CALIBRATION OF CEV MARKET MODEL

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ABSTRACT. Generalized CEV process is an extension to the lognormal process for the standard market model (Brace *et al* (1997) and Jamshidian (1997)). The great advantage of the CEV model is its capacity to produce the *volatility skew* that is pronounced in the swaption prices. In this paper we introduce an efficient and robust methodology to calibrate the CEV model to market prices of swaptions as well as historical correlation of LIBOR rates. We first translate the input swaption prices for any specific swap rate into a pair of numbers: elasticity and implied CEV volatility, and then, consecutively, fit the local volatility coefficients to the historic correlation and the implied CEV volatilities. Regularization is adopted and the calibration is cast into minimization-maximization problems by the method of Lagrange multiplier. By utilizing the quadratic functional form of both objective function and constraints, we solve the inner maximization problems by a single matrix eigenvalue decomposition, which renders the efficiency of our approach. The outer minimization problems, meanwhile, are nicely subdued by gradient-based descending methods due to the convexity of the objective functions. The well-posedness of the Lagrange multiplier problems and the convergence of the descending methods are rigorously justified. Numerical results show that we have achieved very quality calibration.

Key words: CEV market model, non-parametric calibration, constrained optimization, Lagrange multiplier method.

Mathematics Subject Classification (2000): 60J60, 90C47

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1. Introduction

The standard *market model* is based on the assumption of lognormal dynamics for the observable forward rates (Brace, Gatarek and Musiela (1997), Jamshidian (1997) and Miltersen, Sandmann and Sondermann (1997)). It has two attractive features. First, it prices caplets and swaptions in closed-form, which enables efficient calibration of the model. Second, it is an multi-factor model and thus has enough degree of freedom to calibrate simultaneously the prices of benchmark instruments, namely, caplets and swaptions. Because of these features the standard market model is now playing a predominant role in the interest-rate derivative markets of various currencies. Calibration of the market model has been one of the focuses in recent research. A comprehensive calibration technology was recently developed by Wu (2001) for the standard market model.

While the standard market model enjoys great popularity, its limitation has not gone on unnoticed. It is widely agreed that the standard model can not accommodate the effect of *volatility skew*, and in practice the model is calibrated only to the at-the-money (ATM) caplets/floors and ATM swaptions. The volatility skew means the pattern of decreasing implied Black (1976) volatilities of caplet and swaption prices for increasing strike prices, which indicates a fat tail of empirical forward rate distributions relative to log-normal distributions. Volatility skews exist in US and German markets, and is particularly pronounced in the Japanese market. To cope with the volatility skew, Andersen and Andreasen (2000) extended the standard model to *constant elasticity of variance* (CEV) process for the forward rates. CEV model can essentially capture the volatility skew, and, as an extension to the standard market model, it retains the analytical tractability and renders closed-form formula to caplet and swaption prices.

In this paper we will generalize the methodology of Wu (2001) for standard market model to the calibration of the CEV model. In specifically, we will fit the model parameters to the prices of caplets and swaptions, and, in addition, to the historical correlation of the forward rates. We first generalize the notion of implied volatility to the CEV model, and then decouple the calibration into two sub-problems. The calibration to correlations is performed first, and the outcome is used in the calibration of the CEV-implied volatilities, instead of input prices. The decoupled calibration problems are then formulated as constrained minimization problems and subsequently recast into minimization-maximization (min-max) problems along the approach of Lagrange multiplier. Note that fitting the implied CEV

volatilities, instead of the input prices themselves, yields constraints as quadratic functions. What special in our approach are 1) the inner max-problem can be solved by a single matrix eigenvalue decomposition without iteration, and 2) the outer min-problem has convex objective function and thus renders easily to a gradient-based descending algorithm. Given all these advantages, the calibration is achieved reliably and efficiently.

This paper is organized as follows. In §2 we introduce the background of the CEV market model, and set up the mathematical formulations for the calibration. In §3 we describe the methodology of numerical solution, and offer rigorously justifications of the well-posedness of the formulations and the convergence of the numerical iteration processes. In §4 we present computational results with a practical problem. Finally in §5 we conclude.

Notation. For a square matrix A, we denote by $\mathrm{diag}(A)$ the column vector whose entries are the diagonal entries of A. Conversely, if d is a (column) vector, we define $\mathrm{diag}(d)$ the diagonal matrix whose diagonal entries are the components of d. We use $\|\cdot\|_F$ to denote the Frobenius norm for matrices and $\|\cdot\|_2$ for both the spectrum norm of a matrix and the 2-norm of a vector.

2. Problem formulation

The CEV market model (Andersen and Andreasen (2000)) was based on the assumption of CEV processes for forward London-Interbank-Offer-Rates (LIBOR). Let $f_j(t) = f(t; T_j, T_{j+1})$ be the arbitrage-free forward lending rate seen at time t for the period (T_j, T_{j+1}), then $f_j(t)$ is assumed to follow a CEV process[1]

$$df_j(t) = f_j^{\alpha_j}(t)\gamma_j(t) \cdot [\sigma_{j+1}(t)dt + d\mathbf{Z}(t)],$$

where α_j is a positive constant, $\mathbf{Z}(t)$ is the vector of n-dimensional independent Brownian motions for some properly chosen number n, $\gamma_j(t)$ is the vector of the instantaneous volatility coefficients, and $\sigma_{j+1}(t)$ is the vector of instantaneous volatility coefficients of zero-coupon bond of maturity T_{j+1}. Consider a collection of N forward rates, $f_j, j = 1, 2, \ldots, N$. As in the Heath-Jarrow-Morton model(1992), the drifts of forward term rates in the market model are completely determined by their volatilities. The no-arbitrage condition (Brace *et al.*,

[1]Define percentage volatility $\tilde{\gamma}_j = \frac{f_j^{\alpha_j}(t)\gamma_j(t)}{f_j(t)} = (f_j(t))^{\alpha_j - 1}\gamma_j(t)$. The elasticity is defined as $\frac{\partial \tilde{\gamma}_j}{\partial f_j} / \frac{\tilde{\gamma}_j}{f_j} = \alpha_j - 1 = constant$.

1997) gives rise to their relation

$$\sigma_j(t) = \sum_{k=1}^{j} \frac{\Delta T_{j-k} f_{j-k}^{\alpha_{j-k}}(t)}{1 + \Delta T_{j-k} f_{j-k}^{\alpha_{j-k}}(t)} \gamma_{j-k}(t),$$

where $\Delta T_j = T_{j+1} - T_j$ and $\gamma_j(t) = 0$ for $t \geq T_j$. As a convention we label today by $t = T_0 = 0$. For the mathematical properties of the CEV model we refer to Andersen and Andreasen (2000).

The stochastic evolution of the N forward rates is fully described by the quantities of covariance defined by

$$COV_{jk}^i = \int_{T_{i-1}}^{T_i} \gamma_j(t) \cdot \gamma_k(t) dt, \quad 1 \leq i \leq N.$$

Note that $COV_{jk}^i = 0$ for either $j < i$ or $k < i$ since either f_j or f_k has been reset by the time T_i. The corresponding correlations are

$$C_{jk}^i = \frac{COV_{jk}^i}{\sqrt{COV_{jj}^i} \cdot \sqrt{COV_{kk}^i}}, \quad 1 \leq i \leq N.$$

For fixed i, $\{C_{jk}^i\}$ constitute an $(N - i + 1)$ by $(N - i + 1)$ non-negative symmetric matrix:

$$\mathbf{C}^i = \begin{pmatrix} C_{i,i}^i & C_{i,i+1}^i & \dots & C_{i,N}^i \\ C_{i+1,i}^i & C_{i+1,i+1}^i & \dots & C_{i,N}^i \\ \dots & \dots & \dots & \dots \\ C_{N,i}^i & C_{N,i+1}^i & \dots & C_{N,N}^i \end{pmatrix}, \qquad i = 1, 2, \dots, N. \tag{1}$$

We now introduce the pricing of swaptions. The price formula for caplets follows as a special case. A swaption is an option on swap rate. Denote an annuity

$$B^S(t) = \sum_{j=m}^{n-1} \Delta T_j P(t, T_{j+1}),$$

where $P(t, T_{j+1})$ is the time t price of the zero-coupon bond with maturity T_{j+1} and par value \$1. The fair swap rate for the period (T_m, T_n) seen at time t is defined by

$$R_{m,n}(t) = \frac{P(t, T_m) - P(t, T_n)}{\sum_{j=m}^{n-1} \Delta T_j P(t, T_{j+1})}.$$

The swap rate is the fixed rate with which two parties will agree to swap fixed-rate payments for float-rate payments (indexed to LIBOR) for any notional amount, at times $T_j, j = m + 1, m + 2, \dots, n$. The forward rates relate to the zero-coupon bonds by

$$f_j(t) = \frac{1}{\Delta T_j} \left(\frac{P(t, T_j)}{P(t, T_{j+1})} - 1 \right).$$

Following the arguments of Jamshidian (1997) we can show that, with Ito's lemma, the dynamics of $R_{m,n}(t)$ is

$$dR_{m,n}(t) = \sum_{j=m}^{n-1} \frac{\partial R_{m,n}(t)}{\partial f_j(t)} f_j^{\alpha_j}(t)\gamma_j(t) \cdot dW^S(t), \tag{2}$$

where $W^S(t)$ is the n-dimensional independent Brownian motions under the *forward swap measure* induced by choosing $B^S(t)$ as the numeraire, while

$$\frac{\partial R_{m,n}(t)}{\partial f_j(t)} = \frac{\Delta T_j R_{m,n}(t)}{1+\Delta T_j f_j(t)} \left[\frac{P(t,T_n)}{P(t,T_m)-P(t,T_n)} + \frac{\sum_{k=j}^{n-1} \Delta T_k P(t,T_{k+1})}{B^S(t)} \right].$$

The swap rate process (2) is apparently not a CEV process, yet it can be approximated by one. Anderson and Andreasen (2000) proposed the following approximation by "frozen coefficient":

$$dR_{m,n}(u) = R_{m,n}^{\alpha_{m,n}}(u) \sum_{j=m}^{n-1} w_j(t)\gamma_j(u) \cdot dW^S(u), \quad t \le u < T_m, \tag{3}$$

where

$$w_j(t) = \frac{\partial R_{m,n}(t)}{\partial f_j(t)} \frac{f_j^{\alpha_j}(t)}{R_{m,n}^{\alpha_{m,n}}(t)},$$

and $\alpha_{m,n}$ is the power to be determined by least-squared fitting to the volatility skew of swaptions using formulae developed below. As we shall see that (3) leads to a closed-form solution for European swaptions.

Theorem 2.1 (Andersen and Andreasen, 1998). *Consider a European payer swaption on swap rate $R_{m,n}$ with strike rate K. Assume that the forward rate dynamics are given by the CEV specification (3). Define*

$$\begin{aligned} d &= \frac{K^{2(1-\alpha_{m,n})}}{(1-\alpha_{m,n})^2 \zeta_{m,n}^2(t)}, \quad b = \frac{1}{1-\alpha_{m,n}}, \quad f = \frac{R^{2(1-\alpha_{m,n})}}{(1-\alpha_{m,n})^2 \zeta_{m,n}^2(t)}, \\ g_\pm &= \frac{\ln \frac{R_{m,n}(t)}{K} \pm \frac{1}{2}\zeta_{m,n}^2(t)}{\zeta_{m,n}(t)}, \end{aligned} \tag{4}$$

and

$$\zeta_{m,n}^2(t) = \int_t^{T_m} \left\| \sum_{j=m}^{n-1} w_j(t)\gamma_j(s) \right\|^2 ds = \sum_{j=m}^{n-1}\sum_{k=m}^{n-1} w_j(t)w_k(t) \int_t^{T_m} \gamma_j(s)\cdot\gamma_k(s) ds, \tag{5}$$

then the swaption price at t is given by

a) *For $0 < \alpha_{m,n} < 1$ and an absorbing boundary at the level $R_{m,n} = 0$,*

$$PS(t,T_m,T_n) = B^S(t)[R_{m,n}(t)(1-\chi^2(d,b+2,f)) - K\chi^2(f,b,d)], \tag{6}$$

where $\chi^2(\cdot,\cdot,\cdot)$ is the cumulative distribution function for a non-central χ^2-distributed variable.

b). *For $\alpha_{m,n} = 1$,*

$$PS(t, T_m, T_n) = B^S(t)[R_{m,n}(t)N(g_+) - KN(g_-)], \tag{7}$$

where $N(\cdot)$ is the normal accumulative function.

c). *For $\alpha_{m,n} > 1$,*

$$PS(t, T_m, T_n) = B^S(t)[R_{m,n}(t)(1 - \chi^2(f, -b, d)) - K\chi^2(d, 2-b, f)]. \tag{8}$$

Remarks: 1). The non-central χ^2-distribution function can be evaluated numerically with a procedure developed by, for instance, Ding (1992). 2). A caplet is a special case of swaption, corresponding to $n = m + 1$. Note that forward rates are only special cases of swap rates, i.e., $R_{m,m+1} = f_m$, we will only mention swap rates thereafter.

The first step of calibration is to fit the $\alpha_{m,n}$ and $\zeta_{m,n}$ to the swaption (including caplet) prices by least-squares fitting. In particular, if there is only one input option price for a swap rate, we take the corresponding $\alpha_{m,n} = 1$ and solve for $\zeta_{m,n}$ by root finding. This process will translate all option prices on $R_{m,n}$ into the pairs of $\alpha_{m,n}$ and $\zeta_{m,n}$.

In this paper, we define our calibration problem as following: given $\{\alpha_{m,n}\}$, $\{\zeta_{m,n}\}$, and $\{\mathbf{C}^i\}$, determine from equations (1) and (5) the implied volatility functions $\gamma_j(t), j = 1, \ldots, N$.

We will take the non-parametric approach, looking for the volatilities in the form of piece-wise constant function in t:

$$\gamma_j(t) = \gamma_j^i = s_j^i(a_{j,1}^i, a_{j,2}^i, \ldots, a_{j,n}^i) \equiv s_j^i \mathbf{a}_j^i, \qquad \text{for} \quad T_{i-1} \le t \le T_i, \quad i \le j,$$

with

$$s_j^i = \|\gamma_j^i\|_2, \quad \text{and} \quad \|\mathbf{a}_j^i\|_2 = 1. \tag{9}$$

It will become clear shortly why we single out s_i^j, the norm of γ_j^i, as another variable. The total number of unknowns is proportional to $n \times N^2$, which in practice can be in the magnitude of hundreds and far bigger than the number of input prices and elements of correlation matrices. Hence, we are facing a middle- to large-scale under-determined problems. Luckily, the determination of $\{s_j^i\}$ and $\{\mathbf{a}_j^i\}$ are decoupled in the sense that, while the former depends on both $\{\mathbf{C}^i\}$ and $\{\zeta_{m,n}\}$, the latter depends on $\{\mathbf{C}^i\}$ only. To see this,

suppose the rank of $\mathbf{C}^i$ is less than or equal to n. Perform eigenvalue decomposition on $\mathbf{C}^i$:

$$\mathbf{C}^i = U\Lambda U^T,$$

where Λ is an n by n diagonal matrix with non-negative diagonal elements, and define $\mathbf{a}^i_j$ as the j^{th} row of $U\Lambda^{1/2}$, so we have

$$\mathbf{C}^i = \begin{pmatrix} \mathbf{a}^i_i, \\ \vdots \\ \mathbf{a}^i_N \end{pmatrix} \left((\mathbf{a}^i_i)^T, \ldots, (\mathbf{a}^i_N)^T \right). \tag{10}$$

Then the model correlation so obtained is

$$Corr(\Delta f_j(t_i), \Delta f_k(t_i)) = \frac{\Delta t_i \mathbf{a}^i_j \cdot \mathbf{a}^i_k}{\sqrt{\Delta t_i}\|\mathbf{a}^i_j\|_2 \cdot \sqrt{\Delta t_i}\|\mathbf{a}^i_k\|_2} = C^i_{jk}$$

by (9) and (10), where

$$\Delta f_j(t_i) = f_j^{\alpha_j}(t_i) s^i_j \mathbf{a}^i_j \cdot [\sigma_{j+1}\Delta t_i + \Delta Z(t_i)]$$

for some small Δt_i. Note that the columns of matrix $U\Lambda^{1/2}$ are called *principle components* of the matrix $\mathbf{C}^i$.

The complication in the determination of $\mathbf{a}^i_j, j = 1, \ldots, n$ is that the rank of $\mathbf{C}^i$ is in general much bigger than n. The former is typically equal to $N-i+1$, the number of forward rate "alive". In such case the above procedure for calculating $\mathbf{a}^i_j, j = 1, \ldots, n$ breaks down. Therefore, a preprocessing is in general needed to reduce the ranks of the given correlation matrices. For a given correlation matrix $\mathbf{C}^i$, preprocessing is naturally formulated as the following minimization problem with constraints:

$$\begin{aligned} &\min_{\hat{\mathbf{C}}^i} \|\mathbf{C}^i - \hat{\mathbf{C}}^i\|_F, \\ &\text{s.t.} \quad \hat{\mathbf{C}}^i \geq 0, \quad \text{rank}(\hat{\mathbf{C}}^i) \leq n, \quad \hat{C}^i_{kk} = 1, \quad k = i, \ldots, N. \end{aligned} \tag{11}$$

where the subindex F means the Frobenius's norm, and $\hat{C} \geq 0$ means that $\hat{C}$ is a non-negative matrix. Presumerably we need to solve (11) for $i = 1, 2, \ldots, N$.

Once we have obtained the low-rank approximation of the correlation matrices we can proceed to the determination of the forward rate volatilities s^i_j, subject to the input prices. The number of $\{s^i_j\}$ to be determined is $N(N+1)/2$. This is typically much higher than the number of the input prices. Hence, this is again an under determined problem. For the uniqueness of the solution we must adopt an objective function, which also serves as regularization condition for smoothness and stability of the volatility surface. A rather natural candidate for the objective function is

$$\|\nabla \mathbf{s}\|^2 + \epsilon\|\mathbf{s} - \mathbf{s}_0\|^2 \equiv -(\mathbf{s}, (\nabla \cdot \nabla)\mathbf{s}) + \epsilon\|\mathbf{s} - \mathbf{s}_0\|^2 \quad \text{for some } \epsilon > 0, \tag{12}$$

where $(\nabla \cdot \nabla)$ stands for the discrete Laplacian:

$$(\nabla \cdot \nabla)s_j^i = s_{j-1}^i + s_{j+1}^i + s_j^{i-1} + s_j^{i+1} - 4s_j^i, \tag{13}$$

and in details (12) reads

$$\sum_{i=1}^{N}\sum_{j=i}^{N} s_j^i \left(-s_{j-1}^i - s_{j+1}^i - s_j^{i-1} - s_j^{i+1} + 4s_j^i\right) + \epsilon \sum_{i=1}^{N}\sum_{j=i}^{N}(s_j^i - s_{j,0}^i)^2. \tag{14}$$

Note that $\mathbf{s}_0$ is a *priori* volatility surface prescribed as, say for example, the volatility surface of the previous day. By minimizing (12), we try to achieve a balance between the smoothness and the stability of the volatility surface. In (14), there are some "ghost" variables whose sub- or sup-indeces are out of the designated range. These "ghost" variables are eliminated by using "Neumann boundary condition"[2]:

$$\begin{aligned} &s_j^0 = s_j^1, && j = 1, 2, \ldots, N, \\ &s_{N+1}^i = s_N^i, && \\ &s_i^{i+1} = s_i^i, && i = 1, 2, \ldots, N, \\ &s_{i-1}^i = s_i^i. && \end{aligned}$$

The constraints of prices are expressed in terms of the implied CEV volatilities:

$$\begin{aligned} \zeta_{m,n}^2 &= \sum_{j,k=m,n-1} w_j w_k \sum_{i=1}^{m} s_j^i s_k^i \hat{C}_{j,k}^i \Delta T_{i-1} \\ &= \sum_{i=1}^{m} \Delta T_{i-1} \sum_{j,k=m,n-1} s_j^i s_k^i (w_j w_k \hat{C}_{j,k}^i), \quad \text{for some } m \text{ and } n, \end{aligned} \tag{15}$$

where $\hat{C}_{jk}^i$ in (15) is an element of $\hat{\mathbf{C}}^i$, the low-rank approximation to correlation matrix $\mathbf{C}^i$. When $n = m + 1$ we have, $w_j = 1, w_k = 0, k \neq j$, and condition (15) reduces to

$$\zeta_j^2 = \sum_{i=1}^{j} \Delta T_{i-1} (s_j^i)^2, \quad \text{for some } j, \tag{16}$$

Note that all functions in (14)-(15) are quadratic functions in $\{s_j^i\}$. Such feature, as we shall see later, gives rise to a powerful numerical method.

For practical numerical implementation, we introduce matrix notations for the problem. First we line up the volatilities in a one-dimensional array

$$\mathbf{X} = \begin{pmatrix} \mathbf{s}^1 \\ \mathbf{s}^2 \\ \vdots \\ \mathbf{s}^N \end{pmatrix}$$

[2] Corresponding to the boundary condition of zero normal derivative for partial differential equations. The application here may result in multiple definitions of "ghost" values beyond the range of the indeces, which is however, harmless.

with

$$\mathbf{s}^i = \begin{pmatrix} s_i^i \\ s_{i+1}^i \\ \vdots \\ s_N^i \end{pmatrix}.$$

We then define the matrix corresponding to the discrete Laplacian in (13) as

$$B = diag(-1, -1, 4, -1, -1).$$

Finally we associate each instrument with the following "weight" matrix

$$W_{m,n} = diag(0, \ldots, 0, w_m, \ldots, w_{n-1}, 0, \ldots, 0),$$

and "correlation matrix"

$$G_{m,n} = diag(\Delta T_0 W_{m,n} \hat{C}^1 W_{m,n}, \Delta T_1 W_{m,n} \hat{C}^2 W_{m,n}, \ldots, \Delta T_{m-1} W_{m,n} \hat{C}^m W_{m,n}, 0, \ldots, 0).$$

With the above matrices, the calibration to prices under the objective function (12) can be cast into a neat yet equivalent form

$$\begin{aligned} &\min_X X^T B X + \epsilon (X - X_0)^T (X - X_0), \\ &\text{s.t.} \quad X^T G_{m,n} X = \zeta_{m,n}^2 \quad \text{for some } m \text{ and } n. \end{aligned} \tag{17}$$

The objective function in (17) can be simplified further. Expanding the function we have

$$\begin{aligned} &X^T B X + \epsilon (X - X_0)^T (X - X_0) \\ =&(X + \epsilon (B + \epsilon I)^{-1} X_0)^T (B + \epsilon I)(X + \epsilon (B + \epsilon I)^{-1} X_0) \\ &+ \epsilon (X_0)^T (I + (B + \epsilon I)^{-1}) X_0, \end{aligned}$$

where the last term is a constant and thus can be ignored for optimization purpose. Introducing

$$A = B + \epsilon I,$$

which is a positive-definite matrix, we finally formulate the problem of price calibration into

$$\begin{aligned} &\min_X (X - \tilde{X}_0)^T A (X - \tilde{X}_0), \\ &\text{s.t.} \quad X^T G_{m,n} X = \zeta_{m,n}^2 \quad \text{for some } m \text{ and } n. \end{aligned} \tag{18}$$

The recast form (18) highlights that the objection function is in quadratic form with a positive definite matrix. Nevertheless, the Lagrange multiplier problem corresponding to (18) may not be well-defined in the sense that, for a set of finite multipliers, the maximum

of the inner maximization problem can be infinity. For this reason, we superimpose a convex function to the objective function:

$$\begin{aligned} &\min_X U\left((X-\tilde{X}_0)^T A(X-\tilde{X}_0)\right), \\ &\text{s.t.} \quad X^T G_{m,n} X = \zeta_{m,n}^2 \quad \text{for some } m \text{ and } n, \end{aligned} \tag{19}$$

where $U(y)$ is superlinear and monotonically increasing function for $y \geq 0$, for examples, $U(y) = y^2, y\ln y$ or e^y. Problem (19) then shares the same constrained minimum(s) with (18). We will proceed next to solve the problem with the usual approach of descend.

3. Solution Methodology

In this section, we will develop numerical methods to solve the constrained minimization problems (11) and (19). Roughly speaking, our methodology is the combinations of the method of Lagrange multiplier and steepest descend. In developing the numerical methods, we have taken full advantages of the special structure of the objective functions and constraints. Moreover, we have justified rigorously the well-posedness of the Lagrange multiplier problems and obtained the convergence of the numerical methods.

3.1. Eigen-decomposition-based rank reduction algorithm.

For a given non-negative symmetric N by N matrix C, we define a low-rank approximation as the solution to the following problem

$$\begin{aligned} &\min_X \|C-X\|_F, \\ &\text{s.t.} \quad \text{rank}(X) \leq n < N, \quad \text{diag}(X) = \text{diag}(C). \end{aligned} \tag{20}$$

We denote any one solution to problem (20) by C^*, and the feasible set of solutions by

$$\mathcal{F} = \{\, X \in \mathcal{R}^{N\times N} \;\big|\; \text{rank}(X) \leq n,\ \text{diag}(X) = \text{diag}(C) \,\}.$$

For applications in the market model, C^* will serve subsequently as a correlation matrix and thus is expected to be a non-negative definite matrix. It may seem that the feasible set of the optimal problem should be $\mathcal{F}^+$, the subset of $\mathcal{F}$ that consists of only positive semi-definite matrices. Inevitably, adding explicitly such constraint will increase the difficulty of the problem. Fortunately, it was proved later in Zhang and Wu (2001) that the solutions to (20) are automatically positive semi-definite, given C a positive semi-definite matrix. Hence the explicit imposition of the extra constraint becomes unnecessary.

Our approach for solving the constrained optimal approximation problem is to transform it to an equivalently min-max problem by the method of Lagrange multiplier. Let $\mathcal{R}_n$

be the set of $N \times N$ matrices with rank less or equal to n. The Lagrange multiplier problem corresponding to (20) is defined as the following problem:

$$\min_{d} \max_{X \in \mathcal{R}_n} L(X, d), \tag{21}$$

with the Lagrange function:

$$L(X, d) = - \|C - X\|_F^2 - 2d^T \mathrm{diag}(C - X), \tag{22}$$

where d is the vector of the multipliers. Note that $L(X, d)$ is linear in d in the following sense:

$$L(X, td + (1-t)\hat{d}) = tL(X, d) + (1-t)L(X, \hat{d}). \tag{23}$$

We will rigorously justify later, which may not be necessary to all readers, that the constrained problem (21) is equivalent to the problem (20).

Numerically the min-max problem (21) is treated as an minimization problem of the form

$$\min_{d} V(d), \tag{24}$$

with the objective function defined by

$$V(d) = \max_{X \in \mathcal{R}_n} L(X, d). \tag{25}$$

It is a matter then to look for efficient methods separately for the maximization problem (25) and minimization problem (24).

For the maximization problem (25), it is crucial to observe that the Lagrange function can be written into

$$L(X, d) = -\|C + D - X\|_F^2 + \|d\|_2^2, \tag{26}$$

where D is the diagonal matrix of d: $D = \mathrm{diag}(d)$. For fixed d, obviously, the maximizer to (24) can be obtained by the eigenvalue decomposition of matrix $C + D$ (which is symmetric). Let

$$C + D = U \Lambda U^T \tag{27}$$

be the eigenvalue decomposition with orthogonal matrix U and eigenvalue matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N).$$

Note that both U and Λ depend on the multiplier vector d, hence they are also denoted by $U(d)$ and $\Lambda(d)$ when highlighting the dependence is necessary. We assume that the diagonal

elements are put in the decreasing order in magnitude:

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_N|.$$

The solutions to the problem (25), the best rank-n approximations of $C + D$, are obviously given by

$$C(d) \equiv C_n(d) = U_n \Lambda_n U_n^T, \tag{28}$$

where U_n is the matrix consisting of the first n columns of U, and $\Lambda_n = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is the principle submatrix of Λ of degree n. Consequently we have

$$V(d) = -\sum_{j=n+1}^{N} \lambda_j^2 + \|d\|_2^2. \tag{29}$$

Clearly, when $|\lambda_n| > |\lambda_{n+1}|$, the solution to (24) is unique. In the case $|\lambda_n| = |\lambda_{n+1}|$, the number of solutions becomes nonunique or even infinite. The complication for the case of $|\lambda_n| = |\lambda_{n+1}|$ has been treated by Zhang and Wu (2001). Throughout this paper we limit ourselves to the case of $|\lambda_n| > |\lambda_{n+1}|$.

Regarding the existence of solution(s) of the min-max problem (24,25), we have

Theorem 3.1. *There exists at least one solution to (24,25), and any local minimum to (24,25) is also a global minimum.*

PROOF: To prove existence we use the method of contradiction. Suppose there is no solution to (24,25), then there exists a sequence of $d^{(j)} = \{d_i^{(j)}\} \to \infty$ while $V(d^{(j)})$ decreases. Write $D^{(j)} = \text{diag}(d^{(j)}) = D_1^{(j)} + D_2^{(j)}$ as a direct sum of two diagonal matrices with $\text{rank}(D_1^{(j)}) \leq n$ and $\|D_1^{(j)}\|_\infty = \|D^{(j)}\|_\infty \longrightarrow +\infty$. Since

$$V(d^{(j)}) = \max_{X \in \mathcal{R}_n} L(X, d^{(j)}) \geq L(D_1^{(j)}, d^{(j)}),$$

we have, with $L(X, d)$ in the form of (26),

$$\begin{aligned} V(d^{(j)}) &\geq -\|C + D_2^{(j)}\|_F^2 + \|D^{(j)}\|_F^2 \\ &= -\|C\|_F^2 - 2tr(CD_2^{(j)}) + \|D_1^{(j)}\|_F^2 \\ &\geq -\|C\|_F^2 + \Big(\|D^{(j)}\|_\infty - 2tr(|C|)\Big)\|D^{(j)}\|_\infty \longrightarrow +\infty, \end{aligned}$$

which contradicts to the assumption that $V(d^{(j)})$ decreases. Here $|C|$ denotes the matrix with entries of C in absolute values. The existence of solution(s) follows.

The property that any local minimum must be at the same time a global minimum is due to the convexity of $V(d)$. To see this, we consider any two points $d^{(1)}$ and $d^{(2)}$. For any

$t \in (0,1)$ we have

$$\begin{aligned}
V(td^{(1)} + (1-t)d^{(2)}) =& \max_X -\|C - X\|_F^2 - 2(td^{(1)} + (1-t)d^{(2)})^T \mathrm{diag}(C - X) \\
=& \max_X t\left(-\|C - X\|_F^2 - 2(d^{(1)})^T \mathrm{diag}(C - X)\right) \\
& + (1-t)\left(-\|C - X\|_F^2 - 2(d^{(2)})^T \mathrm{diag}(C - X)\right) \\
\leq& t \max_X -\|C - X\|_F^2 - 2(d^{(1)})^T \mathrm{diag}(C - X) \\
& + (1-t) \max_X -\|C - X\|_F^2 - 2(d^{(2)})^T \mathrm{diag}(C - X) \\
=& tV(d^{(1)}) + (1-t)V(d^{(2)}).
\end{aligned}$$

Suppose there are two local minimums d^* and d^{**} such that $V(d^*) > V(d^{**})$, we consider $d(t) = td^* + (1-t)d^{**}$ for $t \in (0,1)$. By the convexity we have

$$\begin{aligned}
V(d(t)) =& V(td^* + (1-t)d^{**}) \\
\leq& tV(d^*) + (1-t)V(d^{**}) \\
<& tV(d^*) + (1-t)V(d^{**}) = V(d^*).
\end{aligned} \tag{30}$$

Let t approach 1, then $d(t)$ approaches d^*, and the inequality in (30) is in contradiction to the assumption that d^* is a local minimum. Hence we arrive at the second conclusion $\square$.

For the analytical properties of the functions involved we have

Theorem 3.2. *When* $|\lambda_n(C+D)| > |\lambda_{n+1}(C+D)|$,

1. *the optimal solution* $C(d)$ *to (25) is unique and differentiable in* d;
2. $V(d)$ *is second-order continuously differentiable in a neighborhood of* d;
3. *if* $\nabla V(d) = 0$, *then* d *must be a global minimizer of* $V(d)$.

PROOF: The uniqueness is obtained by construction, as all solutions must be in the form (28), where Λ_n is unique when $|\lambda_n(C+D)| > |\lambda_{n+1}(C+D)|$, so is $C(d)$. Since C is symmetric and D is only a diagonal matrix, Gerschgorin Theorem in linear algebra (see for instance, Steward and Sun (1990)) implies that all eigenvalues (not necessarily distinct from each other) and eigenvectors are differentiable in a neighborhood of d, so are $V(d)$ and $C(d)$. The partial derivatives of $V(d)$ and $C(d)$ are related by the chain rule:

$$\frac{\partial V(d)}{\partial d_k} = \sum_{ij} \frac{\partial L(C(d),d)}{\partial X_{ij}} \frac{\partial X_{ij}}{\partial d_k} + \frac{\partial L(C(d),d)}{\partial d_k}, \quad 1 \leq k \leq N.$$

For fixed d, the optimality of $C(d)$ implies that

$$\frac{\partial L(C(d),d)}{\partial X_{ij}} = 0, \qquad \text{for all } i \text{ and } j. \tag{31}$$

Consequently we have

$$\frac{\partial V(d)}{\partial d_k} = -2(C_{kk} - C_{kk}(d)). \tag{32}$$

Differentiating (32) with respect to d_l yields

$$\frac{\partial^2 V(d)}{\partial d_k \partial d_l} = 2\frac{\partial C_{kk}(d)}{\partial d_l}, \quad \forall k \text{ and } l,$$

whose continuity follows from that of $\frac{\partial C_{kk}(d)}{\partial d_l}$.

Next we will show that any critical points must be the global minimum. In fact if $\nabla V(d) = 0$ but d is not a global minimum, we must have another point $\hat{d}$ such that $V(\hat{d}) < V(d)$. From convexity property we have

$$V(t\hat{d} + (1-t)d) \leq tV(\hat{d}) + (1-t)V(d), \qquad \text{for any } t \in (0,1).$$

The above equation can be rewritten into

$$\frac{V(t\hat{d} + (1-t)d) - V(d)}{t} \leq V(\hat{d}) - V(d) < 0.$$

Let $t \to 0$ we than have

$$\frac{(\hat{d} - d)}{\|\hat{d} - d\|_2} \cdot \nabla_d V(d) \leq V(\hat{d}) - V(d) < 0, \tag{33}$$

which is contradict to the zero-gradient condition $\nabla_d V(d) = 0$ for critical point d. The lemma is thus proved $\square$

Based on the existence of minimizer and differentiability of the value function, we can establish the equivalence between the constrained minimization problem (20) and the Lagrange multiplier problem (21).

Theorem 3.3. *Let d^* be any minimizer of the Lagrange multiplier problem (21). If $|\lambda_n(C + D^*)| > |\lambda_{n+1}(C + D^*)|$ and $diag(C) > 0$, then d^* is the unique minimizer and $C(d^*)$ solves the constrained minimization problem (20).*

PROOF: Assume to the contrary that there exists $d^{**} \neq d^*$ such that $V(d^{**}) = V(d^*)$. Denote $d(t) = d^* + t(d^{**} - d^*)$. The convexity property of $V(d)$ yields $V(d(t)) = V(d^*)$. Denote $C(t)$ as $C(d(t))$. Due to the linearity of $L(X, d)$ in d, we have that for $t \in [0, 1]$

$$\begin{aligned} V(d^*) &= V(d(t)) = L(C(t), d(t)) \\ &= (1-t)L(C(t), d^*) + tL(C(t), d^{**}) \\ &\leq (1-t)\max_{X \in \mathcal{F}} L(X, d^*) + t\max_{X \in \mathcal{F}} L(X, d^{**}) \\ &= (1-t)V(d^*) + tV(d^{**}) = V(d^*). \end{aligned}$$

From the above equalities we get $L(C(t), d^*) = V(d^*)$, or

$$\|C + D^* - C(t)\|_F = \min_{X \in \mathcal{R}_n} \|C + D^* - X\|_F.$$

It follows that, by the uniqueness of $C(d^*)$,

$$C(t) = C(0), \quad t \in [0, 1]$$

is the optimal solution corresponding to $d(t)$. The eigen-decomposition of matrix $C + D(t)$ is then

$$C + D(t) = U(t)\Lambda(t)U(t)^T = C(0) + E(t),$$

where

$$C(0) = U_n(0)\Lambda_n(0)U_n^T(0),$$

$$E(t) = G_n(t)\Theta_n(t)G_n^T(t),$$

and $G_n(t)$ consists of the last $N - n$ columns of $U(t)$. Since the columns of $G_n(0)$ and $G_n(t)$ form two orthogonal bases of the null space of $C(0)$, there exists an orthogonal matrix $W(t)$ such that $G_n(t) = G_n(0)W(t)$, and thus

$$E(t) = G_n(0) * (W(t)\Theta_n(t)W^T(t)G_n(0).$$

Substituting the above expression into the equality

$$E(t) - E(0) = t(D - D^*)$$

we have

$$t(D - D^*) = G_n(0) * (W(t)\Theta_n(t)W^T(t) - \Theta_n) * G_n^T(0),$$

or

$$(D - D^*) = G_n(0) * H(t) * G_n^T(0) \quad \text{for some } H(t).$$

Post-multiplying $U_n(0)$ to the above equation and recalling the orthogonality between $G_n(0)$ and $U_n(0)$, we have

$$(D - D^*)U_n(0) = 0.$$

Clearly, if the i^{th} diagonal of $D - D^*$ is not zero, then the i^{th} row of $U_n(0)$ must be zero. Consequently, this would enforce the i^{th} row of $C(0)$, including the diagonal $C_{ii}(0)$, to be zero. This is however contradict to the condition $diag(C(d)) = diag(C) > 0$ and hence there can not be more than one minimizer.

If d^* solve the min-max problem (21), then, due to the differentiability of $V(d)$, d^* must be a critical point of $V(d)$ and its gradient vanishes, that is,

$$0 = \frac{\partial V(d)}{\partial d_k} = -2(C_{kk} - C_{kk}(d)), \quad 1 \leq k \leq N$$

from Theorem 3.2. Hence for any other matrix $\tilde{C} \in \mathcal{F}$ we have

$$\begin{aligned} V(d^*) &= -\|C - C(d^*)\|_F^2 \\ &= \max_{X \in \mathcal{F}} -\|C - X\|_F^2 \\ &\geq -\|C - \tilde{C}\|_F^2, \end{aligned}$$

meaning that $C(d^*)$ is the only solution to the constrained minimization problem (21) □

The next theorem characterizes the minimizer of the Lagrange multiplier problem (21).

Theorem 3.4. *Assume matrix C is positive semi-definite and $diag(C) > 0$. If the n-th eigenvalue of matrix $C + diag(d^*)$ is single for any minimizer d^*, then 1) d^* is nonnegative, and 2) the solution $C^* = C(d^*)$ is also positive semi-definite.*

PROOF: The positive semi-definiteness of C^* follows directly from $d^* \geq 0$, since $C + diag(d^*)$ is positive semi-definite. We prove the nonnegativeness of d^* using the method of contradiction.

Assume on the contrary that d^* has a negative component. Write $C + D^* = C(d^*) + E$ with E orthogonal to $C(d^*)$. Furthermore, without loss of generality, we assume that the diagonal entries of D^* are nondecreasing so that the first diagonal entry $d^*(1)$ of $D^* = \text{diag}(d^*)$ is the smallest one. By construction we have $\text{diag}(C) = \text{diag}(C(d^*))$. So we have $d^* = \text{diag}(E)$ and the matrix E must have at least one negative eigenvalue. Let μ be the smallest eigenvalue of E and x be the corresponding eigenvector with unit 2-norm. It is easy to verify by the orthogonality that $C(d^*)x = 0$ because $\mu \neq 0$. Hence we have that

$$d^*(1) \leq x^T D^* x = \mu - x^T C x \leq \mu.$$

On the other hand, recalling that the diagonals of a symmetric matrix are bounded by its 2-norm (see for example Golub (1996)), we have that $\mu \leq d^*(1)$. Therefore

$$d^*(1) = \mu$$

and the first few of diagonals of $E - \mu I$ must be zeros. Note that $E - \mu I$ is a positive semi-definite matrix. Thus the first few rows and columns of $E - \mu I$ should be zero, too,

i.e., E is a block diagonal matrix,

$$E = \begin{pmatrix} \mu I & \\ & E_0 \end{pmatrix}.$$

By the orthogonality between $C(d^*)$ and E, we have also

$$C(d^*) = \begin{pmatrix} 0 & \\ & C_0 \end{pmatrix}.$$

Hence the first row/column of matrix $C = C(d^*) + E - diag(d^*)$ must be zero, which contradicts the assumption $diag(C) > 0$. □

We have shown that the inner maximization problem (25) can be solved nicely by an eigenvalue decomposition. The outer minimization (24) then is dealt with the method of steepest descend. The convexity property of $V(d)$ renders great advantage to the efficiency of the descending method. The algorithm is given below.

Algorithm: Take $D^{(0)}$ to be a null matrix, and repeat the following steps:

1. Compute the eigen-decomposition of $C + D^{(k)}$: $C + D^{(k)} = U^{(k)}\Lambda^{(k)}(U^{(k)})^T$; set $\alpha^{(k)} = 1$ and
   $\nabla V(d^{(k)}) = -2\text{diag}(C - U_n^{(k)}\Lambda_n^{(k)}(U_n^{(k)})^T)$;
2. Define $d^{(k+1)} = d^{(k)} - \alpha^{(k)}\nabla V(d^{(k)})$;
3. If $V(d^{(k+1)}) > V(d^{(k)}) - \frac{\alpha^{(k)}}{2}\|\nabla V(d^{(k)})\|^2$, take $\alpha^{(k)} := \alpha^{(k)}/2$, go back to step 2;
4. if $\|d^{(k+1)} - d^{(k)}\|_2 > tol$, go to step 1;
5. Take $d^* = d^{(k+1)}$ and $C^* = U_n^{(k)}\Lambda_n^{(k)}(U_n^{(k)})^T$.

We conclude this section with the convergence of the descending method.

Theorem 3.5. *The sequence $\{d^{(k)}\}$ is bounded and hence accumulation points of $\{d^{(k)}\}$ exist. Let d^* be an accumulation point such that $|\lambda_n(C + D^*)| > |\lambda_{n+1}(C + D^*)|$, then d^* is the unique global minimizer, and $\{d_i^{(k)}\}$ converges to d^*.*

PROOF: The boundedness of the sequence comes from the monotonical decreasing of the function $V(d^{(k)})$. If the boundedness is not true, then there must be a subsequence with index $k \in K_0$ such that $\|d^{(k)}\|_2 \to +\infty$ for $k \in K_0$. Repeating the relevant arguments in Theorem 3.1, we would generate a contradiction to the decreasing property of $V(d^{(k)})$. Since $d^{(k)}$ are bounded sequence, there must be at least one accumulation point.

Let d^* be an accumulation point such that $d^{(k)} \to d^*$ for k in some index set K_0. Under the condition $|\lambda_n(C + D^*)| > |\lambda_{n+1}(C + D^*)|$ we will show that d^* must be a critical point of $V(d)$ such that $\nabla_d V(d^*) = 0$. Suppose that this is not the case. According to the line

search and the continuity of $\nabla V(d)$ around d^* we must have

$$\lim_{k\in K_0, k\to\infty} \alpha^{(k)} = 0,$$

and

$$V(d^{(k)} - \hat{\alpha}^{(k)}\nabla V(d^{(k)})) > V(d^{(k)}) - \frac{\hat{\alpha}^{(k)}}{2}\|\nabla V(d^{(k)})\|^2, \tag{34}$$

here $\hat{\alpha}^{(k)} = 2\alpha^{(k)}$. Note that there is

$$\lim_{\hat{\alpha}^{(k)}\to 0} \frac{V(d^* - \hat{\alpha}^{(k)}\nabla_d V(d^*)) - V(d^*)}{\hat{\alpha}^{(k)}} = -\|\nabla_d V(d^*)\|^2.$$

From the continuity of the second-order derivatives we have, for sufficiently large $k \in K_0$,

$$\begin{aligned} \frac{V(d^{(k)} - \hat{\alpha}^{(k)}\nabla_d V(d^{(k)})) - V(d^{(k)})}{\hat{\alpha}^{(k)}} &= -\|\nabla_d V(d^{(k)})\|^2 + O(\alpha^2) \\ &\le -\frac{1}{2}\|\nabla_d V(d^{(k)})\|^2. \end{aligned} \tag{35}$$

Clearly, (35) is contradictory to (34). Hence there must be $\nabla_d V(d^*) = 0$. The solution d^* is a global minimizer follows from Theorem 3.2 $\square$

3.2. **Eigenvalue problem for calibration of input prices.** The calibration to input prices has been formulated in the concise form of (19). The corresponding Lagrange multiplier problem is

$$\min_d \max_X L(X, d), \tag{36}$$

where

$$L(X, d) = -\left((X - X_0)^T A (X - X_0)\right)^2 + 2\sum_{i=1}^{N} d_i (X^T G_i X - h_i). \tag{37}$$

Note that for simplicity we have used $\{G_i, h_i\}$ in place of $\{G_{m,n}, \zeta^2_{m,n}\}$. To facilitate discussions we denote the value function for the outer minimization problem by

$$V(d) = \max_X L(X, d), \tag{38}$$

and feasible set of solutions by

$$\mathcal{F} = \{X | X^T G_i X = h_i, i = 1, \ldots, N\}.$$

Due to the positive-definiteness of the matrix A, the maximizer of $L(X, d)$ is finite for fixed d, and $V(d)$ therefore exists for all d. It is obvious that the Lagrange function is smooth

in X. Any solutions to the maximization problem (38) hence must be a critical point of the Lagrange function, satisfying the following first-order condition

$$\begin{aligned}\left((X-X_0)^T A(X-X_0)\right) A(X-X_0) &= (\sum d_i G_i)X \\ &= (\sum d_i G_i)(X-X_0) + (\sum d_i G_i)X_0.\end{aligned} \tag{39}$$

For simplicity we denote $B_d = \sum d_i G_i$ and $Y = X - X_0$. Equation (39) then reads

$$[(Y^T AY)A - B_d]Y = B_d X_0. \tag{40}$$

The above equation can be solved through eigenvalue decomposition. Let $(\lambda_i, \mathbf{u}_i)$ be the eigenpairs of (B_d, A) such that

$$B_d \mathbf{u}_i = \lambda_i A \mathbf{u}_i, \tag{41}$$

$$\mathbf{u}_i^T A \mathbf{u}_i = 1, \quad \mathbf{u}_i^T B_d \mathbf{u}_i = \lambda_i, \tag{42}$$

$$\lambda_i \geq \lambda_{i+1}, \quad i = 1, 2, \ldots, N-1, \tag{43}$$

and define by U the A-orthogonormal eigenvector matrix

$$U = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N). \tag{44}$$

To solve for Y, we let $\alpha = Y^T AY$ and pre-multiply U^T to equation (40), we then arrive at

$$[\alpha I - \Lambda]U^{-1}Y = U^T B_d X_0, \tag{45}$$

where

$$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_N)$$

is the eigenvalue matrix. From (45) we obtain the solution to (40) as

$$Y \equiv \tilde{Y}(\alpha) = U[\alpha I - \Lambda]^{-1} U^T B_d X_0,$$

where the scalar α is subject to the nonlinear equation

$$\tilde{Y}^T(\alpha) A \tilde{Y}(\alpha) = \alpha,$$

which can be solved with a few steps of iteration. When $X_0 = 0$, in particular, (40) becomes an eigenvalue problem and thus admits multiple solutions, and they are given by

$$Y_i = \sqrt{\max\{\lambda_i, 0\}}\mathbf{u}_i, \quad i = 1, 2, \ldots, N.$$

To fix idea we will from now on concentrate on the special case of $X_0 = 0$. The case $X_0 \neq 0$ is the trivial extension to this special case. When $X_0 = 0$, function $L(X, d)$ achieves it maximum at

$$Y = Y_1, \tag{46}$$

and consequently

$$\begin{aligned} V(d) &= \max_{j,\lambda_j \geq 0} -(Y_j^T A Y_j)^2 + 2\sum_{i=1}^{N} d_i (Y_j^T G_i Y_j - h_i) \\ &= \max_{\lambda_j \geq 0} \lambda_j^2 - 2\sum_{i=1}^{N} d_i h_i \\ &= \lambda_1^2 - 2\sum_{i=1}^{N} d_i h_i. \end{aligned}$$

It is well-known in matrix theory that the eigenvector corresponding to the largest eigenvalue of a positive semi-definite matrix is the smoothest one amongst all eigenvectors. Very encouragingly, the result in (46) establishes the connection between the smoothest fit of volatility surface and the smoothest eigenvector of a generalized eigenvalue problem.

For the analytical properties of $\lambda_1(d), Y(d)(= Y_1(d))$ and $V(d)$ we have

Theorem 3.6. *If $\lambda_1(d) > \lambda_2(d)$, then*

1. *$(\lambda_1(d), Y(d))$ is differentiable with respect to d locally;*
2. *$V(d)$ is differentiable in d locally;*
3. *the gradient of $V(d)$ is*

$$\nabla_d V(d) = 2 \begin{pmatrix} Y^T G_1 Y - h_1 \\ Y^T G_2 Y - h_2 \\ \vdots \\ Y^T G_N Y - h_N \end{pmatrix};$$

4. *the elements of the Hessian matrix are given by*

$$H_{ij}(d) \equiv \frac{\partial^2 V}{\partial d_i \partial d_j} = 2 Y^T G_i U \Phi^{-1} U^T G_j Y, \tag{47}$$

where

$$\Phi = \lambda_1 I + 2 \begin{pmatrix} \lambda_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} - \Lambda, \tag{48}$$

and the Hessian is a non-negative definite matrix.

PROOF: The proof for statements 1-3 is similar to that of Theorem 3.2, so we only prove the last statement.

The solution Y satisfies the equation (40) for $X_0 = 0$. Differentiate both sides of the equation with respect to d_i we have

$$(Y^T A Y) A \frac{\partial Y}{\partial d_i} + 2 A Y Y^T A \frac{\partial Y}{\partial d_i} = G_i Y + B_d \frac{\partial Y}{\partial d_i}.$$

From the above equation we can solve for $\frac{\partial Y}{\partial d_i}$:

$$\left[\lambda_1 A + 2AYY^T A - B_d\right] \frac{\partial Y}{\partial d_i} = G_i Y,$$

where we have put $Y^T AY = \lambda_1$. Pre-multiply both sides by U^T we then have

$$\begin{aligned}
&\left[\lambda_1 U^T AU + 2U^T AYY^T AU - U^T B_d U\right] U^{-1} \frac{\partial Y}{\partial d_i} \\
=&\left[\lambda_1 I + 2 \begin{pmatrix} \lambda_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} - \Lambda\right] U^{-1} \frac{\partial Y}{\partial d_i} \\
=&U^T G_i Y,
\end{aligned}$$

i.e.,

$$\Phi U^{-1} \frac{\partial Y}{\partial d_i} = U^T G_i Y,$$

for the Φ defined in (48). When $\lambda_1 > \lambda_2 \geq \lambda_i, i \geq 3$, Φ is a positive definite matrix, and it follows that

$$\frac{\partial Y}{\partial d_i} = U \Phi^{-1} U^T G_i Y. \tag{49}$$

Component-wise the third statement reads

$$\frac{\partial V}{\partial d_i} = 2(Y^T G_i Y - h_i), \qquad i = 1, 2, \ldots, N.$$

Differentiating this equation with respect to d_j produces the elements of Hessian:

$$\begin{aligned}
\frac{\partial^2 V}{\partial d_i \partial d_j} &= 4Y^T G_i \frac{\partial Y}{\partial d_j} \\
&= 4Y^T G_i U \Phi^{-1} U^T G_j Y.
\end{aligned}$$

Denote

$$G = [G_1, G_2, \ldots, G_N],$$

and define

$$G \otimes Y = [G_1 Y, G_2 Y, \ldots, G_N Y].$$

The non-negative definiteness of the Hessian follows from expression

$$H(d) = 4(G \otimes Y)^T U \Phi^{-1} U^T (G \otimes Y).$$

This completes the proof $\square$

We make the following remarks for future references. For a calibrated model we will have

$$0 = \frac{\partial V(d^*)}{\partial d_i} = 2(Y^T G_i Y - h_i), \quad i = 1, 2, \ldots, N.$$

From the above relation we can treat the input h_i's as the functions of d^*:

$$h_i = h_i(d^*) \equiv Y^T G_i Y \Big|_{d=d^*}.$$

We can differentiate h_i with respect to d^*, which gives rise to

$$\begin{aligned} \frac{\partial h_i}{\partial d_j^*} &= 2Y^T G_i \frac{\partial Y}{\partial d_j}\Big|_{d=d^*} \\ &= \frac{1}{2}\frac{\partial^2 V(d^*)}{\partial d_i \partial d_j} = \frac{1}{2} H_{ij}(d^*). \end{aligned} \tag{50}$$

This relation indicates that, for a calibrated model, the Hessian defines the sensitivities of the input prices with respect to the Lagrange multipliers. It makes more sense to know the opposite, that is, the sensitivities of Lagrange multipliers with respect to the input prices. To obtain such sensitivities it is a matter of computing matrix inverse, as (Rockafellar, 1970)

$$\frac{\partial d_j^*}{\partial h_i} = \left(\left(\frac{\partial h_i}{\partial d_j^*} \right)^{-1} \right)_{ij} = 2(H^{-1}(d^*))_{ij}. \tag{51}$$

The above result will be used later to calculate the sensitivities of a derivative instrument with respect to the benchmark instruments.

To ensure that min-max problem has at least one solution, we introduce the concept of *non-arbitrageable prices*.

Definition 3.1. *We call $h = \{h_i\}$ a set of non-arbitrageable prices if there is an $\epsilon_h > 0$ such that for any $\{\epsilon_i\}$ with $\epsilon_i \leq \epsilon_h$, there is at least one solution X to*

$$X^T G_i X = h_i + \epsilon_i, \quad i = 1, 2, \ldots, N.$$

We should understand the above concept from the viewpoint of price-volatility correspondence. We anticipate that, for a set of realistic prices of market instruments, the market model with a reasonable number of driving factors should be able to "rationalize" the prices through generating a corresponding volatility surface. Furthermore, we want to see that small changes in the prices will be accommodated by the proper variation of the volatility surface. If such accommodation does not happen, then either the model is suggesting the existence of an arbitrage opportunity, or the model simply has no enough dimensions, driving factors for example, to describe the reality.

For the existence of the global minimizer we have

Theorem 3.7. *If $h = \{h_i\}$ is a set of no-arbitrage prices, then there is at least one solution to problem $V(d)$. Also, any local minimum is a global minimum.*

PROOF: Given non-empty $\mathcal{F}$, we have

$$V(d) \geq \max_{X \in \mathcal{F}} -(X^T A X) \equiv V^*$$

for some bounded value V^* due to the positiveness of A. Assume that there exist $d^{(j)} \to \infty$ such that $V(d^{(j)}) \to V(d^*)$ monotonically from above, then, since $\mathcal{F}$ is not empty we can choose a sequence of $X^{(j)}$ such that

$$(X^{(j)})^T G_i X^{(j)} - h_i = \epsilon^* \text{sign}(d^{(j)}),$$

for some fixed $\epsilon^* > 0$. Consequently we will have

$$V(d^{(j)}) \to +\infty,$$

contradicting to the assumption of monotonic decreasing of $V(d^{(j)})$.

The conclusion that any local minimum is also a global minimum follows from the convexity of $V(d)$ $\square$

For the uniqueness of the solution we have

Theorem 3.8. *Let d^* be a minimizer of $V(d)$. If $\lambda_1(d^*) > \lambda_2(d^*)$ and $H(d^*)$ is positive definite, then d^* is the unique minimizer of $V(d)$ and $Y(d^*)$ solves the constrained minimization problem (36).*

PROOF: If $\lambda_1(d^*) > \lambda_2(d^*)$, then according to Theorem 3.3 $V(d)$ is differentiable near d^* and $\nabla_d V(d^*) = 0$. Moreover, the positive-definiteness of $H(d^*)$ implies that d^* must be the only local minimizer in its immediate neighbourhood. Assume there is another minimizer, say, $d^{**} \neq d^*$, then by Theorem 3.5 and the convexity of $V(d)$, we have $V(d(t)) = V(d^*), d(t) = td^* + (1-t)d^{**}$ for all $t \in (0,1)$. That means $V(d^*)$ is not the only minimum in its immediate neighbourhood, which is a contradiction. $\square$

Similar to the previous section, we can show the convergence of the gradient-based algorithm. Yet, when Hessian is not expensive to obtain, we should definitely use a Hessian-based algorithm for the numerical solution.

4. Numerical results

We consider a practical example taken from Brace *et al.* (1997). In this example, we want to calibrate the market model for the Sterling Pound to the prices of a set of benchmark instruments and the correlation matrix of Sterling Pound forward rates. Note that from historical data we can only estimate the *spot* correlations (the correlations seen at the moment), instead of both *spot* and *forward* correlations (the correlation seen at a future date). It is very much a convention that we define the *forward* correlations by *time homogeneity*, namely, the correlation between any two forward rates depends only on the time to reset. This convention allows us to define the market correlations of all forward rates. The prices of a set of ATM caplets and swaptions are listed in Table 1, where the caplet prices were obtained through an *bootstrapping* procedure. The correlation matrix is given in Table 2 (which was calculated with the one-year data of 1994). The first row and column of the table show the maturities of the forward rates.

First let us take a look at the low-rank approximations to the market correlation matrix. We have calculated the rank- one, two, three, six and ten approximations and the outcomes are visualized in Figure 1-6, where the first figure is the original market correlation surface. By vision we shall agree that the approximation improves with rank increasing. Numerically the trend of convergence with respect to the rank increasing is given in Figure 7. For each number of factors the calibration requires less than seven functional valuation (of $V(d)$). In fact this is a very small scale problem for the method developed in this paper.

With calculated low-rank approximations of the correlation matrix, we proceed to calculate the forward rate volatilities from the input implied CEV volatilities of caplets and swaptions. The results are plot as volatility surfaces from Figure 8 to 12. The magnitude of calibration error relative to the input volatility (squares) is of order

$$\frac{\|Y^T G_{m,n} Y - \zeta^2_{m,n}\|_2}{\|\zeta^2_{m,n}\|_2} = O(10^{-4}).$$

For the Black volatility under 20%, this corresponds to less than 0.2% of error in model volatility, which is much less than one *kappa*, the usual market bid/ask spread defined as the change in present value for 1% change in volatility. The volatility surfaces for different correlation matrices look incredibly close yet, as Figure 13 shows, they are not identical. The reason is that caplet prices do not depend on correlation, and the swaption prices in our data set depend only on elements near the diagonal of the correlation matrix, which are close

across correlation matrices of various ranks. A positive implication of the closeness is that the calculated forward rate volatilities is not sensitive to the changes in correlation matrix. The algorithm is implemented with a Hessian-based unconstrained minimization function in MATLAB ("fminunc" in specific). Again each calibration to prices takes about seven functional valuations. The entire calibration (to both correlations and prices) is finished within ten seconds. Note that there is noticeable difference in the speed of convergence between the Hessian-based and gradient-based minimization with "fminunc". The former is much faster and thus highly recommended.

5. Conclusion

We have developed an efficient methodology to calibrate the CEV market model to a collection of caplet/swaption prices and exogenously given correlation matrices. Both the well-posedness of our mathematical formulation and the convergence of our algorithm are rigorously justified. Numerical studies confirm the efficiency and reliability of the method. From the viewpoint of financial engineering, we have solved the important problem of calibrating to the volatility skews in the LIBOR derivative markets. From mathematical point of view, we have invented a powerful new technique for general constrained minimization problem with quadratic objective function and constraints.

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Table 1. Stripped Caplet Prices and Swaption Prices

Contract type	Option maturity ×Swap length	ATM Strike (%)	Black Vol. (%)	Market Price(bp)
Caplet	0.25× 0.25	7.88	0.15	1.59
Caplet	0.5× 0.25	7.88	0.15	13.20
Caplet	1× 0.25	8.39	0.19	19.26
Caplet	2× 0.25	8.64	0.18	25.36
Caplet	3× 0.25	8.69	0.17	24.61
Caplet	4× 0.25	8.79	0.18	29.92
Caplet	5× 0.25	8.90	0.14	29.20
Caplet	7× 0.25	8.89	0.13	21.88
Caplet	9× 0.25	8.89	0.13	17.40
Swaption	0.25×2	8.59	0.16	50.00
Swaption	0.25×3	8.79	0.16	73.00
Swaption	1×4	0.0910	0.16	172.00
Swaption	0.25×5	8.95	0.15	103.00
Swaption	0.25×7	9.04	0.14	123.00
Swaption	0.25×10	9.02	0.13	151.00
Swaption	1×9	9.14	0.13	271.00
Swaption	2×8	9.18	0.13	312.00

Table 2. Historical Correlation Matrix for the GBP Forward Rates

	0.25	0.5	1	1.5	2	2.5	3	4	5	7	9
0.25	1.0000	0.8415	0.6246	0.6231	0.5330	0.4287	0.3274	0.4463	0.2439	0.3326	0.2625
0.5	0.8415	1.0000	0.7903	0.7844	0.7320	0.6346	0.4521	0.5812	0.3439	0.4533	0.3661
1	0.6246	0.7903	1.0000	0.9967	0.8108	0.7239	0.5429	0.6121	0.4426	0.5189	0.4251
1.5	0.6231	0.7844	0.9967	1.0000	0.8149	0.7286	0.5384	0.6169	0.4464	0.5233	0.4299
2	0.5330	0.7320	0.8108	0.8149	1.0000	0.9756	0.5676	0.6860	0.4969	0.5734	0.4771
2.5	0.4287	0.6346	0.7239	0.7286	0.9756	1.0000	0.5457	0.6583	0.4921	0.5510	0.4581
3	0.3274	0.4521	0.5429	0.5384	0.5676	0.5457	1.0000	0.5942	0.6078	0.6751	0.6017
4	0.4463	0.5812	0.6121	0.6169	0.6860	0.6583	0.5942	1.0000	0.4845	0.6452	0.5673
5	0.2439	0.3439	0.4426	0.4464	0.4969	0.4921	0.6078	0.4845	1.0000	0.6015	0.5200
7	0.3326	0.4533	0.5189	0.5233	0.5734	0.5510	0.6751	0.6452	0.6015	1.0000	0.9889
9	0.2625	0.3661	0.4251	0.4299	0.4771	0.4581	0.6017	0.5673	0.5200	0.9889	1.0000

Forward rates were assumed constant on the intervals between the terms.

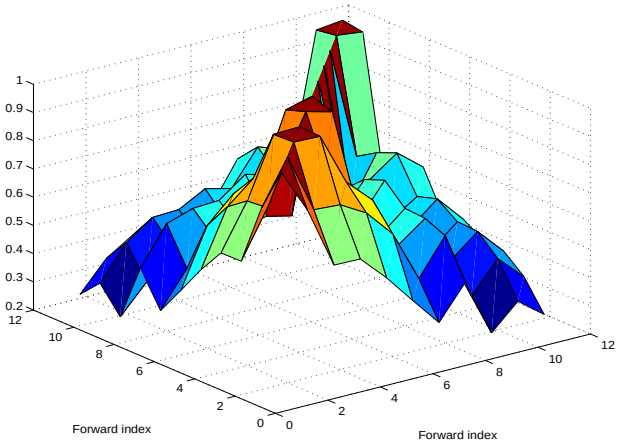

FIGURE 1. Market correlation surface

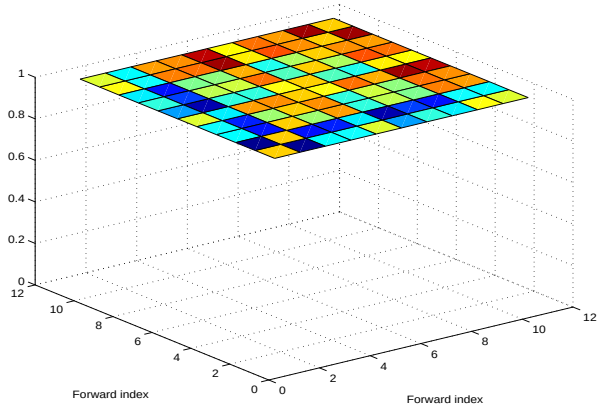

FIGURE 2. Correlation surface of one-factor model

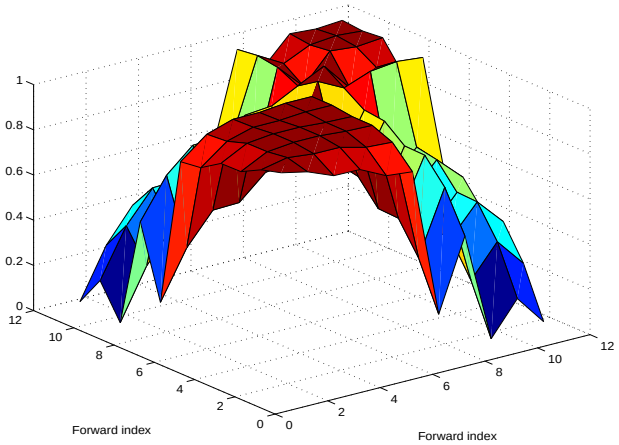

FIGURE 3. Correlation surface of two-factor model

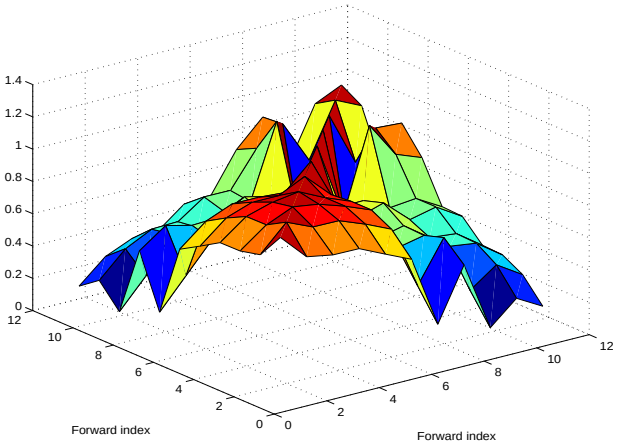

FIGURE 4. Correlation surface of three-factor model

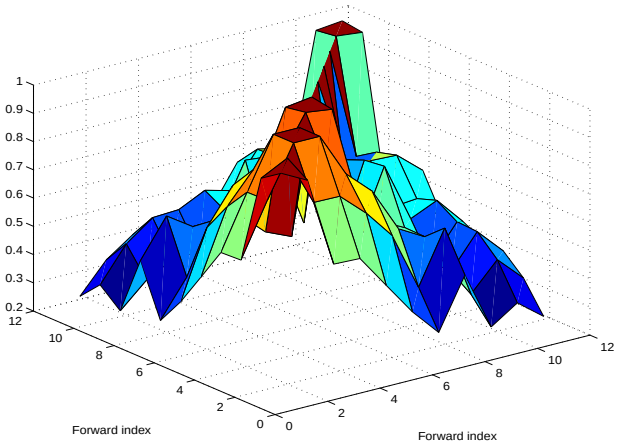

FIGURE 5. Correlation surface of six-factor model

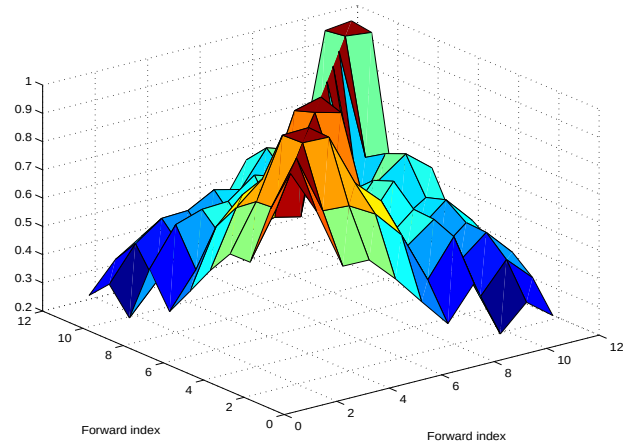

FIGURE 6. Correlation surface of ten-factor model

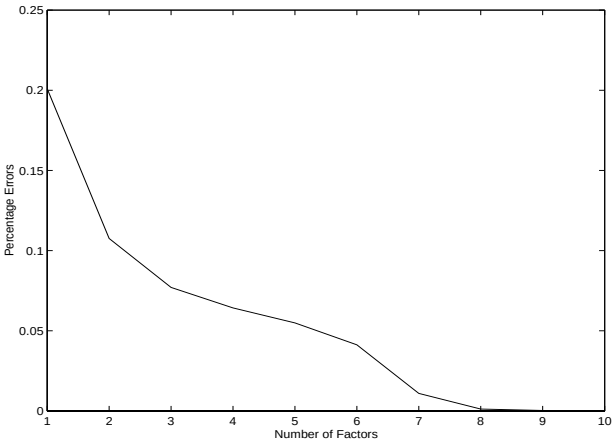

FIGURE 7. Trend of Convergence with increasing Rank

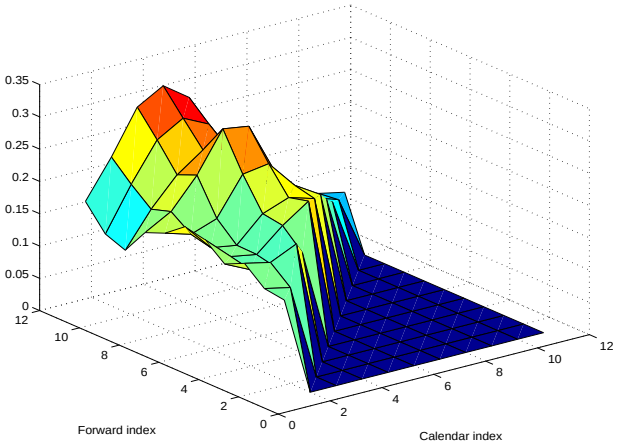

FIGURE 8. Volatility surface of one-factor model

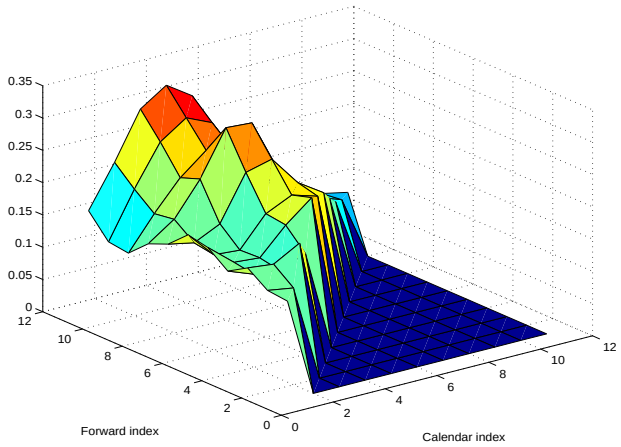

FIGURE 9. Volatility surface of two-factor model

FIGURE 10. Volatility surface of three-factor model

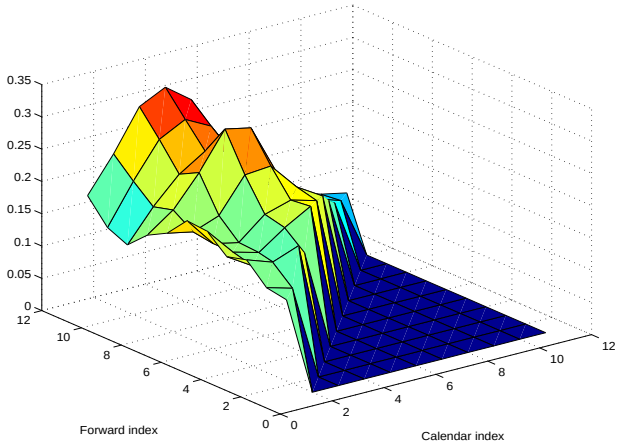

FIGURE 11. Volatility surface of six-factor model

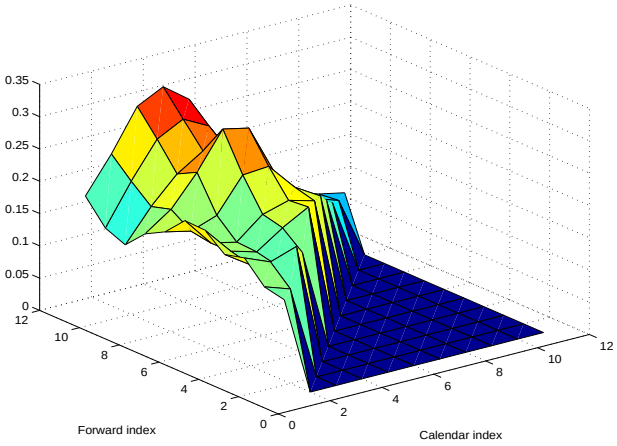

FIGURE 12. Volatility surface of ten-factor model

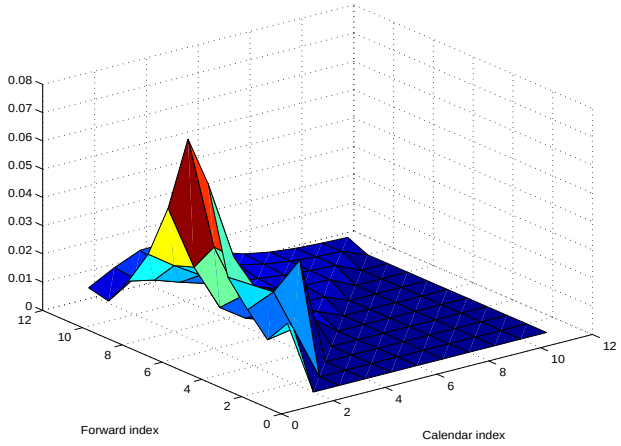

FIGURE 13. Difference between one- and ten-factor models