	13.02 (0.158)
	121.94 (0.000)	107.10 (0.143)	79.51 (0.156)	67.33 (0.155)	56.48 (0.154)	39.03 (0.154)	26.70 (0.154)	18.33 (0.155)	12.71 (0.156)
	0.45 (—)	0.46 (0.025)	0.45 (0.005)	0.44 (0.004)	0.42 (0.003)	0.38 (0.002)	0.34 (0.002)	0.29 (0.002)	0.25 (0.002)
Exp.	**Tenor=1 year**								
1	250.32 (0.294)	203.80 (0.267)	112.95 (0.227)	72.54 (0.214)	40.94 (0.207)	10.34 (0.214)	2.70 (0.230)	0.81 (0.245)	0.27 (0.258)
	250.13 (0.000)	203.61 (0.000)	112.66 (0.221)	72.29 (0.211)	40.89 (0.207)	10.27 (0.213)	2.59 (0.228)	0.77 (0.244)	0.27 (0.258)
	0.51 (—)	0.52 (—)	0.50 (0.007)	0.46 (0.002)	0.38 (0.000)	0.22 (0.001)	0.12 (0.002)	0.07 (0.002)	0.04 (0.001)
5	256.21 (0.206)	217.92 (0.194)	145.98 (0.177)	114.62 (0.173)	87.80 (0.170)	49.08 (0.169)	27.01 (0.171)	15.25 (0.175)	8.99 (0.179)
	255.83 (0.000)	217.53 (0.177)	145.66 (0.175)	114.37 (0.172)	87.66 (0.169)	49.18 (0.169)	27.10 (0.171)	15.28 (0.175)	9.04 (0.179)
	0.84 (—)	0.85 (0.017)	0.83 (0.002)	0.79 (0.001)	0.74 (0.001)	0.62 (-0.000)	0.50 (-0.000)	0.40 (-0.000)	0.32 (-0.000)
10	243.52 (0.173)	214.16 (0.166)	159.29 (0.158)	134.95 (0.156)	113.24 (0.154)	78.19 (0.153)	53.38 (0.153)	36.52 (0.154)	25.27 (0.155)
	242.80 (0.000)	213.43 (0.139)	158.56 (0.153)	134.22 (0.152)	112.48 (0.152)	77.43 (0.151)	52.66 (0.151)	35.88 (0.152)	24.69 (0.153)
	0.86 (—)	0.87 (0.027)	0.86 (0.005)	0.85 (0.003)	0.82 (0.003)	0.74 (0.002)	0.65 (0.002)	0.56 (0.002)	0.48 (0.002)

Table 2b. Swaption prices (bps) by MC and FFT methods, $\rho_j = 0.0$

	Strikes																	
	0.015		0.020		0.030		0.035		0.040		0.050		0.060		0.070		0.080	
Exp.	Tenor=5 year																	
1	1270.91	(0.259)	1057.23	(0.230)	631.81	(0.191)	427.18	(0.176)	246.78	(0.165)	49.91	(0.165)	8.77	(0.181)	1.78	(0.196)	0.43	(0.209)
	1270.08	(0.000)	1056.40	(0.000)	630.80	(0.173)	425.87	(0.171)	245.72	(0.163)	49.83	(0.165)	8.53	(0.180)	1.74	(0.195)	0.42	(0.209)
	1.83	(—)	1.86	(—)	1.89	(0.018)	1.83	(0.005)	1.62	(0.002)	0.89	(0.000)	0.40	(0.001)	0.19	(0.000)	0.10	(0.000)
5	1262.43	(0.185)	1086.66	(0.172)	745.69	(0.155)	589.04	(0.150)	449.57	(0.146)	240.00	(0.144)	120.76	(0.145)	61.16	(0.149)	32.26	(0.153)
	1260.63	(0.000)	1084.82	(0.000)	743.74	(0.150)	587.09	(0.147)	447.94	(0.145)	239.42	(0.143)	120.59	(0.145)	60.99	(0.148)	32.20	(0.153)
	3.08	(—)	3.15	(—)	3.19	(0.005)	3.11	(0.003)	2.95	(0.002)	2.44	(0.000)	1.88	(0.000)	1.41	(0.000)	1.06	(0.000)
10	1165.19	(0.159)	1032.01	(0.153)	777.02	(0.145)	660.32	(0.142)	554.11	(0.141)	378.64	(0.139)	252.47	(0.138)	167.13	(0.139)	111.22	(0.140)
	1163.32	(0.000)	1030.03	(0.110)	774.82	(0.140)	657.89	(0.139)	551.45	(0.138)	375.55	(0.137)	249.59	(0.137)	164.55	(0.138)	108.98	(0.139)
	3.21	(—)	3.30	(0.043)	3.38	(0.005)	3.35	(0.003)	3.26	(0.002)	2.97	(0.002)	2.59	(0.001)	2.19	(0.001)	1.84	(0.001)
Exp.	Tenor=10 year																	
1	2520.39	(0.265)	2138.82	(0.210)	1376.20	(0.174)	998.53	(0.160)	636.92	(0.146)	138.52	(0.138)	18.50	(0.150)	2.83	(0.165)	0.51	(0.178)
	2518.98	(0.000)	2137.37	(0.000)	1374.63	(0.000)	996.63	(0.147)	635.25	(0.143)	138.39	(0.137)	18.35	(0.149)	2.76	(0.165)	0.49	(0.177)
	2.68	(—)	2.77	(—)	2.92	(—)	2.96	(0.013)	2.83	(0.003)	1.73	(0.000)	0.69	(0.000)	0.28	(0.001)	0.12	(0.001)
5	2427.10	(0.175)	2116.19	(0.163)	1503.82	(0.145)	1212.55	(0.140)	943.35	(0.135)	513.11	(0.131)	252.82	(0.131)	121.72	(0.134)	60.32	(0.138)
	2424.60	(0.000)	2113.60	(0.000)	1500.91	(0.138)	1209.52	(0.136)	940.41	(0.133)	511.09	(0.130)	251.67	(0.131)	120.75	(0.134)	59.50	(0.138)
	4.67	(—)	4.85	(—)	5.10	(0.007)	5.09	(0.003)	4.95	(0.002)	4.23	(0.001)	3.24	(0.000)	2.36	(0.000)	1.71	(0.001)
10	2170.52	(0.154)	1938.07	(0.147)	1486.51	(0.138)	1274.99	(0.136)	1078.76	(0.134)	745.36	(0.131)	498.03	(0.131)	327.58	(0.131)	215.24	(0.132)
	2169.17	(0.000)	1936.53	(0.124)	1484.67	(0.135)	1272.60	(0.134)	1075.71	(0.132)	740.78	(0.130)	492.65	(0.129)	322.26	(0.130)	210.15	(0.130)
	4.88	(—)	5.07	(0.023)	5.34	(0.003)	5.37	(0.002)	5.31	(0.002)	4.94	(0.002)	4.34	(0.002)	3.67	(0.001)	3.04	(0.002)

Table 3a. Swaption prices (bps) by FFT and MC methods, $\rho_j = -0.5$.

	Strikes								
	0.015	0.020	0.030	0.035	0.040	0.050	0.060	0.070	0.080
Exp.	Tenor=0.5 year								
1	124.74 (0.346)	101.33 (0.314)	56.31 (0.260)	36.46 (0.237)	20.30 (0.217)	3.63 (0.192)	0.46 (0.191)	0.06 (0.198)	0.01 (0.206)
	124.84 (0.420)	101.43 (0.337)	56.31 (0.260)	36.41 (0.236)	20.40 (0.218)	3.85 (0.196)	0.50 (0.194)	0.07 (0.199)	0.02 (0.211)
	0.26 (-0.074)	0.26 (-0.023)	0.24 (0.000)	0.21 (0.001)	0.17 (-0.001)	0.08 (-0.004)	0.03 (-0.003)	0.01 (-0.001)	0.01 (-0.005)
5	128.40 (0.241)	109.39 (0.224)	73.99 (0.198)	58.37 (0.188)	44.63 (0.179)	23.67 (0.166)	11.18 (0.158)	4.91 (0.153)	2.11 (0.150)
	128.50 (0.251)	109.47 (0.227)	74.00 (0.198)	58.37 (0.188)	44.64 (0.180)	23.74 (0.167)	11.35 (0.159)	5.10 (0.154)	2.26 (0.153)
	0.40 (-0.009)	0.40 (-0.003)	0.38 (-0.000)	0.36 (-0.000)	0.33 (-0.000)	0.26 (-0.000)	0.18 (-0.001)	0.13 (-0.002)	0.09 (-0.002)
10	122.56 (0.200)	108.01 (0.189)	80.96 (0.175)	68.84 (0.169)	57.86 (0.164)	39.51 (0.156)	25.93 (0.150)	16.48 (0.146)	10.24 (0.143)
	122.26 (0.164)	107.69 (0.178)	80.62 (0.170)	68.50 (0.166)	57.54 (0.162)	39.27 (0.155)	25.79 (0.150)	16.44 (0.146)	10.29 (0.143)
	0.42 (0.036)	0.42 (0.012)	0.41 (0.004)	0.40 (0.003)	0.38 (0.002)	0.33 (0.001)	0.28 (0.001)	0.23 (0.000)	0.19 (-0.000)
Exp.	Tenor=1 year								
1	250.34 (0.338)	203.95 (0.306)	114.25 (0.253)	74.24 (0.230)	41.33 (0.210)	7.08 (0.185)	0.81 (0.184)	0.11 (0.191)	0.02 (0.199)
	250.54 (0.419)	204.15 (0.332)	114.25 (0.254)	74.15 (0.230)	41.49 (0.211)	7.52 (0.189)	0.89 (0.187)	0.11 (0.193)	0.02 (0.205)
	0.49 (-0.081)	0.49 (-0.026)	0.46 (-0.000)	0.41 (0.001)	0.33 (-0.001)	0.15 (-0.004)	0.05 (-0.003)	0.02 (-0.002)	0.01 (-0.006)
5	256.57 (0.237)	218.84 (0.220)	148.23 (0.194)	116.90 (0.184)	89.25 (0.175)	46.95 (0.162)	21.80 (0.154)	9.33 (0.149)	3.90 (0.146)
	256.72 (0.246)	218.96 (0.223)	148.23 (0.194)	116.88 (0.184)	89.24 (0.175)	47.05 (0.163)	22.12 (0.155)	9.69 (0.150)	4.18 (0.149)
	0.77 (-0.008)	0.77 (-0.003)	0.73 (-0.000)	0.69 (0.000)	0.63 (0.000)	0.49 (-0.000)	0.35 (-0.001)	0.24 (-0.002)	0.16 (-0.002)
10	243.95 (0.197)	215.12 (0.187)	161.37 (0.172)	137.21 (0.166)	115.26 (0.162)	78.52 (0.154)	51.30 (0.148)	32.40 (0.144)	19.96 (0.140)
	243.31 (0.142)	214.45 (0.173)	160.65 (0.167)	136.49 (0.163)	114.58 (0.159)	78.00 (0.153)	51.00 (0.147)	32.29 (0.143)	20.05 (0.140)
	0.81 (0.055)	0.81 (0.014)	0.79 (0.005)	0.77 (0.003)	0.73 (0.002)	0.65 (0.001)	0.55 (0.001)	0.45 (0.000)	0.36 (-0.000)

Table 3b. Swaption prices (bps) by FFT and MC methods, $\rho_j = -0.5$.

	Strikes																	
	0.015		0.020		0.030		0.035		0.040		0.050		0.060		0.070		0.080	
Exp.	Tenor=5 year																	
1	1270.92	(0.295)	1057.33	(0.265)	634.28	(0.218)	432.98	(0.197)	253.24	(0.177)	38.80	(0.148)	2.52	(0.145)	0.17	(0.152)	0.01	(0.160)
	1271.51	(0.415)	1057.92	(0.317)	634.62	(0.221)	432.90	(0.196)	253.34	(0.177)	40.86	(0.151)	2.89	(0.148)	0.23	(0.156)	0.03	(0.167)
	1.78	(-0.120)	1.81	(-0.052)	1.80	(-0.003)	1.70	(0.000)	1.45	(-0.000)	0.66	(-0.003)	0.18	(-0.003)	0.05	(-0.004)	0.02	(-0.006)
5	1263.09	(0.215)	1088.81	(0.199)	753.89	(0.174)	599.73	(0.164)	459.81	(0.156)	237.79	(0.142)	102.99	(0.133)	38.86	(0.128)	13.79	(0.126)
	1262.32	(0.170)	1088.01	(0.191)	752.89	(0.172)	598.68	(0.163)	458.88	(0.155)	238.01	(0.142)	104.36	(0.134)	40.51	(0.129)	15.06	(0.128)
	2.95	(0.045)	2.99	(0.008)	2.95	(0.002)	2.83	(0.001)	2.64	(0.001)	2.08	(-0.000)	1.45	(-0.001)	0.92	(-0.002)	0.57	(-0.002)
10	1166.33	(0.185)	1035.00	(0.175)	785.51	(0.161)	670.88	(0.155)	565.15	(0.150)	384.83	(0.142)	248.67	(0.137)	153.53	(0.132)	91.45	(0.128)
	1161.75	(0.000)	1030.23	(0.122)	780.44	(0.152)	665.75	(0.149)	560.22	(0.146)	380.75	(0.140)	246.01	(0.135)	152.36	(0.131)	91.47	(0.129)
	3.09	(—)	3.16	(0.053)	3.18	(0.009)	3.12	(0.006)	3.02	(0.004)	2.69	(0.002)	2.28	(0.001)	1.84	(0.001)	1.45	(-0.000)
Exp.	Tenor=10 year																	
1	2520.39	(0.289)	2138.85	(0.245)	1377.56	(0.201)	1003.41	(0.181)	648.36	(0.163)	130.06	(0.132)	6.01	(0.120)	0.29	(0.129)	0.02	(0.140)
	2520.58	(0.373)	2139.06	(0.290)	1377.88	(0.205)	1003.89	(0.183)	649.08	(0.164)	133.89	(0.134)	7.79	(0.125)	0.39	(0.133)	0.02	(0.137)
	2.69	(-0.084)	2.77	(-0.045)	2.90	(-0.004)	2.86	(-0.002)	2.65	(-0.001)	1.44	(-0.003)	0.36	(-0.006)	0.08	(-0.004)	0.01	(0.003)
5	2427.65	(0.206)	2118.31	(0.190)	1514.76	(0.166)	1229.49	(0.156)	963.78	(0.148)	521.08	(0.134)	231.57	(0.124)	86.03	(0.118)	28.81	(0.115)
	2423.17	(0.000)	2113.86	(0.000)	1510.78	(0.159)	1225.84	(0.153)	960.89	(0.146)	520.90	(0.134)	233.99	(0.125)	89.00	(0.120)	30.97	(0.117)
	4.60	(—)	4.75	(—)	4.87	(0.007)	4.78	(0.003)	4.56	(0.002)	3.74	(0.000)	2.66	(-0.001)	1.68	(-0.001)	1.00	(-0.002)
10	2171.88	(0.180)	1942.00	(0.170)	1499.81	(0.156)	1293.14	(0.150)	1099.88	(0.146)	762.93	(0.138)	500.62	(0.132)	312.17	(0.127)	186.54	(0.123)
	2160.78	(0.000)	1930.60	(0.000)	1487.89	(0.140)	1281.16	(0.141)	1088.17	(0.139)	752.57	(0.134)	492.75	(0.129)	307.48	(0.125)	184.62	(0.123)
	4.76	(—)	4.94	(—)	5.12	(0.016)	5.10	(0.009)	5.00	(0.006)	4.56	(0.004)	3.92	(0.002)	3.20	(0.001)	2.51	(0.001)

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Received December 2005; Revised February 2006

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