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# Fast swaption pricing under the market model with a square-root volatility process 

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#### Abstract

In this paper we study a correlation-based LIBOR market model with a square-root volatility process. This model captures downward volatility skews through taking negative correlations between forward rates and the multiplier. An approximate pricing formula is developed for swaptions, and the formula is implemented via fast Fourier transform. Numerical results on pricing accuracy are presented, which strongly support the approximations made in deriving the formula.


Keywords: LIBOR model; Stochastic volatility; Square-root process; Swaptions; Fast Fourier transform (FFT)

## 1. Introduction

This paper introduces a LIBOR market model with stochastic volatility. Over the past decade, the standard market model (Brace et al. 1997, Jamshidian 1997, Miltersen et al. 1997) has established itself as the benchmark model for interest-rate derivatives. Several virtues of the market model are responsible for its popularity. First, state variables of the model, LIBOR, are directly observable. Second, the model justifies the use of Black's formula for caplets and even swaptions (the so-called benchmark derivative instruments). The closed-form pricing of the benchmark derivatives enables efficient calibration of the model, making it possible to implement the model in real time. Third, as a multi-factor model, the market model can conveniently incorporate exogenous forward-rate correlations. Nonetheless, the standard market model suffers from insufficient capacity: it cannot generate implied volatility smiles or skews $\ddagger$ (for those benchmark derivatives), which have become a very pronounced reality of LIBOR markets. For more consistent pricing and more effective hedging, market participants have been seeking
for extensions of the standard model that address, in particular, the issue of volatility smiles and skews.

Extensions to the standard market model have been made largely through adding at least one of the following features or ingredients: level-dependent volatilities, stochastic volatilities, and jumps. Andersen and Andreasen (2000) adopt constant-elasticity-variance (CEV) processes. On top of the CEV model, Andersen and Brotherton-Ratcliffe (2005) superimpose an independent square-root volatility process, which effectively produces additional curvature to the otherwise monotonic volatility skews. Zhou (2003) develops a parallel theory using some unconventional specifications of volatility processes. Under these models, swaptions can be priced in closedform or, in some cases, with the Fourier transformation method (Andersen and Andreasen 2002). In the other line of research, Glasserman and Kou (2003) develop a comprehensive term structure theory with the jumpdiffusion dynamics. Glasserman and Merener (2003) derive approximate closed-form formulae for caplets and swaptions. A theory, parallel to Glasserman and Kou (2003) and Glasserman and Merener (2003), was extended to a LIBOR model based on general Lévy

[^0]processes by Eberlein and Özkan (2004). Jarrow et al. (2003) combine the two types of models, and examine empirically the model's ability to fit actual caplet volatility smiles/skews. Other interesting developments include a model based on displaced-diffusion (DD) processes (Joshi and Rebonato 2003), a model based on mixed lognormal densities for LIBOR (Brigo and Mercurio 2003), and in particular, the popular SABR model (Hagen et al. 2002) for equity derivatives, which is also applicable for managing smile or skew risk of either caplets or swaptions. In the above models, the primary mechanism for volatility skews is either state-dependent volatility or jump. Yet it has been a popular belief among participants of fixedincome derivatives markets that stochastic volatilities should be the primary mechanism. Such a belief has not been reflected in those models.

In this paper, we develop a genuine correlation-based model for LIBOR derivatives. We adopt a set-up similar to that of Andersen and Brotherton-Ratcliffe (2005), yet, contrary to their approach, we exclude state-dependent diffusions but include correlations between forward rates and a stochastic multiplier (hereafter rate-multiplier correlations). The model so developed can be regarded as the LIBOR version of Heston's model (1993), which has been one of the most popular equity option models with stochastic volatility. $\dagger$ Several more reasons are also behind this extension. First, the time series data of interest rates suggests the randomness of interest-rate volatilities (Chen and Scott 2001), and it is a popular belief that the stochastic volatilities are the primary factors behind the leptokurtic feature $\ddagger$ of empirical interest-rate distributions. With the Heston-type model we can nicely capture the leptokurtic feature. Second, Heston's model establishes a direct correspondence between a downward skew to a negative correlation between the state variable and its stochastic volatility.§ Third, among stochastic volatility models (see e.g. Lewis 2000), Heston's model carries nice analytical tractability that renders exact closed-form pricing for equity options. Ironically, such a correlation has been deemed counter-productive for the Heston-type model in the context of LIBOR, as it causes dependence of the volatility process on the forward rates after we change from a risk neutral measure to any forward measure, which is a kind of circular dependence that spoils analytical tractability. A key question for developing a Heston-type model for LIBOR is whether it is possible to relegate such dependence without compromising the accuracy on derivative pricing.

Our answer to the above question is positive. We have observed that, under its corresponding forward swap measure, a swap-rate process retains the formalism of Heston's model, with however state-dependent coefficients for its volatility process. Yet, the time variability of those coefficients is rather small. This crucial observation has motivated us to get rid of the circular dependance through 'freezing coefficients',
and eventually to obtain a 'closed-form' formula for swaptions in terms of a fast Fourier transform (FFT).

To a large extent, the theory developed in this paper is a product of re-engineering guided by the insight that a swap-rate process under its corresponding forward swap measure is close to a Heston's process. We have taken a number of models or techniques as building blocks, especially Heston's (1993) model for equity option, the volatility modelling technique of Chen and Scott (2001) and Andersen and Brotherton-Ractliffe (2005), the lognormal approximation of swap-rate processes under the standard market model of Andersen and Andreasen (2000), and the FFT evaluation technique of Carr and Madan (1998). The new model highlights the role of the rate-factor correlations in the formation of volatility smiles/skews, which is appealing in finance. In addition, the new model lends itself for further extensions, for example, to incorporate jump risk under the general framework of time-changed Lévy processes.

The remaining part of the paper is organized as follows. In section 2 we set up the LIBOR market model with stochastic volatility, and develop an approximate caplet formula following Heston's approach. Section 3 is for swaption pricing, where we introduce necessary treatments/approximations to retain analytical tractability, present analytical moment-generating function for piecewise constant model parameters, and describe a transformation method for numerical option valuation. In section 4 we make pricing comparisons between our transformation method and the Monte Carlo simulation method, and demonstrate the correspondence between the rate-multiplier correlations and the smiles/skews. Finally in section 5 we conclude. Most technical details are put in the appendix.

## 2. The Market model with stochastic volatility

Let $P(t, T)$ be the price of a zero-coupon Treasury bond maturing at $T(\geq t)$ with par value $\$ 1$, and let $B(t)$ be the money market account under discrete compounding:

$$
\begin{aligned}
B(t)= & \left(\prod_{j=0}^{\eta(t)-2}\left(1+f_{j}\left(T_{j}\right) \Delta T_{j}\right)\right) \\
& \times\left(1+f_{\eta(t)-1}\left(T_{\eta(t)-1}\right)\left(t-T_{\eta(t)-1}\right)\right)
\end{aligned}
$$

where $\Delta T_{j}=T_{j+1}-T_{j}$ and $\eta(t)$ is the smallest integer such that $T_{\eta(t)} \geq t$. We assume the risk-neutralized process for the discount price of $P(t, T)$ to be

$$
\begin{equation*}
\mathrm{d}\left(\frac{P(t, T)}{B(t)}\right)=\left(\frac{P(t, T)}{B(t)}\right) \sigma(t, T) \cdot \mathrm{d} \mathbf{Z}_{t} . \tag{1}
\end{equation*}
$$

Here, $\boldsymbol{\sigma}(t, T)$ is the volatility vector of $P(t, T)$, and $\mathbf{Z}_{t}$ is a finite dimensional Brownian motion under the risk-neutral measure, which we denote by $\mathbb{Q}$, and ' $\because$ ' is

[^1]the usual vector product. The volatility function satisfies the boundedness condition, $E\left[\int_{0}^{t} \sigma^{2}(s, T) \mathrm{d} s\right]<\infty, \forall t<T$.

Let $f_{j}(t)=f\left(t ; T_{j}, T_{j+1}\right)$ be the arbitrage-free forward lending rate (for simple compounding) seen at time $t$ for the period $\left(T_{j}, T_{j+1}\right)$, which relates to zero-coupon bond prices through

$$
f_{j}(t)=\frac{1}{\Delta T_{j}}\left(\frac{P\left(t, T_{j}\right)}{P\left(t, T_{j+1}\right)}-1\right) .
$$

Using Ito's lemma, we can derive that

$$
\begin{equation*}
\mathrm{d} f_{j}(t)=f_{j}(t) \gamma_{j}(t) \cdot\left[\mathrm{d} \mathbf{Z}_{t}-\sigma\left(t, T_{j+1}\right) \mathrm{d} t\right], \quad 1 \leq j \leq N \tag{2}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma_{j}(t)=\frac{1+\Delta T_{j} f_{j}(t)}{\Delta T_{j} f_{j}(t)}\left[\sigma\left(t, T_{j}\right)-\sigma\left(t, T_{j+1}\right)\right] \tag{3}
\end{equation*}
$$

i.e. the forward-rate volatility can be treated as a function of zero-coupon bond volatilities.

The LIBOR market model (or simply market model), instead, begins with the prescription of $\left\{\gamma_{j}(t)\right\}$. The volatilities of zero-coupon bonds, conversely, become functions of forward-rate volatilities:

$$
\begin{equation*}
\sigma\left(t, T_{j+1}\right)=-\sum_{k=\eta(t)}^{j} \frac{\Delta T_{k} f_{k}(t)}{1+\Delta T_{k} f_{k}(t)} \gamma_{k}(t)+\sigma\left(t, T_{\eta(t)}\right) \tag{4}
\end{equation*}
$$

Under usual regularity conditions on $\left\{\gamma_{j}(t)\right\}$, Brace et al. (1997) prove that $f_{j}(t)$ does not blow up. In addition, one can put $\sigma\left(t, T_{\eta(t)}\right)=0$ for $T_{\eta(t)-1} \leq t \leq T_{\eta(t)}$ without causing trouble. To summarize, equations (2) and (3) constitute the market model of interest rates, and, roughly speaking, the stochastic evolution of the $N$ forward rates is governed by their covariance defined by $\dagger$

$$
\begin{align*}
\operatorname{Cov}_{j k}^{i}= & \int_{T_{i-1}}^{T_{i}} \gamma_{j}(t) \cdot \gamma_{k}(t) \mathrm{d} t, \\
& i \leq j, k \leq N, \quad 1 \leq i \leq N . \tag{5}
\end{align*}
$$

With the market model, one can conveniently build in desirable correlation structures for forward rates.

To model volatility smiles/skews, we, following Chen and Scott (2001) and Andersen and Brotherton-Ratcliffe (2005), adopt a stochastic multiplier to the risk neutralized processes of the forward rates:

$$
\begin{align*}
\mathrm{d} f_{j}(t) & =f_{j}(t) \sqrt{V(t)} \gamma_{j}(t) \cdot\left[\mathrm{d} \mathbf{Z}_{t}-\sqrt{V(t)} \sigma_{j+1}(t) \mathrm{d} t\right] \\
\mathrm{d} V(t) & =\kappa(\theta-V(t)) \mathrm{d} t+\epsilon \sqrt{V(t)} \mathrm{d} W_{t} \tag{6}
\end{align*}
$$

Here, $\kappa, \theta$ and $\epsilon$ are state-independent variables, $\psi$ and $W_{t}$ is an additional 1D Brownian motion under the
risk-neutral measure. As a distinct feature of our modelling, we allow correlations between the stochastic multiplier and forward rates:

$$
\begin{equation*}
E^{Q}\left[\left(\frac{\gamma_{j}(t)}{\left\|\gamma_{j}(t)\right\|} \cdot \mathrm{d} \mathbf{Z}_{t}\right) \cdot \mathrm{d} W_{t}\right]=\rho_{j}(t) \mathrm{d} t, \quad \text { with }\left|\rho_{j}(t)\right| \leq 1 \tag{7}
\end{equation*}
$$

Here, $\left.\left(\gamma_{j}(t) /\left\|\gamma_{j}(t)\right\|\right\} \cdot \mathrm{d} \mathbf{Z}_{t}\right)$ is equivalent to (the differential of) a single Brownian motion that drives $f_{j}(t)$. The correlation coefficients, $\left\{\rho_{j}(t)\right\}$, will play an essential role to capture volatility smiles/skews.§ Technically, adopting a uniform volatility multiplier for all rates rather than one multiplier for each rate renders great advantages for analytical swaption pricing, and in addition, has very positive implications for model calibration. Note that for the model above, (3) remains as the no-arbitrage condition.

We now consider a pricing caplet under the extended LIBOR model (6). A caplet is a call option on a forward rate. Assume that the notional value of a caplet is one dollar, then the payoff of the caplet on $f_{j}\left(T_{j}\right)$ is

$$
\Delta T_{j}\left(f_{j}\left(T_{j}\right)-K\right)^{+} \triangleq \Delta T_{j} \max \left\{f_{j}\left(T_{j}\right)-K, 0\right\}
$$

To price the caplet we choose, in particular, $P\left(t, T_{j+1}\right)$ to be the numeraire and let $\mathbb{Q}^{j+1}$ denote the corresponding forward measure (i.e. the martingale measure corresponding to numeraire $P\left(t, T_{j+1}\right)$ ). The next proposition establishes the relationship between Brownian motions under the risk-neutral measure and under the forward measure.
Proposition 1: Let $\mathbf{Z}_{t}$ and $W_{t}$ be Brownian motions under $\mathbb{Q}$, then $\mathbf{Z}_{t}^{j+1}$ and $W_{t}^{j+1}$, defined by

$$
\begin{align*}
\mathrm{d} \mathbf{Z}_{t}^{j+1} & =\mathrm{d} \mathbf{Z}_{t}-\sqrt{V(t)} \sigma_{j+1}(t) \mathrm{d} t \\
\mathrm{~d} W_{t}^{j+1} & =\mathrm{d} W_{t}+\xi_{j}(t) \sqrt{V(t)} \mathrm{d} t \tag{8}
\end{align*}
$$

are Brownian motions under $\mathbb{Q}^{j+1}$, where

$$
\xi_{j}(t)=\sum_{k=1}^{j} \frac{\Delta T_{k} f_{k}(t) \rho_{k}(t)\left\|\gamma_{k}(t)\right\|}{1+\Delta T_{k} f_{k}(t)}
$$

In terms of $\mathbf{Z}_{t}^{j+1}$ and $W_{t}^{j+1}$, the extended market model (6) becomes

$$
\begin{gather*}
\mathrm{d} f_{j}(t)=f_{j}(t) \sqrt{V(t)} \gamma_{j}(t) \cdot \mathrm{d} \mathbf{Z}_{t}^{j+1}  \tag{9}\\
\mathrm{~d} V(t)=\left[\kappa \theta-\left(\kappa+\epsilon \xi_{j}(t)\right) V(t)\right] \mathrm{d} t+\epsilon \sqrt{V(t)} \mathrm{d} W_{t}^{j+1} \tag{10}
\end{gather*}
$$

In formalism, the multiplier process remains a square-root process under $\mathbb{Q}^{j+1}$. Yet part of the coefficients, $\xi_{j}(t)$, depends on forward rates, and such

[^2]dependence prohibits analytical option valuation. The time variability of $\xi_{j}(t)$, however, is small. In fact, we can write
\[

$$
\begin{align*}
\xi_{j}(t)= & \sum_{k=1}^{j} \frac{\Delta T_{k} f_{k}(0) \rho_{k}(t)\left\|\gamma_{k}(t)\right\|}{1+\Delta T_{k} f_{k}(0)} \\
& +\frac{\rho_{k}(t)\left\|\gamma_{k}(t)\right\| \Delta T_{k}}{\left(1+\Delta T_{k} f_{k}(0)\right)^{2}}\left(f_{k}(t)-f_{k}(0)\right) \\
& +O\left(\rho_{k}(t)\left\|\gamma_{k}(t)\right\| \Delta T_{k}^{2}\left(f_{k}(t)-f_{k}(0)\right)^{2}\right) \tag{11}
\end{align*}
$$
\]

In light of the martingale property $E^{Q^{i+1}}\left[f_{j}(t) \mid \mathcal{F}_{0}\right]=f_{j}(0)$, we see that

$$
\begin{aligned}
E^{Q^{j+1}}\left[\xi_{j}(t) \mid \mathcal{F}_{0}\right]= & \sum_{k=1}^{j} \frac{\Delta T_{k} f_{k}(0) \rho_{k}(t)\left\|\gamma_{k}(t)\right\|}{1+\Delta T_{k} f_{k}(0)} \\
& +O\left(\rho_{k}(t)\left\|\gamma_{k}(t)\right\| \operatorname{Var}\left(\Delta T_{k} f_{k}(t)\right)\right) \\
\operatorname{Var}\left(\xi_{j}(t) \mid \mathcal{F}_{0}\right) \approx & \left(\rho_{k}(t)\left\|\gamma_{k}(t)\right\|\right)^{2} \operatorname{Var}\left(\Delta T_{k} f_{k}(t)\right)
\end{aligned}
$$

According to the model, $\operatorname{Var}\left(\Delta T_{k} f_{k}(t)\right) \sim$ $\Delta T_{k} f_{k}^{2}(t)\left\|\gamma_{k}(t)\right\|^{2} V(t) t$. Since $\Delta T_{k} f_{k}(t)$ is mostly under $5 \%$, the expansion in (11) is dominated by the first term. Hence, to remove the dependence of $V(t)$ on $f_{j}(t)$ 's, we choose to ignore higher order terms in (11) and consider the approximation

$$
\begin{equation*}
\xi_{j}(t) \approx \sum_{k=1}^{j} \frac{\Delta T_{k} f_{k}(0) \rho_{k}(t)\left\|\gamma_{k}(t)\right\|}{1+\Delta T_{k} f_{k}(0)} \tag{12}
\end{equation*}
$$

This is close to the technique of 'freezing coefficients'. For notational simplicity we denote

$$
\tilde{\xi}_{j}(t)=1+\frac{\epsilon}{\kappa} \xi_{j}(t)
$$

and thus retain a neat equation for the process of $V(t)$ :

$$
\begin{equation*}
\mathrm{d} V(t)=\kappa\left[\theta-\tilde{\xi}_{j}(t) V(t)\right] \mathrm{d} t+\epsilon \sqrt{V(t)} \mathrm{d} W_{t}^{j+1} \tag{13}
\end{equation*}
$$

For the processes joint by (9) and (13), caplets can be priced along the approach pioneered by Heston (1993). According to arbitrage pricing theory (APT) (Harrison and Pliska, 1981), the price of the caplet on $f_{j}\left(T_{j}\right)$ can be expressed as

$$
\begin{aligned}
C_{l e t}(0)= & P\left(0, T_{j+1}\right) \Delta T_{j} E^{Q^{j+1}}\left[\left(f_{j}\left(T_{j}\right)-K\right)^{+} \mid \mathcal{F}_{0}\right] \\
= & P\left(0, T_{j+1}\right) \Delta T_{j} f_{j}(0) \\
& \times\left(E^{Q^{j+1}}\left[\mathrm{e}^{X\left(T_{j}\right)} \mathbf{1}_{X\left(T_{j}\right)>k} \mid \mathcal{F}_{0}\right]-\mathrm{e}^{k} E_{0}^{Q^{j+1}}\left[\mathbf{1}_{X\left(T_{j}\right)>k} \mid \mathcal{F}_{0}\right]\right),
\end{aligned}
$$

where $\quad X(t)=\ln f_{j}(t) / f_{j}(0) \quad$ and $\quad k=\operatorname{In} K / f_{j}(0)$. The two expectations above can be valuated using the moment generating function of $X\left(T_{j}\right)$, defined by

$$
\phi(X(t), V(t), t ; z) \triangleq E\left[\mathrm{e}^{z X\left(T_{j}\right)} \mid \mathcal{F}_{t}\right], \quad z \in C
$$

In terms of $\phi_{T}(z) \triangleq \phi(0, V(0), 0 ; z)$, we have that (see e.g. Kendall (1994) or more recently Duffie et al. (2000))

$$
\begin{align*}
& E^{Q^{j+1}}\left[\mathbf{1}_{X\left(T_{j}\right)>k} \mid \mathcal{F}_{0}\right] \\
& \quad=\frac{\phi_{T}(0)}{2}+\frac{1}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im}\left\{\mathrm{e}^{-\mathrm{i} u k} \phi_{T}(\mathrm{i} u)\right\}}{u} \mathrm{~d} u, \\
& E^{Q^{j+1}}\left[\mathrm{e}^{X\left(T_{j}\right)} \mathbf{1}_{X\left(T_{j}\right)>k} \mid \mathcal{F}_{0}\right]  \tag{14}\\
& \quad=\frac{\phi_{T}(1)}{2}+\frac{1}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im}\left\{\mathrm{e}^{-\mathrm{i} u k} \phi_{T}(1+\mathrm{i} u)\right\}}{u} \mathrm{~d} u .
\end{align*}
$$

The integrals above can then be evaluated numerically. For later reference we call this approach the Heston method.

When the Brownian motions $\mathbf{Z}_{t}^{j+1}$ and $W_{t}^{j+1}$ are independent, one can work out the moment generating function directly. In general, one can solve for $\phi(x, V, t ; z)$ from the Kolmogorov backward equation corresponding to the joint processes:

$$
\begin{align*}
& \frac{\partial \phi}{\partial t}+\kappa\left(\theta-\tilde{\xi}_{j} V\right) \frac{\partial \phi}{\partial V}-\frac{1}{2}\left\|\gamma_{j}(t)\right\|^{2} V \frac{\partial \phi}{\partial x}+\frac{1}{2} \epsilon^{2} V \frac{\partial^{2} \phi}{\partial V^{2}} \\
& \quad+\epsilon \rho_{j} V\left\|\gamma_{j}(t)\right\| \frac{\partial^{2} \phi}{\partial V \partial x}+\frac{1}{2}\left\|\gamma_{j}(t)\right\|^{2} V \frac{\partial^{2} \phi}{\partial x^{2}}=0 \tag{15}
\end{align*}
$$

subject to terminal condition

$$
\begin{equation*}
\phi\left(x, V, T_{j} ; z\right)=\mathrm{e}^{z x} . \tag{16}
\end{equation*}
$$

It is known that the solution is of the form

$$
\begin{equation*}
\phi(x, V, t ; z)=\mathrm{e}^{A(t, z)+B(t, z) V+z x} \tag{17}
\end{equation*}
$$

and $A$ and $B$ are available analytically for constant coefficients (Heston 1993). The analytical solutions can be extended to the case of piece-wise coefficients through recursion, as is also pointed out in Andersen and Andreasen (2002). The proof of the next proposition is provided in the appendix for completeness.
Proposition 2: For piece-wise constant coefficients, $A$ and $B$ are given by recursive expressions

$$
\left\{\begin{array}{rl}
A(t, z)= & A\left(T_{k}, z\right)+\frac{\kappa \theta}{\epsilon^{2}}\left\{\left(a_{k}+d_{k}\right)\left(T_{k}-t\right)\right. \\
& \left.-2 \ln \left[\frac{1-g_{k} \mathrm{e}^{d_{k}\left(T_{k}-t\right)}}{1-g_{k}}\right]\right\}, \\
B(t, z)= & \left.B\left(T_{k}, z\right)+\frac{\left(a_{k}+d_{k}-\epsilon^{2} B\left(T_{k}, z\right)\right)\left(1-\mathrm{e}^{d_{k}\left(T_{k}-t\right)}\right)}{\epsilon^{2}\left(1-g_{k} \mathrm{e}_{k}\left(T_{k}-t\right)\right.}\right) \tag{18}
\end{array},\right.
$$

where

$$
\begin{aligned}
& a_{k}=\kappa \xi-\rho_{j}\left(T_{k}\right) \epsilon\left\|\gamma_{j}\left(T_{k}\right)\right\| z, \\
& d_{k}=\sqrt{a^{2}-\left\|\gamma_{j}\left(T_{k}\right)\right\|^{2} \epsilon^{2}\left(z^{2}-z\right)}, \\
& g_{k}=\frac{a_{k}+d_{k}-\epsilon^{2} B\left(T_{k}, z\right)}{a_{k}-d_{k}-\epsilon^{2} B\left(T_{k}, z\right)} .
\end{aligned}
$$

## 3. Swaption pricing

The equilibrium swap rate for a period $\left(T_{m}, T_{n}\right)$ is defined by

$$
R_{m, n}(t)=\frac{P\left(t, T_{m}\right)-P\left(t, T_{n}\right)}{B^{\mathrm{S}}(t)}
$$

where

$$
B^{\mathrm{S}}(t)=\sum_{j=m}^{n-1} \Delta T_{j} P\left(t, T_{j+1}\right)
$$

is an annuity. The payoff of a swaption on $R_{m, n}\left(T_{m}\right)$ can be expressed as

$$
B^{\mathrm{S}}\left(T_{m}\right) \cdot \max \left(R_{m, n}\left(T_{m}\right)-K, 0\right)
$$

where $K$ is the strike rate.
The swap rate can be regarded as the price of a tradable portfolio relative to the price of the annuity $B^{\mathrm{S}}(t)$. This portfolio consists of one long $T_{m}$-maturity zero-coupon bond and one short $T_{n}$-maturity zero-coupon bond. According to APT, the swap rate is a martingale under the measure corresponding to the numeraire $B^{\mathrm{S}}(t)$. This measure is called the forward swap measure (Jamshidian 1997) and is denoted by $\mathbb{Q}^{S}$ in this paper. Similar to pricing under a forward measure, we need to characterize the Brownian motions under the forward swap measure.

Proposition 3: Let $\mathbf{Z}_{t}$ and $W_{t}$ be Brownian motions under $\mathbb{Q}$, then $\mathbf{Z}_{t}^{\mathrm{S}}$ and $W_{t}^{\mathrm{S}}$, defined by

$$
\begin{align*}
\mathrm{d} \mathbf{Z}_{t}^{\mathrm{S}} & =\mathrm{d} \mathbf{Z}_{t}-\sqrt{V(t)} \sigma^{\mathrm{S}}(t) \mathrm{d} t \\
\mathrm{~d} W_{t}^{\mathrm{S}} & =\mathrm{d} W_{t}+\sqrt{V(t)} \xi^{\mathrm{S}}(t) \mathrm{d} t \tag{19}
\end{align*}
$$

are Brownian motions under $\mathbb{Q}^{S}$, where

$$
\begin{equation*}
\sigma^{\mathrm{S}}(t)=\sum_{j=m}^{n-1} \alpha_{j} \sigma\left(t, T_{j+1}\right), \quad \xi^{\mathrm{S}}(t)=\sum_{j=m}^{n-1} \alpha_{j} \xi_{j}, \tag{20}
\end{equation*}
$$

with weights

$$
\alpha_{j}=\alpha_{j}(t)=\frac{\Delta T_{j} P\left(t, T_{j+1}\right)}{B^{\mathrm{S}}(t)}
$$

Using Ito's lemma one can show that, under the forward swap measure, the swap rate process becomes

$$
\begin{align*}
\mathrm{d} R_{m, n}(t) & =\sqrt{V(t)} \sum_{j=m}^{n-1} \frac{\partial R_{m, n}(t)}{\partial f_{j}(t)} f_{j}(t) \gamma_{j}(t) \cdot \mathrm{d} \mathbf{Z}^{\mathrm{S}}(t),  \tag{21}\\
\mathrm{d} V(t) & =\kappa\left[\theta-\tilde{\xi}^{\mathrm{S}}(t) V(t)\right] \mathrm{d} t+\epsilon \sqrt{V(t)} \mathrm{d} W^{\mathrm{S}}(t) .
\end{align*}
$$

Here

$$
\tilde{\xi}^{\mathrm{S}}(t)=1+\frac{\epsilon}{\kappa} \xi^{\mathrm{S}}(t)
$$

For the partial derivatives of the swap rate with respect to forward rates, we have

## Proposition 4: Let

$$
R_{m, n}(t)=\sum_{k=m}^{n-1} \alpha_{k} f_{k}, \quad \alpha_{k}=\frac{\Delta T_{k} P\left(t, T_{k+1}\right)}{B^{\mathrm{S}}(t)}
$$

then there is

$$
\begin{aligned}
\frac{\partial R_{m, n}(t)}{\partial f_{j}(t)}= & \alpha_{j}+\frac{\Delta T_{j}}{1+\Delta T_{j} f_{j}(t)}\left[\sum_{l=m}^{j-1} \alpha_{l}\left(f_{l}-R_{m, n}(t)\right)\right] \\
& m \leq j \leq n-1
\end{aligned}
$$

Parallel to swaption pricing under the standard market model (e.g. Andersen and Andreasen 2000, Sidennius 2000), we approximate the swap rate process by a lognormal process with a stochastic volatility:

$$
\begin{align*}
\mathrm{d} R_{m, n}(t)= & R_{m, n}(t) \sqrt{V(t)} \sum_{j=m}^{n-1} w_{j}(0) \gamma_{j}(t) \cdot \mathrm{d} \mathbf{Z}^{\mathrm{S}}(t), \\
& 0 \leq t<T_{m}  \tag{22}\\
\mathrm{~d} V(t)= & \kappa\left[\theta-\tilde{\xi}_{0}^{\mathrm{S}}(t) V(t)\right] \mathrm{d} t+\epsilon \sqrt{V(t)} \mathrm{d} W^{\mathrm{S}}(t)
\end{align*}
$$

where

$$
\begin{aligned}
w_{j}(t) & =\frac{\partial R_{m, n}(t)}{\partial f_{j}} \frac{f_{j}(t)}{R_{m, n}(t)} \\
\xi_{0}^{S}(t) & =\sum_{j=m}^{n-1} \alpha_{j}(0) \xi_{j}(t)
\end{aligned}
$$

In the above approximations, we have removed the dependence of $\xi_{0}^{S}(t)$ on forward rates through taking full advantage of the negligible time variability of $w_{j}(t)$ and $\alpha_{j}(t)$ (compared with that of forward rates). As a result, the approximate swap-rate process has moment generating function in closed form, and we thus regain the analytical tractability of the model under the forward swap measure. This is the key treatment in this paper, which works well for the market model with the square-root volatility dynamics but may not work for general volatility dynamics. Note that when $n=m+1$, $R_{m, m+1}(t)=f_{m}(t)$ and $B^{\mathrm{S}}(t)=\Delta T_{m} P\left(t, T_{m+1}\right)$, i.e. the swap rate reduces to a forward rate, and the swaption reduces to a caplet. $\dagger$ Theoretically, we can treat a caplet as a special case of swaptions.

Instead of following Heston's approach for numerical pricing, we adopt a transformation method developed by Carr and Madan (1998). Under the forward swap measure, we have the following expression for swaptions

$$
\begin{equation*}
P S(0)=B^{\mathrm{S}}(0) R_{m, n}(0) E^{\mathrm{S}}\left[\left(\mathrm{e}^{X\left(T_{m}\right)}-\mathrm{e}^{k}\right)^{+} \mid \mathcal{F}_{0}\right] \tag{23}
\end{equation*}
$$

where $E^{\mathrm{S}}[\cdot]$ stands for expectation under the forward swap measure, $X\left(T_{m}\right)=\ln R_{m, n}\left(T_{m}\right) / R_{m, n}(0)$ and

[^3]$k=\ln K / R_{m, n}(0)$. Let $G(k)=E^{\mathrm{S}}\left[\left(\mathrm{e}^{X\left(T_{m}\right)}-\mathrm{e}^{k}\right)^{+} \mid \mathcal{F}_{0}\right]$. Carr and Madan (1998) relate the Laplace transform of $G(k)$ to the moment-generating function of $X\left(T_{m}\right)$ :
$$
\psi(u)=\frac{\phi_{T_{m}}(1+a+\mathrm{i} u)}{(a+\mathrm{i} u)(1+a+\mathrm{i} u)} \quad \text { for } a>0
$$
where $\phi_{T_{m}}(v)=\phi(0, V(0), 0, v)$, which is characterized in (17) and detailed in Proposition 2 (with $m, \rho^{\mathrm{S}}$ and $\gamma_{m, n}$ in places of $j, \rho_{j}$ and $\gamma_{j}$ ). The prices of swaptions on $R_{m, n}\left(T_{m}\right)$ then follows from an inverse Laplace transform
\[

$$
\begin{equation*}
G_{T}(k)=\frac{1}{\pi} \int_{0}^{\infty} \mathrm{e}^{-(a+\mathrm{i} u) k} \psi(u) \mathrm{d} u \tag{24}
\end{equation*}
$$

\]

which can be evaluated numerically using FFT. For details the reader is referred to Carr and Madan (1998). For later reference we call the above method the FFT method.

Some remarks are in order. First, a rigorous error analysis of the lognormal approximation poses an open challenge. For the standard market model (without stochastic volatility), a recent paper by Brigo et al. (2004) studies the quality of the approximation using entropy distance, but an error estimation for option pricing remains beyond reach. The analysis may be applicable to the approximation (22) for the case of zero rate-multiplier correlation. In this paper, we resort to numerical pricing comparisons in order to gauge the pricing accuracy of the FFT method.

Second, fast calibration is another challenge. A calibration procedure may proceed with the following steps. First of all, we can decouple the calibration of the multiplier process from that of the forward-rate processes. We may, for instance, first estimate the multiplier process using the time series data of implied Black's volatilities of at-the-money caplets, as is suggested in Chen and Scott (2001). Once the process of $V(t)$ is specified, we can proceed to determine the pair of $\left(\left\|\gamma_{j}\right\|, \rho_{j}\right)$ through matching to the smile or skew of $T_{j}$-maturity caplets. This will lead to a bi-variate optimization problem, which is easily manageable. Once $\rho_{j}^{\prime}$ s are obtained, we can proceed to calibrate the model to ATM swaptions by taking time-dependent $\left\|\gamma_{j}\right\|$ 's. In our preliminary studies, the calibration is time consuming. If, furthermore, one wants to calibrate to swaption smiles/skews, then he/she may have to let $\rho_{j}$ 's be time dependent as well. This will result in a middle-scale optimization problem, which may be challenging to solve. As a matter of fact, for Heston's type model with time-dependent parameters, efficient calibration remains an outstanding problem.

Specifically, taking the rate-multiplier correlations into account, we can derive the following equation for $\left(\gamma_{j} /\left\|\gamma_{j}\right\|\right)$ )s,

$$
\left(1-\rho_{j}\left(t_{i}\right) \rho_{k}\left(t_{i}\right)\right)\left(\frac{\gamma_{j}}{\left\|\gamma_{j}\right\|}\right) \cdot\left(\frac{\gamma_{k}}{\left\|\gamma_{k}\right\|}\right)+\rho_{j}\left(t_{i}\right) \rho_{k}\left(t_{i}\right)=C_{j k}\left(t_{i}\right),
$$

where $C_{j k}$ is the historical correlation between the time series data of $f_{j}$ and $f_{k}$. The existence of $\left(\gamma_{j} /\left\|\gamma_{j}\right\|\right)$ 's requires that the matrix with components

$$
\begin{equation*}
\frac{C_{j k}\left(t_{i}\right)-\rho_{j}\left(t_{i}\right) \rho_{k}\left(t_{i}\right)}{1-\rho_{j}\left(t_{i}\right) \rho_{k}\left(t_{i}\right)}, \quad i \leq j \wedge k \tag{25}
\end{equation*}
$$

be non-negative definite. Intuitively, (25) represents the correlation between the two forward rates after the factor of stochastic volatility is removed. An eigenvalue decomposition for the matrix with elements given in (25) will produce $\left\{\gamma_{j}\right\} /\left\|\gamma_{j}\right\|$. For details we refer to Wu (2003).

## 4. Numerical results

In this section we present results on swaption pricing by the FFT method. Under scrutiny are two issues: pricing accuracy and capability to generate volatility smiles and skews. For given forward-rate and multiplier processes, we compute and compare swaption prices (including caplets as special cases) obtained by both the FFT method and the Monte Carlo (MC) simulation method. We will check on the accuracy of the FFT method under both weak and strong effects of stochastic volatility. As will be seen, the differences in implied volatilities are mostly under $1 \%$, the bid/ask spread often seen in the markets. This suggests a high accuracy of the analytical approximation method.

Let us briefly describe the Monte Carlo simulation method for the extended market model. The MC method is implemented under the risk neutral measure. To build in the correlation between the forward rates and the stochastic factor, we recast the equation for the forward rates into

$$
\begin{align*}
\frac{\mathrm{d} f_{j}(t)}{f_{j}(t)}= & -V(t) \gamma_{j}(t) \cdot \sigma\left(t, T_{j+1}\right) \mathrm{d} t \\
& +\sqrt{V(t)}\left(\sqrt{1-\rho_{j}^{2}(t)} \gamma_{j}(t) \cdot \mathrm{d} \hat{\mathbf{Z}}_{t}+\rho_{j}(t)\left\|\gamma_{j}(t)\right\|_{2} \mathrm{~d} W_{t}\right) \tag{26}
\end{align*}
$$

where $\left(\hat{\mathbf{Z}}_{t}, W_{t}\right)$ is a vector of independent Brownian motions. Treated as a lognormal process, $f_{j}(t)$ is advanced by the so-called log-Euler scheme:

$$
f_{j}(t+\Delta t)=f_{j}(t) \mathrm{e}^{-V(t)\left(\gamma_{j}(t) \cdot \sigma\left(t, T_{j+1}\right)+\frac{1}{2}\left\|\gamma_{j}(t)\right\|^{2}\right) \Delta t+\sqrt{V(t)}\left(\sqrt{1-\rho_{j}^{2}(t)} \gamma_{j}(t) \cdot \Delta \hat{\mathbf{Z}}_{t}+\rho_{j}(t)\left\|\gamma_{j}(t)\right\|_{2} \Delta W_{t}\right)} .
$$

Once $\left\{\left\|\gamma_{j}\right\|\right\}$ are obtained, the determination of $\left\{\gamma_{j}\right\}$ is subject to forward-rate correlations. Given, for instance, historical forward-rate correlations, we can solve for $\gamma_{j}$ by matching model correlations to the historical correlations.

The evolution of volatility, meanwhile, takes a step-wise moment-matched log-normal scheme (Andersen and Brotherton-Ratcliffe 2005). The use of the lognormal time-stepping scheme avoids the possible breakdown


Figure 1. Implied volatilities of 1-year swaptions; $\rho=0$.


Figure 2. Implied volatilities of 5-year swaptions; $\rho=0$.


Figure 3. Implied volatilities of 10-year swaptions; $\rho=0$.


Figure 4. Implied volatilities of 1-year swaptions; $\rho=-0.5$.


Figure 5. Implied volatilities of 5-year swaptions; $\rho=-0.5$.


Figure 6. Implied volatilities of 10 -year swaptions; $\rho=-0.5$.
Table 1. Swaption prices (bps) by FFT and MC methods, $\rho_{j}=-0.5$.

|  |  |  |  |  |  |  |  |  |  | ikes |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | . 015 |  |  |  |  |  |  |  | . 040 |  | 050 |  | 060 |  | 070 |  | . 080 |
| Exp. |  |  |  |  |  |  |  |  | Teno | . 5 year |  |  |  |  |  |  |  |  |
|  | 124.74 | (0.346) | 101.33 | (0.314) | 56.31 | (0.260) | 36.46 | (0.237) | 20.30 | (0.217) | 3.63 | (0.192) | 0.46 | (0.191) | 0.06 | (0.198) | 0.01 | (0.206) |
| 1 | 124.74 | (0.346) | 101.32 | (0.312) | 56.23 | (0.258) | 36.41 | (0.236) | 20.45 | (0.218) | 3.88 | (0.197) | 0.51 | (0.194) | 0.07 | (0.200) | 0.01 | (0.208) |
|  | 0.00 | (0.000) | 0.01 | (0.002) | 0.05 | (0.003) | 0.09 | (0.001) | 0.14 | $(-0.002)$ | 0.08 | (-0.005) | 0.03 | (-0.003) | 0.01 | (-0.002) | 0.00 | (-0.002) |
| 5 | 128.40 | (0.241) | 109.39 | (0.224) | 73.99 | (0.198) | 58.37 | (0.188) | 44.63 | (0.179) | 23.67 | (0.166) | 11.18 | (0.158) | 4.91 | (0.153) | 2.11 | (0.150) |
|  | 128.40 | (0.241) | 109.37 | (0.223) | 73.89 | (0.197) | 58.26 | (0.187) | 44.56 | (0.179) | 23.84 | (0.167) | 11.48 | (0.160) | 5.18 | (0.155) | 2.31 | (0.153) |
|  | 0.02 | (0.000) | 0.04 | (0.001) | 0.09 | (0.001) | 0.13 | (0.001) | 0.18 | (0.000) | 0.26 | $(-0.001)$ | 0.19 | (-0.002) | 0.13 | (-0.003) | 0.09 | (-0.003) |
| 10 | 122.56 | (0.200) | 108.01 | (0.189) | 80.96 | (0.175) | 68.84 | (0.169) | 57.86 | (0.164) | 39.51 | (0.156) | 25.93 | (0.150) | 16.48 | (0.146) | 10.24 | (0.143) |
|  | 122.55 | (0.199) | 107.98 | (0.188) | 80.90 | (0.174) | 68.78 | (0.168) | 57.79 | (0.164) | 39.53 | (0.156) | 25.67 | (0.149) | 16.36 | (0.145) | 10.22 | (0.142) |
|  | 0.02 | (0.001) | 0.04 | (0.001) | 0.09 | (0.001) | 0.12 | (0.001) | 0.16 | (0.000) | 0.24 | (-0.000) | 0.28 | (0.001) | 0.23 | (0.001) | 0.19 | (0.000) |
| Exp. |  |  |  |  |  |  |  |  | Teno | $=1$ year |  |  |  |  |  |  |  |  |
| 1 | 250.34 | (0.338) | 203.95 | (0.306) | 114.25 | (0.253) | 74.24 | (0.230) | 41.33 | (0.210) | 7.08 | (0.185) | 0.81 | (0.184) | 0.11 | (0.191) | 0.02 | (0.199) |
|  | 250.34 | (0.338) | 203.94 | (0.305) | 114.10 | (0.251) | 74.13 | (0.229) | 41.58 | (0.212) | 7.59 | (0.190) | 0.91 | (0.187) | 0.12 | (0.193) | 0.02 | (0.201) |
|  | 0.01 | (-0.000) | 0.02 | (0.002) | 0.10 | (0.003) | 0.17 | (0.001) | 0.26 | $(-0.002)$ | 0.15 | (-0.005) | 0.06 | (-0.003) | 0.02 | (-0.002) | 0.01 | (-0.001) |
| 5 | 256.57 | (0.237) | 218.84 | (0.220) | 148.23 | (0.194) | 116.90 | (0.184) | 89.25 | (0.175) | 46.95 | (0.162) | 21.80 | (0.154) | 9.33 | (0.149) | 3.90 | (0.146) |
|  | 256.57 | (0.237) | 218.80 | (0.219) | 148.05 | (0.193) | 116.68 | (0.183) | 89.11 | (0.175) | 47.24 | (0.163) | 22.34 | (0.155) | 9.83 | (0.151) | 4.28 | (0.149) |
|  | 0.03 | (0.000) | 0.07 | (0.001) | 0.18 | (0.001) | 0.25 | (0.001) | 0.34 | (0.001) | 0.50 | $(-0.001)$ | 0.36 | (-0.002) | 0.25 | $(-0.002)$ | 0.17 | (-0.003) |
| 10 | 243.95 | (0.197) | 215.12 | (0.187) | 161.37 | (0.172) | 137.21 | (0.166) | 115.26 | (0.162) | 78.52 | (0.154) | 51.30 | (0.148) | 32.40 | (0.144) | 19.96 | (0.140) |
|  | 243.92 | (0.196) | 215.06 | (0.186) | 161.25 | (0.171) | 137.08 | (0.166) | 115.14 | (0.161) | 78.57 | (0.154) | 50.75 | (0.147) | 32.13 | (0.143) | 19.89 | (0.140) |
|  | 0.03 | (0.001) | 0.07 | (0.001) | 0.17 | (0.001) | 0.23 | (0.001) | 0.31 | (0.000) | 0.46 | (-0.000) | 0.54 | (0.001) | 0.45 | (0.001) | 0.36 | (0.000) |

Table 2. Swaption prices (bps) by FFT and MC methods, $\rho_{j}=-0.5$.

of an Euler type scheme (e.g. Kloeden and Platen 1992) for the square-root process. In order to achieve higher accuracy, we have taken a small time-step size $(\Delta t=1 / 12)$ and a large number of paths $(50000)$ for the simulation method, and have applied antithetic variates technique (e.g. Boyle et al. 1997). Following a popular market practice, we only calculate at-the-money and out-of-the-money call or put options (payer's or receiver's swaptions). The prices for in-the-money options are obtained through put-call parity.

Example 1: The term structure of interest-rates, for-ward-rate processes and multiplier process are described below.

- Spot forward-rate curve: $\Delta T_{j}=0.5, \quad f_{j}(0)=$ $0.04+0.00075 j$, for all $j$.
- Volatility term structure of a two-factor model

$$
\begin{aligned}
\gamma_{j}\left(T_{k}\right)= & \left(0.08+0.1 \mathrm{e}^{-0.05(j-k)}, 0.1-0.25 \mathrm{e}^{-0.1(j-k)}\right) \\
& k \leq j .
\end{aligned}
$$

- Multiplier process: $V(0)=\theta=\kappa=1$ and $\epsilon=1.5$.
- Rate-multiplier correlations: $\rho_{j}=-0.5$ or 0 for all $j$.
This volatility term structure corresponds to a short rate volatility of about $25 \%$. The initial interest-rate term structure and multiplier dynamics are taken from Andersen and Brotherton-Ratcliffe (2005), which may have a practical background. In this example, $\kappa=1$ corresponds to a half life of mean reversion equal to $\ln (2) / \kappa=0.69$ (year), which represents a strong mean reversion that results in a weak effect of stochastic volatility for a long horizon.

Figures 1-6 display the implied Black's volatilities of swaptions, where ' $x$ ' is for the FFT method and ' $o$ ' is for the MC method. It can be seen that the two sets of implied volatilities mostly overlap each other. We can also see that the smiles and skews become flattened as maturity gets longer, which reflects the weakened stochasticity of volatilities. To get a complete picture of pricing accuracy, we also present dollar prices of the swaptions. Tables 1 and 2 detail swaption prices in basis points (bps) for the case $\rho=-0.5$. Also included in the tables are implied Black's volatilities, the difference between the implied volatilities and the radius (or half of the width) of $95 \%$ confidence interval (CI) for the Monte Carlo prices. These quantities are presented for pairs of maturity and strike under the format given in table 3.

For the FFT method we have taken dampening parameter $\alpha=2$, truncation range $A=50$, and number of divisions $N=100$. These selections were made after several trials. Figures 7 and 8 display the real and imaginary parts of $\psi_{T}(u)$ for the in-1-to-1 swaption with notional value equal to one dollar. As can be seen in the plots, beyond $A=50$, both real and imaginary parts are well under the magnitude of $10^{-4}$. Laplace transforms of other swaptions look similar. We have also computed the prices using Heston's method, and found that, under the same resolution of discretization, the Heston prices are almost indistinguishable from their FFT

Table 3. The ordering of cell entries.

|  | Strike |  |
| :--- | :--- | :---: |
| Maturity | FFT Price | (Implied Vol.) |
|  | MC Price |  |
| Radius of 95\% CI | (Implied Vol.) <br> (Difference of <br> implied volatilities.) |  |



Figure 7. Real part of $\phi_{T}(u)$ for the in-1-to-1 swaption.


Figure 8. Imaginary part of $\phi_{T}(u)$ for the in-1-to-1swaption.

Table 4. CPU times of the three methods with $\rho=-0.5$.

|  | FFT | Heston | Monte Carlo |
| :---: | :---: | :---: | :---: |
| CPU(seconds) | 0.93 | 1.3 | 39486 |

counterparts. We thus omit the Heston prices in the presentation.

In table 4, we report the CPU times for FFT, Heston's and the Monte Carlo methods. The computations are done under MATLAB-5.3 on a PC with a 1.1 GHz Intel Celeron CPU. Note that each execution of FFT method


Figure 9. Implied volatilities of one-year swaptions; $\rho=-0.5$.


Figure 10. Implied volatilities of five-year swaptions; $\rho=-0.5$.


Figure 11. Implied volatilities of ten-year swaptions; $\rho=-0.5$.
produces $N=100$ prices for about $71 \%$ of the CPU time taken by Heston's method (which produces only one price). The ratio of $71 \%$ is consistent with the fact Heston's method evaluates two integrals, while the FFT method evaluates only one.

Example 2: We redo the calculation with the same input data of Example 1 except $\kappa=0.15$. This kappa corresponds to a half-life of mean reversion of $\ln (2) / \kappa=4.6$ years, and it represents a stronger effect of stochastic volatility for a longer time horizon. For brevity, we only report the implied Black's volatilities for the case of negative rate-multiplier correlations, $\rho_{j}=-0.5$. Figures 9-11 again demonstrate the close agreement between two sets of implied volatilities, with however exceptions amongst 10 -year maturity caplets and in-10-to- 1 swaptions. We notice that for these options with a long maturity yet short tenors, the MC results remain the same for a direct valuation or indirect valuation through put-call parity. On the other hand, the percentage price differences are very small for the deeply in-the-money swaptions. We thus can still conclude that the pricing accuracy of the FFT method under strong stochastic volatility remains very high. The high accuracy suggests the robustness of the FFT method with regard to the strength of stochastic volatilities. Comparing figure 11 with figure 6 , we see the former has obviously steeper skews for in-5-to-10 and in-10-to-10 swaptions, due to the stronger stochastic volatility.

From the modelling point of view, it is interesting to understand the impact of stochastic local volatility on the


Figure 12. Volatility skew versus swap-rate volatility.
level of an implied Black's volatility curve. Figures 12 and 13 show the implied volatility curves of swaptions across strikes produced by the market models (with and without stochastic volatility) that share the same terminal swaprate variance, $\operatorname{Var}\left(X\left(T_{m}\right)\right)$. For the stochastic volatility model, we have taken $\rho_{j}=-0.5$. These plots suggest that the implied volatility curve for the stochastic volatility model hangs in the same level as that of the flat implied volatility curve for the deterministic volatility model, while tilts near at-the-money strike. This desirable feature may not exist in other models, e.g. the displaced diffusion


Figure 13. Volatility skew versus swap-rate volatility.


Figure 14. Volatility smile and skews for one-year caplets.
model, for which the level of implied volatility curves changes with the displacement.

Finally, we take another look at the role of ratemultiplier correlations on the formation of volatility smiles or skews, through examining the variation of volatility smiles/skews in response to changes in the correlations. Figure 14 is for caplets, where the downward sloping skew corresponds to a negative correlation of $\rho=-0.5$, the upward sloping skew corresponds to a positive correlation of $\rho=0.5$, and the nearly symmetric smile corresponds to zero correlation, $\rho=0$. Unsurprisingly, similar correspondence exists in swaptions, as is depicted in figure 15. These figures show that through the extended model we can attribute volatility smiles/skews directly to the 'leveraging effect'. This is a very plausible feature to many practitioners.

## 5. Conclusion

This paper introduces a correlation-based extension of the market model. By adopting a multiplicative volatility


Figure 15. Volatility smile and skews for one-year swaptions.
multiplier that follows a square-root process, we develop a LIBOR version of Heston's model. With such a model, we can generate either volatility smiles or skews by taking appropriate correlation between the stochastic multiplier and forward rates. Approximate swaption pricing is achieved through an inverse Laplace transform, and the high accuracy of the transformation method is confirmed through pricing comparisons. The outcomes of the comparisons are strongly supportive of the entire treatment. The preliminary success of the model introduces other interesting problems, including a rigorous accuracy analysis for the approximations made in pricing, and the calibration of the model.

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## Appendix A: details of some derivations

Proof of Proposition 1: The Radon-Nikodym derivative of $\mathbb{Q}^{j+1}$ with respect to $\mathbb{Q}$ is

$$
\begin{aligned}
\frac{\mathrm{d} \mathbb{Q}^{j+1}}{\mathrm{~d} \mathbb{Q}} & =\frac{P\left(t, T_{j+1}\right) / P\left(0, T_{j+1}\right)}{B(t)} \\
& =\exp \left[\int_{0}^{t}-\frac{1}{2} V(\tau) \sigma_{j+1}^{2}(\tau) \mathrm{d} \tau+\sqrt{V(\tau)} \sigma_{j+1} \cdot \mathrm{~d} \mathbf{Z}_{t}\right] \\
& \triangleq m_{j+1}(t), \quad t \leq T_{j+1} .
\end{aligned}
$$

Clearly we have

$$
\mathrm{d} m_{j+1}(t)=m_{j+1}(t) \sqrt{V(t)} \sigma_{j+1}(t) \cdot \mathrm{d} \mathbf{Z}_{t} .
$$

Let $\langle\cdot, \cdot\rangle$ denote covariance. By the Girsanov theorem (e.g. Hunt and Kennedy 2000), we obtain the Brownian motions under $\mathbb{Q}^{j+1}$ :

$$
\begin{aligned}
\mathrm{d} \mathbf{Z}_{t}^{j+1}= & \mathrm{d} \mathbf{Z}_{t}-\left\langle\mathrm{d} \mathbf{Z}_{t}, \mathrm{~d} m_{j+1}(t) / m_{j+1}(t)\right\rangle \\
= & \mathrm{d} \mathbf{Z}_{t}-\sqrt{V(t)} \sigma_{j+1}(t) \mathrm{d} t \\
\mathrm{~d} W_{t}^{j+1}= & \mathrm{d} W_{t}-\left\langle\mathrm{d} W_{t}, \mathrm{~d} m_{j+1}(t) / m_{j+1}(t)\right\rangle \\
= & \mathrm{d} W_{t}-\left\langle\mathrm{d} W_{t}, \sqrt{V(t)} \sigma_{j+1}(t) \cdot \mathrm{d} \mathbf{Z}_{t}\right\rangle \\
= & \mathrm{d} W_{t}+\sqrt{V(t)} \sum_{k=1}^{j} \frac{\Delta T_{k} f_{k}(t)\left\|\gamma_{k}(t)\right\|}{1+\Delta T_{k} f_{k}(t)} \\
& \times\left\langle\mathrm{d} W_{t}, \frac{\gamma_{k}(t)}{\left\|\gamma_{k}(t)\right\|} \cdot \mathrm{d} \mathbf{Z}_{t}\right\rangle \\
= & \mathrm{d} W_{t}+\sqrt{V(t)} \sum_{k=1}^{j} \frac{\Delta T_{k} f_{k}(t)\left\|\gamma_{k}(t)\right\|}{1+\Delta T_{k} f_{k}(t)} \rho_{k}(t) \mathrm{d} t
\end{aligned}
$$

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Proof of Proposition 2: For clarity we let $\tau=T-t$ and $l=\left\|\gamma_{j}\right\|$. Substituting the formal solution (17) to (15), we obtain the following equations for the undetermined coefficient:

$$
\begin{align*}
& \frac{\mathrm{d} A}{\mathrm{~d} \tau}=a \theta B, \\
& \frac{\mathrm{~d} B}{\mathrm{~d} \tau}=b_{2} B^{2}+b_{1} B+b_{0}, \tag{A1}
\end{align*}
$$

where
$a=\kappa \theta, \quad b_{0}=\frac{1^{2}}{2}\left(z^{2}-z\right), \quad b_{1}=(\rho \epsilon l z-\kappa \xi), \quad b_{2}=\frac{1}{2} \epsilon^{2}$.
Now consider (A1) with constant coefficients and general initial conditions

$$
A(0)=A_{0}, \quad B(0)=B_{0} .
$$

Since $B$ is independent of $A$, it is solved first. In the special case when

$$
b_{2} B_{0}^{2}+b_{1} B_{0}+b_{0}=0
$$

we have an easy solution

$$
\begin{align*}
& B(\tau)=B_{0} \\
& A(\tau)=A_{0}+a_{0} B_{0} \tau \tag{A2}
\end{align*}
$$

Otherwise, let $Y_{1}$ be the solution to

$$
b_{2} Y^{2}+b_{1} Y+b_{0}=0
$$

Assume $b_{2} \neq 0$, then

$$
\begin{equation*}
Y_{1}=\frac{-b_{1} \pm d}{2 b_{2}}, \quad \text { with } d=\sqrt{b_{1}^{2}-4 b_{0} b_{2}} \tag{A3}
\end{equation*}
$$

Without making any difference we take the ' + ' sign for Letting $Y_{1}$. We then consider the difference between $Y_{1}$ and $B$ :

$$
Y_{2}=B-Y_{1} .
$$

Clearly, $Y_{2}$ satisfies

$$
\begin{align*}
\frac{\mathrm{d} Y_{2}}{\mathrm{~d} \tau} & =\frac{\mathrm{d}\left(Y_{1}+Y_{2}\right)}{\mathrm{d} \tau} \\
& =b_{2}\left(Y_{1}+Y_{2}\right)^{2}+b_{1}\left(Y_{1}+Y_{2}\right)+b_{0}  \tag{A4}\\
& =b_{2} Y_{2}^{2}+\left(2 b_{2} Y_{1}+b_{1}\right) Y_{2} \\
& =b_{2} Y_{2}^{2}+d Y_{2}
\end{align*}
$$

with initial condition

$$
Y_{2}(0)=B_{0}-Y_{1} .
$$

Note in the last equality of (A4) we have used equation (A3). Equation (A4) belongs to the class of Bernoulli equations which can be solved explicitly. One can verify that the solution is

$$
\begin{equation*}
Y_{2}=\frac{d}{b_{2}} \frac{g \mathrm{e}^{\mathrm{d} \tau}}{\left(1-g \mathrm{e}^{\mathrm{d} \tau}\right)}, \quad \text { with } \quad g=\frac{-b_{1}+d-2 B_{0} b_{2}}{-b_{1}-d-2 B_{0} b_{2}} . \tag{A5}
\end{equation*}
$$

It follows that

$$
\begin{aligned}
B(\tau) & =Y_{1}+Y_{2} \\
& =\frac{-b_{1}+d}{2 b_{2}}+\frac{d}{b_{2}} \frac{g \mathrm{e}^{\mathrm{d} \tau}}{\left(1-g \mathrm{e}^{\mathrm{d} \tau}\right)} \\
& =B_{0}+\frac{\left(-b_{1}+d-2 b_{2} B_{0}\right)\left(1-\mathrm{e}^{\mathrm{d} \tau}\right)}{2 b_{2}\left(1-g \mathrm{e}^{\mathrm{d} \tau}\right)} .
\end{aligned}
$$

Having obtained $B$, we integrate the first equation of (A1) to get $A$ :

$$
\begin{aligned}
A(\tau)= & A_{0}+a_{0} \int_{0}^{\tau} B(s) \mathrm{d} s \\
= & A_{0}+a_{0} B_{0} \tau+\frac{a_{0}\left(-b_{1}+d-2 b_{2} B_{0}\right)}{2 b_{2}} \int_{0}^{\tau} \frac{1-\mathrm{e}^{\mathrm{d} \tau}}{1-g \mathrm{e}^{\mathrm{d} \tau}} \mathrm{~d} \tau \\
= & A_{0}+a_{0} B_{0} \tau+\frac{a_{0}\left(-b_{1}+d-2 b_{2} B_{0}\right)}{2 b_{2}} \\
& \times\left[\tau-\int_{0}^{\tau} \frac{(1-g) \mathrm{e}^{\mathrm{d} \tau}}{1-g \mathrm{e}^{\tau} \tau} \mathrm{d} \tau\right] \\
= & A_{0}+\frac{a_{0}\left(-b_{1}+d\right) \tau}{2 b_{2}}-\frac{a_{0}\left(-b_{1}+d-2 b_{2} B_{0}\right)}{2 b_{2} d} \\
& \times \int_{1}^{\mathrm{e}^{\mathrm{d} \tau}} \frac{(1-g)}{1-g u} \mathrm{~d} u \\
= & A_{0}+\frac{a_{0}\left(-b_{1}+d\right) \tau}{2 b_{2}}-\frac{a_{0}\left(-b_{1}+d-2 b_{2} B_{0}\right)}{2 b_{2} d} \frac{(g-1)}{g} \\
& \times \ln \left(\frac{1-g \mathrm{e}^{\mathrm{d} \tau}}{1-g}\right) \\
= & A_{0}+\frac{a_{0}}{2 b_{2}}\left[\left(-b_{1}+d\right) \tau-2 \ln \left(\frac{1-g \mathrm{e}^{\mathrm{d} \tau}}{1-g}\right)\right] .
\end{aligned}
$$

$$
\begin{aligned}
& A_{0}=A\left(\tau_{j}, z\right), \\
& B_{0}=B\left(\tau_{j}, z\right),
\end{aligned}
$$

and replacing $\tau$ by $\tau-\tau_{j}$, we arrive at (18). The solution $\phi(z)$ so obtained belongs to $\mathbb{C}^{1}$ and hence is a weak solution to (15)

Proof of Proposition 3: Denote the forward swap measure by $\mathbb{Q}^{\mathrm{S}}$. The Radon-Nikodym derivative for $\mathbb{Q}^{S}$ is

$$
\begin{aligned}
\frac{\mathrm{d} \mathbb{Q}^{\mathrm{S}}}{\mathrm{~d} \mathbb{Q}}= & \frac{B^{\mathrm{S}}(t) / B^{\mathrm{S}}(0)}{B(t)} \\
= & \frac{1}{B^{\mathrm{S}}(0)} \sum_{j=m}^{n-1} \Delta T_{j} P\left(0, T_{j+1}\right) \\
& \times \exp \left[\int_{0}^{t}-\frac{1}{2} V(\tau) \sigma_{j+1}^{2}(\tau) \mathrm{d} \tau+\sqrt{V(\tau)} \sigma_{j+1} \cdot \mathrm{~d} \mathbf{Z}_{t}\right] \\
\triangleq & m_{\mathrm{S}}(t), \quad t \leq T_{m} .
\end{aligned}
$$

There is

$$
\begin{aligned}
\mathrm{d} m_{\mathrm{S}}(t)= & \frac{1}{B^{\mathrm{S}}(0)} \sum_{j=m}^{n-1} \Delta T_{j} P\left(0, T_{j+1}\right) \mathrm{e}^{\int_{0}^{t}-\frac{1}{2} V(t) \sigma_{j+1}^{2}(\tau) \mathrm{d} \tau+\sqrt{V(t)} \sigma_{j+1} \cdot \mathrm{~d} \mathbf{Z}_{t}} \\
& \times \sqrt{V(t)} \sigma_{j+1}(t) \cdot \mathrm{d} \mathbf{Z}_{t} \\
= & \frac{1}{B^{\mathrm{S}}(0) B(t)} \sum_{j=m}^{n-1} \Delta T_{j} P\left(t, T_{j+1}\right) \sqrt{V(t)} \sigma_{j+1}(t) \cdot \mathrm{d} \mathbf{Z}_{t} \\
= & m_{\mathrm{S}}(t) \sum_{j=m}^{n-1} \alpha_{j} \sqrt{V(t)} \sigma_{j+1}(t) \cdot \mathrm{d} \mathbf{Z}_{t} .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathrm{d} \mathbf{Z}_{t}^{\mathrm{S}}= & \mathrm{d} \mathbf{Z}_{t}-\left\langle\mathrm{d} \mathbf{Z}_{t}, \mathrm{~d} m_{\mathrm{S}}(t) / m_{\mathrm{S}}(t)\right\rangle \\
= & \mathrm{d} \mathbf{Z}_{t}-\sqrt{V(t)} \sum \alpha_{j} \sigma_{j+1}(t) \mathrm{d} t, \\
= & \mathrm{d} \mathbf{Z}_{t}-\sqrt{V(t)} \sigma^{\mathrm{S}}(t) \mathrm{d} t, \\
\mathrm{~d} W_{t}^{\mathrm{S}}= & \mathrm{d} W_{t}-\left\langle\mathrm{d} W_{t}, \mathrm{~d} m_{\mathrm{S}}(t) / m_{\mathrm{S}}(t)\right\rangle \\
= & \mathrm{d} W_{t}-\left\langle\mathrm{d} W_{t}, \sqrt{V(t)} \sum \alpha_{j} \sigma_{j+1}(t) \cdot \mathrm{d} \mathbf{Z}_{t}\right\rangle \\
= & \mathrm{d} W_{t}+\sqrt{V(t)} \sum_{j=m}^{n-1} \alpha_{j} \sum_{k=1}^{j} \frac{\Delta T_{k} f_{k}(t)\left\|\gamma_{k}(t)\right\|}{1+\Delta T_{k} f_{k}(t)} \\
& \times\left\langle\mathrm{d} W_{t}, \frac{\gamma_{k}(t)}{\left\|\gamma_{k}(t)\right\|} \cdot \mathrm{d} \mathbf{Z}_{t}\right\rangle \\
= & \mathrm{d} W_{t}+\sqrt{V(t)} \sum_{j=m}^{n-1} \alpha_{j} \sum_{k=1}^{j} \frac{\Delta T_{k} f_{k}(t)\left\|\gamma_{k}(t)\right\|}{1+\Delta T_{k} f_{k}(t)} \rho_{k}(t) \mathrm{d} t \\
= & \mathrm{d} W_{t}+\sqrt{V(t)} \sum_{j=m}^{n-1} \alpha_{j} \xi_{j}(t) \mathrm{d} t \\
= & \mathrm{d} W_{t}+\sqrt{V(t)} \xi^{\mathrm{S}}(t) \mathrm{d} t
\end{aligned}
$$

Proof of Proposition 4: Differentiating the swap rate where $H(\cdot)$ is the Heaviside function defined such that with respect to a forward rate, we literally have $H(x)=1$ for $x \geq 0$ and $H(x)=0$ otherwise. Using the above derivatives as building blocks, we have

$$
\begin{aligned}
\frac{\partial \alpha_{k}}{\partial f_{j}} & =\Delta T_{k} \cdot\left(\frac{\partial P\left(t, T_{k+1}\right)}{\partial f_{j}} B^{\mathrm{S}}(t)-P\left(t, T_{k+1}\right) \frac{\partial B^{\mathrm{S}}(t)}{\partial f_{j}}\right) /\left(B^{\mathrm{S}}(t)\right)^{2} \\
& =\Delta T_{k} \cdot\left(\frac{-\Delta T_{j}}{1+\Delta T_{j} f_{j}} \cdot P\left(t, T_{k+1}\right) \cdot H(k-j) B^{\mathrm{S}}(t)-P\left(t, T_{k+1}\right) \sum_{l=m}^{n-1} \Delta T_{l} \frac{-\Delta T_{j}}{1+\Delta T_{j} f_{j}} \cdot P\left(t, T_{l+1}\right) \cdot H(l-j)\right) /\left(B^{\mathrm{S}}(t)\right)^{2} \\
& =\frac{-\Delta T_{j}}{1+\Delta T_{j} f_{j}} \alpha_{k}\left(H(k-j)-\sum_{l=j}^{n-1} \alpha_{l}\right) \\
& =\frac{\Delta T_{j}}{1+\Delta T_{j} f_{j}} \alpha_{k}\left(1-H(k-j)-\sum_{l=m}^{j-1} \alpha_{l}\right)
\end{aligned}
$$

$$
\frac{\partial R_{m, n}(t)}{\partial f_{j}}=\alpha_{j}+\sum_{k=m}^{n-1} \frac{\partial \alpha_{k}}{\partial f_{j}} f_{k}
$$

From the price-yield relation we obtain

$$
P\left(t, T_{k+1}\right)=\frac{P\left(t, T_{k}\right)}{1+\Delta T_{k} f_{k}}=\ldots=\frac{P\left(t, T_{m}\right)}{\prod_{l=m}^{k}\left(1+\Delta T_{l} f_{l}\right)}
$$

Apparently

$$
\begin{aligned}
\frac{\partial P\left(t, T_{k+1}\right)}{\partial f_{j}} & = \begin{cases}\frac{-\Delta T_{j}}{1+\Delta T_{j} f_{j}} \cdot \frac{P\left(t, T_{m}\right)}{\Pi_{l=m}^{k}\left(1+\Delta T_{l} f_{l}\right)}, & k \geq j \\
0, & k<j\end{cases} \\
& =\frac{-\Delta T_{j}}{1+\Delta T_{j} f_{j}} \cdot P\left(t, T_{k+1}\right) \cdot H(k-j),
\end{aligned}
$$

Substituting the above expression into equation (A6), we end up with

$$
\begin{align*}
\frac{\partial R_{m, n}(t)}{\partial f_{j}}= & \alpha_{j}+\frac{\Delta T_{j}}{1+\Delta T_{j} f_{j}} \sum_{k=m}^{n-1} \alpha_{k}\left(1-H(k-j)-\sum_{l=m}^{j-1} \alpha_{l}\right) f_{k}  \tag{A6}\\
= & \alpha_{j}+\frac{\Delta T_{j}}{1+\Delta T_{j} f_{j}}\left\{\sum_{k=m}^{n-1} \alpha_{k} f_{k}[1-H(k-j)]\right.  \tag{A7}\\
& \left.-\left(\sum_{l=m}^{j-1} \alpha_{l}\right)\left(\sum_{k=m}^{n-1} \alpha_{k} f_{k}\right)\right\} \\
= & \alpha_{j}+\frac{\Delta T_{j}}{1+\Delta T_{j} f_{j}} \sum_{k=m}^{j-1} \alpha_{k}\left(f_{k}-R_{m, n}(t)\right) \tag{A10}
\end{align*}
$$


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    $\dagger$ Current address: Department of Financial Mathematics, Peking University, China.
    \$In the literature both 'skew' and 'smirk' are used to name a slanted smile.

[^1]:    $\dagger$ A lognormal process whose volatility follows a square-root process (Cox et al. 1985).
    $\$$ Higher peak and fatter tails than that of a normal distribution.
    §Such a correspondence in fact also exists in other stochastic volatility models, e.g. Zhou (2003).

[^2]:    $\dagger$ Note that $\gamma_{j}(t)=0$ for $t \geq T_{j}$ since $f_{j}$ is fixed from the time $T_{j}$ becomes 'dead'.
    $\ddagger$ The distributional properties of $V(t)$ are well understood (e.g. Avellaneda and Laurence 2000). When $2 \kappa \theta>\epsilon^{2}$, in particular, $V(t)$ has a stationary distribution and stays strictly positive.
    §The empirical results of Chen and Scott (2001) suggest zero rate-multiplier correlation only for the nearest-term forward rate. In early versions of this paper, we had included plots for implied caplet volatilities of USD for the date of 5 July 2002. While the implied volatility curve of longer maturities appear like downward skews, the implied volatility curve of the six-month caplets is a smile, which is consistent with the finding of zero correlation between the stochastic volatility and the near-term forward rate. The plots are omitted for brevity.

[^3]:    $\dagger$ For convenience we have taken the same $\Delta T$ for both caps and swaptions. Note that in reality caps and swaptions can have different intervals between cash flows. In such a case, we may take the smallest interval for $\Delta T$.

