# "Market Model" vs. "Foreign Currency Analogy"

To clarify the notion of "market models" for inflation derivatives, Wu redefines inflation forward rates using arbitrage arguments and rebuilds the practitioners' market model. The rebuilt market model possesses the clarity and simplicity of the LIBOR market model, and is found consistent with the framework of "foreign currency analogy".

There is certain degree of disorder in inflation derivatives modeling. Early developments of the field have been associated to the concept of "foreign currency analogy" (Jarrow and Yildirim, 2003), yet later developments have been not so correlated. Now there are at least three "market models" (Beldgrade-Benhamou-Koehlar, 2004; Mercurio-Moreni, 2006; the practitioners' model<sup>1</sup>), and at least two versions of "inflation forward rates", causing a distinction between models based on zero-coupon inflation-indexed swaps (ZCIIS) and models based on year-on-year inflation indexed swaps (YYIIS). Recently, smile models have started to emerge, which are based on either diffusion (Kenyon, 2008) or displaced-diffusion dynamics (Mercurio and Moreni, 2009) of "inflation forward rates", or developed along the approach of currency analogy with jump-diffusion dynamics (Hinnerich, 2008).

With this article we hope to sort out the field. We redefine the notion of inflation forward rate as the fair rate for a forward contract on inflation rate, which is shown to be replicable statically and thus is unique. We then justify lognormal martingale dynamics for displaced inflation forward rates, and thus rigorously rebuild the practitioners' model, with which the notion of market model should be clarified. Moreover, we establish a Heath-Jarrow-Morton (HJM) type equation for instantaneous inflation forward rates and, by also making use of the classic HJM equation for nominal forward rates, re-derive the HJM type equation for real forward rates established by Jarrow and Yildirim (2003), along with a correction that the notion of the "volatility of the Consumer Price Index" is flawed and useless for modeling.

This article has several important implications. First, we show that the ZCIIS- and YYIIS-based market models are identical and the use of "convexity adjustment", a common practice, is redundant. Second, we unify the closed-form pricing of inflation caplets, floorlets and swaptions and pave the

<sup>&</sup>lt;sup>1</sup>The practitioners' model is not seen in literature available to public.

way to quoting these derivatives by "Black's implied volatilities". Finally, we provide a proper platform for developing smile models.

#### Nominal and Real Discount Bonds

Our model construction takes nominal and real zero-coupon bonds, two strings of replicable securities, as model primitives. While the prices of nominal discount bonds can be constructed out of LIBOR and swap-rate curves<sup>2</sup>, the prices of real discount bonds are given almost directly by the quotes of ZCIIS, as explained below.

The ZCIIS is a swap contract between two parties with a single exchange of payments. Consider a contract that is initiated at time t and will be expired at T. At expiry, the two parties exchange payments according to the following scheme:

$$Not. \times \left(\frac{I(T)}{I(t)} - 1\right) \iff Not. \times \left((1 + K(t, T))^{T-t} - 1\right),$$

where Not. is the notional value of the contract, I(t) is the Consumer Price Index<sup>3</sup> (CPI) at time t, and K(t,T) is the quote of the contract. Because the value of the ZCIIS is zero at initiation, from the quote we can calculate the so-called real zero-coupon bond which pays inflation adjusted principal<sup>4</sup> at time T:

$$P_{R}(t,T) = E^{Q} \left[ e^{-\int_{t}^{T} r_{s} ds} \frac{I(T)}{I(t)} \middle| \mathcal{F}_{t} \right] = P(t,T) \left( 1 + K(t,T) \right)^{T-t}.$$
 (1)

Here, P(t,T) is the nominal discount factor from T back to t. For the real zero-coupon bond with the same maturity date T but a fixed issuance date, say,  $T_0 \leq t$ , the price is

$$P_R(t, T_0, T) = E^Q \left[ e^{-\int_t^T r_s ds} \frac{I(T)}{I(T_0)} \middle| \mathcal{F}_t \right] = \frac{I(t)}{I(T_0)} P_R(t, T).$$
(2)

Note that  $P_R(t, T_0, T)$  instead of  $P_R(t, T)$  is treated as the price of a tradable security.

<sup>&</sup>lt;sup>2</sup>Alternatively, they can be calculated from benchmark Treasury bonds.

<sup>&</sup>lt;sup>3</sup>There is a two-month time lag.

 $<sup>{}^{4}</sup>P_{R}(t,T)$  is treated as the price of a zero-coupon bond of a virtue "foreign currency" in Jarrow and Yildirim (2003).

Unlike in Jarrow-Yildirim model, the CPI has no role to play in our modeling, yet about it there is an important point needs to be clarified. In terms of the instantaneous inflation rate,  $i_t$ , the CPI index can be expressed as

$$I(t) = I(0)e^{\int_0^t i_s ds}.$$
 (3)

It follows that

$$dI(t) = i_t I(t) dt, \tag{4}$$

meaning that as a lognormal variable I(t) has no volatility, and it behaves like a money market account instead of an exchange rate.

### Inflation Discount Bonds and Inflation Forward Rates

The cash flows of several major inflation-indexed instruments, including YYIIS, inflation caplets and floorlets, are expressed in term rates of inflation (or simple inflation rates). For pricing and hedging we need to define inflation forward rates. We begin with

**Definition 1**: The discount bond associated to inflation rate is defined by

$$P_I(t,T) \stackrel{\triangle}{=} \frac{P(t,T)}{P_R(t,T)}.$$
(5)

Here, " $\stackrel{\triangle}{=}$ " means "being defined by".

We define inflation forward rates as the *returns implied by the inflation discount bonds*.

**Definition 2**: The forward inflation rate for a future period  $[T_1, T_2]$  seen at time  $t \leq T_2$  is defined by

$$f^{(I)}(t, T_1, T_2) \stackrel{\triangle}{=} \frac{1}{(T_2 - T_1)} \left( \frac{P_I(t, T_1)}{P_I(t, T_2)} - 1 \right).$$
(6)

There is a slight problem with the above definition: the forward inflation rate is fixed at  $t = T_2$ , beyond the life of the  $T_1$ -maturity bond, so we need to define  $P_I(t, T_1)$  for  $t > T_1$ . In view of (2), we have

$$P_I(t,T_1) = \frac{I(t)}{I(T_0)} \frac{P(t,T_1)}{P_R(t,T_0,T_1)}.$$
(7)

The second ratio on the right-hand side of (7) is the relative price between two traded bonds with an identical maturity date, and thus its value beyond  $T_1$  can be defined by constant extrapolation, yielding

$$P_I(t, T_1) = \frac{I(t)}{I(T_0)} \frac{I(T_0)}{I(T_1)} = \frac{I(t)}{I(T_1)}, \quad \forall t \ge T_1.$$
(8)

Given (8), we have the value of the forward rate at its fixing date to be

$$f^{(I)}(T_2, T_1, T_2) = \frac{1}{T_2 - T_1} \left( \frac{I(T_2)}{I(T_1)} - 1 \right), \tag{9}$$

so the inflation forward rate converges to inflation spot rate at maturity.

Next, we will argue that  $f^{(I)}(t, T_1, T_2)$  so defined is the fair rate seen at time t for a forward contract on inflation over  $[T_1, T_2]$ . We rewrite (6) into

$$f^{(I)}(t, T_1, T_2) = \frac{1}{(T_2 - T_1)} \left( \frac{F_R(t, T_1, T_2) P(t, T_1)}{P(t, T_2)} - 1 \right),$$
(10)

where

$$F_R(t, T_1, T_2) \stackrel{\triangle}{=} \frac{P_R(t, T_0, T_2)}{P_R(t, T_0, T_1)} \tag{11}$$

is the relative price of two tradable securities. The following result is the corner stone of our theory. Its proof is given in the appendix.

**Proposition 1**: Let  $t \leq T_1 \leq T_2$ . The  $T_1$ -forward price of a real bond with maturity  $T_2$  seen at time t is  $F_R(t, T_1, T_2)$ .

In view of (10), we can treat  $f^{(I)}(t, T_1, T_2)$  as the  $T_2$ -forward price for the payoff of  $f^{(I)}(T_2, T_1, T_2)$  at  $T_2$ , and thus have proven

**Proposition 2**: The forward inflation rate  $f^{(I)}(t, T_1, T_2)$  is the unique arbitrage-free rate seen at the time t for a  $T_1$ -expiry forward contract on the inflation rate over the future period  $[T_1, T_2]$ .

Proposition 2 should help to end the situation of the coexistence of multiple definitions of forward inflation rates. Note that our definition (6) coincides with one of the definitions,  $Y_i(t)$ , given in Mercurio and Moreni (2009).

## The Consistency Condition

We now proceed to the construction of dynamic models for inflation forward rates of both simple and instantaneous compounding. We model the uncertain economy by a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,\tau]}, Q)$  for some  $\tau > 0$ , where Q is the risk neutral probability measure for the uncertain economical environment, which can be defined in a usual way in an arbitrage-free market (Harrison and Krep, 1979; Harrison and Pliska, 1981), and the filtration  $\{\mathcal{F}_t\}_{t\in[0,\tau]}$  is generated by a *d*-dimensional *Q* Brownian motion  $\mathbf{Z}_t, t \geq 0$ .

Under the risk neutral measure Q, P(t,T) and  $P_R(t,T_0,T)$  are assumed to follow the lognormal processes

$$dP(t,T) = P(t,T) \left( r_t dt + \Sigma(t,T) \cdot d\mathbf{Z}_t \right), dP_R(t,T_0,T) = P_R(t,T_0,T) \left( r_t dt + \Sigma_R(t,T) \cdot d\mathbf{Z}_t \right),$$
(12)

where  $r_t$  is the risk-free nominal (stochastic) interest rate,  $\Sigma(t, T)$  and  $\Sigma_R(t, T)$ are *d*-dimensional  $\mathcal{F}_t$ -adaptive volatility functions of P(t, T) and  $P_R(t, T_0, T)$ , respectively<sup>5</sup>, and "·" means scalar product. The volatility function are assumed sufficiently regular in *t* and *T* so that the SDE (12) admits a unique strong solution, and their partial derivatives with respect to *T* exist and have finite  $L_2$  norms w.r.t. *t*. Moreover, the volatility functions must satisfy<sup>6</sup>

$$\Sigma(t,t) = \Sigma_R(t,t) = 0.$$

Note that the dynamics of  $P_R(t,T)$  follows from those of  $P_R(t,T_0,T)$  and I(t):

$$dP_R(t,T) = P_R(t,T) \left( (r_t - i_t)dt + \Sigma_R(t,T) \cdot d\mathbf{Z}_t \right).$$
(13)

Being a  $T_1$ -forward price of a tradable security,  $F(t, T_1, T_2)$  should be a lognormal martingale under the  $T_1$ -forward measure whose volatility is the difference between those of  $P_R(t, T_0, T_1)$  and  $P_R(t, T_0, T_2)$ , i.e.,

$$\frac{dF_R(t, T_1, T_2)}{F_R(t, T_1, T_2)} = \left(\Sigma_R(t, T_2) - \Sigma_R(t, T_1)\right)^T (d\mathbf{Z}_t - \Sigma(t, T_1)dt).$$
(14)

Note that  $d\mathbf{Z}_t - \Sigma(t, T_1)dt$  is (the differential of) a Brownian motion under the  $T_1$ -forward measure,  $Q_{T_1}$ , defined by the Radon-Nikodym derivative

$$\left. \frac{dQ_{T_1}}{dQ} \right|_{\mathcal{F}_t} = \frac{P(t, T_1)}{B(t)P(0, T_1)},$$

where  $B(t) = \exp(\int_0^t r_s ds)$  is the unit price of money market account.

<sup>&</sup>lt;sup>5</sup>It is not hard to see that the volatility of  $P_R(t, T_0, T)$  does not depend on  $T_0$ . <sup>6</sup>Note that both  $P_R(t, t)$  and I(t) have no volatility.

There is an important implication by (14). By Ito's lemma, we also have

$$\frac{dF_R(t, T_1, T_2)}{F_R(t, T_1, T_2)} = \left(\Sigma_R(t, T_2) - \Sigma_R(t, T_1)\right)^T (d\mathbf{Z}_t - \Sigma_R(t, T_1)dt).$$
(15)

The coexistence of equations (14) and (15) poses a constraint on the volatility functions of the real zero-coupon bonds.

**Proposition 3 (Consistency condition)**: For arbitrage pricing, the volatility functions of the real bonds must satisfy the following condition:

$$(\Sigma_R(t, T_2) - \Sigma_R(t, T_1)) \cdot (\Sigma(t, T_1) - \Sigma_R(t, T_1)) = 0.$$
(16)

Its differential version is

$$\dot{\Sigma}_R(t,T) \cdot \Sigma_I(t,T) = 0, \qquad (17)$$

where the overhead dots mean partial derivatives with respect to T, the maturity.

Let us try to understand the consistency condition. We know obviously that  $\Sigma_I(t,T_1) \stackrel{\triangle}{=} \Sigma(t,T_1) - \Sigma_R(t,T_1)$  is the percentage volatility of  $P_I(t,T)$ , while  $\Sigma_R(t,T_2) - \Sigma_R(t,T_1)$  is the volatility of the real forward rate defined by

$$f_R(t, T_1, T_2) \stackrel{\triangle}{=} \frac{1}{T_2 - T_1} \left( \frac{P_R(t, T_1)}{P_R(t, T_2)} - 1 \right).$$

Literally, (16) means that the price of inflation discount bond with maturity  $T_1$  must be uncorrelated with real forward rates of any future period beyond  $T_1$ . This sounds reasonable and is nonrestrictive at all.

The differential version of the consistency condition will be used later to derive an HJM type model for inflation rates.

### The Market Model

For generality, we let  $T = T_2$  and  $\Delta T = T_2 - T_1$ , we then can cast (10) into

$$f^{(I)}(t,T-\Delta T,T) + \frac{1}{\Delta T} = \frac{1}{\Delta T} \frac{F_R(t,T-\Delta T,T)P(t,T-\Delta T)}{P(t,T)}.$$

The dynamics of  $f^{(I)}(t, T - \Delta T, T)$  follows readily from those of  $F_R$  and P's.

**Proposition 3**. Under the risk neutral measure, the governing equation for the simple inflation forward rate is

$$d\left(f^{(I)}(t, T - \Delta T, T) + \frac{1}{\Delta T}\right)$$

$$= \left(f^{(I)}(t, T - \Delta T, T) + \frac{1}{\Delta T}\right) \left\{\gamma^{(I)}(t, T) \cdot (d\boldsymbol{Z}_t - \Sigma(t, T)dt)\right\},$$
(18)

where

$$\gamma^{(I)}(t,T) = \Sigma_I(t,T-\Delta T) - \Sigma_I(t,T)$$

is the percentage volatility of the displaced inflation forward rate.

In formalism, equation (18) is the practitioners' model<sup>7</sup>, where  $\gamma^{(I)}(t,T)$  is obtained by calibration instead of being derived from the volatilities of the inflation discount bonds. Let us present the market model for inflation rates in comprehensive terms. The state variables consist of two streams of spanning nominal forward rates (Brace *et al.*, 1997) and forward inflation rates,  $f_j(t) \stackrel{\triangle}{=} f(t, T_j, T_{j+1})$  and  $f_j^{(I)}(t) \stackrel{\triangle}{=} f^{(I)}(t, T_{j-1}, T_j), j = 1, 2, \ldots, N$ , that follow the following dynamics:

$$\begin{cases} df_j(t) = f_j(t)\gamma_j(t) \cdot (d\mathbf{Z}_t - \Sigma_{j+1}(t)dt), \\ d\left(f_j^{(I)}(t) + \frac{1}{\Delta T_j}\right) = \left(f_j^{(I)}(t) + \frac{1}{\Delta T_j}\right)\gamma_j^{(I)}(t) \cdot (d\mathbf{Z}_t - \Sigma_j(t)dt), \end{cases}$$
(19)

where

$$\Sigma_{j+1}(t) = -\sum_{k=\eta_t}^{j} \frac{\Delta T_{k+1} f_k(t)}{1 + \Delta T_{k+1} f_k(t)} \gamma_k(t)$$

and  $\eta_t = \min\{i|T_i > t\}$ . So,  $f_j^{(I)}(t)$  is also a martingale under its own "cash flow measure".

#### The Extended Heath-Jarrow-Morton Model

Analogously to the definition of nominal forward rates, we define the instantaneous inflation forward rates as

$$f^{(I)}(t,T) \stackrel{\triangle}{=} -\frac{\partial \ln P_I(t,T)}{\partial T}, \quad \forall T \ge t,$$
(20)

 $<sup>^7\</sup>mathrm{Practitioners}$  take inflation forward rates from YYIIS, and consider their model a YYIIS-based model.

$$P_I(t,T) = e^{-\int_t^T f^{(I)}(t,s)ds}.$$

By the Ito's lemma, we have

$$-d\ln P_I(t,T) = d\ln\left(\frac{P_R(t,T)}{P(t,T)}\right)$$
  
=  $-\left(i_t + \frac{1}{2}\|\Sigma_I(t,T)\|^2\right)dt - \Sigma_I^T(t,T)\left(d\mathbf{W}_t - \Sigma(t,T)dt\right).$  (21)

Differentiating the above equation with respect to T and making use of the consistency condition (17), we then have

$$df^{(I)}(t,T) = -\dot{\Sigma}_I \cdot \left( d\mathbf{Z}_t - \Sigma(t,T) dt \right).$$
(22)

Equation (22) shows that  $f^{(I)}(t,T)$  is a  $Q_T$ -martingale and its dynamics is fully specified by the volatilities of the nominal and inflation forward rates.

In an HJM context, the volatilities of nominal and inflation forward rates,  $\sigma(t,T) = -\dot{\Sigma}(t,T)$  and  $\sigma^{(I)}(t,T) = -\dot{\Sigma}_I(t,T)$ , are first prescribed, and the volatilities of the zero-coupon bonds follow from

$$\Sigma(t,T) = -\int_t^T \sigma(t,s)ds$$
 and  $\Sigma_I(t,T) = -\int_t^T \sigma^{(I)}(t,s)ds$ .

Then, the extended HJM model with nominal and inflation forward rates is

$$\begin{cases} df(t,T) = \sigma(t,T) \cdot d\mathbf{Z}_t + \sigma(t,T) \cdot \left(\int_t^T \sigma(t,s)ds\right) dt, \\ df^{(I)}(t,T) = \sigma^{(I)}(t,T) \cdot d\mathbf{Z}_t + \sigma^{(I)}(t,T) \cdot \left(\int_t^T \sigma(t,s)ds\right) dt, \end{cases}$$
(23)

which takes the initial term structures of nominal and inflation forward rates as inputs.

If we treat (23) as a framework of no-arbitrage models, then the market model (19) fits in the framework with the volatility function

$$\sigma^{(I)}(t,T) = -\dot{\Sigma}_I(t,T) = \frac{\partial}{\partial T} \left( \sum_{k=0}^{\left[\frac{T-t}{\Delta T}\right]} \gamma^{(I)}(t,T-k\Delta T) \right),$$

or

where [x] is the integer part of x.

## Connection with the Jarrow-Yildirim Model

According to their definitions, nominal, inflation and real forward rates for continuous compounding satisfy the relationship

$$f_R(t,T) = f(t,T) - f^{(I)}(t,T).$$

Subtracting the two equations of (23) and applying the consistency condition, (17), we will arrive at

$$df_R(t,T) = \sigma_R(t,T) \cdot d\mathbf{Z}_t + \sigma_R(t,T) \cdot \left(\int_t^T \sigma_R(t,s)ds\right) dt, \qquad (24)$$

where

$$\sigma_R(t,T) = \sigma(t,T) - \sigma^{(I)}(t,T) = -\dot{\Sigma}_R(t,T).$$

In contrast, under our notations the equation established by Jarrow and Yildirim (2003) for the real forward rates is

$$df_R(t,T) = \sigma_R(t,T) \cdot d\mathbf{Z}_t + \sigma_R(t,T) \cdot \left(\int_t^T \sigma_R(t,s)ds - \sigma_I(t)\right) dt, \quad (25)$$

where  $\sigma_I(t)$  is the volatility of the CPI index. Given that  $\sigma_I(t) \equiv 0$ , the two equations are identical.

Even if the CPI volatility were not zero, we can still re-derive the Jarrow and Yildirim model by recognizing that the volatility of  $P_R(t, T_0, T)$  satisfies  $\Sigma_R(t,t) = \sigma_I(t)$  and redoing the arguments. Based on the above analysis, we claim that market model is consistent with the framework of "foreign currency analogy".

#### **Pricing Inflation Derivatives**

We have established for the first time that inflation forward rates are lognormal martingales under respective forward measures. As a result, the current practices on pricing some inflation derivatives must undergo some changes.

## $\Box$ YYIIS

The price of a YYIIS is the difference in value of the fixed leg and floating

leg. While the fixed leg is priced as an annuity, the floating leg is priced by discounting the expectation of each piece of payment:

$$V_{float}^{(j)}(t) = Not.P(t, T_j)E_t^{Q_{T_j}} \left[ \left( \frac{I(T_j)}{I(T_{j-1})} - 1 \right) \right] = Not.\Delta T_j P(t, T_j)E_t^{Q_{T_j}} \left[ f_j^{(I)}(T_j) \right] = Not.\Delta T_j P(t, T_j)f_j^{(I)}(t),$$
(26)

due to the martingale property of the inflation forward rates. The value of the floating leg is just a summation, and the value of the YYIIS is the difference between the values of the fixed and floating legs.

In the market place, YYIIS are treated as another set of securities parallel to ZCIIS, and the "inflation forward rates" implied by YYIIS and ZCIIS can be different. Our theory, for the first time, suggests that such differences create arbitrage opportunities. In existing literatures, pricing YYIIS using a ZCIIS-based model goes through a procedure of "convexity adjustment", which is unnecessary.

## $\Box$ Caplets

In view of the displaced diffusion processes for simple forward inflation rates, we can price a caplet with \$1 notional value straightforwardly as follows:

$$\Delta T_{j} E_{t}^{Q} \left[ e^{-\int_{t}^{T_{j}} r_{s} ds} (f_{j}^{(I)}(T_{j}) - K)^{+} \right]$$
  
=  $\Delta T_{j} P(t, T_{j}) E_{t}^{Q_{T_{j}}} \left[ \left( \left( f_{j}^{(I)}(T_{j}) + \frac{1}{\Delta T_{j}} \right) - \left( K + \frac{1}{\Delta T_{j}} \right) \right)^{+} \right]$  (27)  
=  $\Delta T_{j} P(t, T_{j}) \{ \mu_{j}(t) \Phi(d_{1}^{(j)}(t)) - \tilde{K}_{j} \Phi(d_{2}^{(j)}(t)) \},$ 

where  $\Phi(\cdot)$  is the standard normal accumulative distribution function, and

$$\mu_j(t) = f_j^{(I)}(t) + 1/\Delta T_j, \quad \tilde{K}_j = K + 1/\Delta T_j,$$
  
$$d_1^{(j)}(t) = \frac{\ln \mu_j / \tilde{K}_j + \frac{1}{2}\sigma_j^2(t)(T_j - t)}{\sigma_j(t)\sqrt{T_j - t}}, \quad d_2^{(j)}(t) = d_1^{(j)}(t) - \sigma_j(t)\sqrt{T_j - t},$$

with  $\sigma_j(t)$  to be the mean volatility of  $\ln(f_j^{(I)}(t) + \frac{1}{\Delta T_j})$ :

$$\sigma_j^2(t) = \frac{1}{T_j - t} \int_t^{T_j} \|\gamma_j^{(I)}(s)\|^2 ds.$$
(28)

Equation (27) is like an old bottle filled with new wine: the input inflation forward rates should be jointly implied by ZCIIS and YYIIS.

#### $\Box$ Swaptions

The discussions on the pricing of inflation swaptions have been rare (Hinnerich, 2008). An inflation swaption is an option to enter into a YYIIS at the option's maturity. Without loss of generality, we consider here an underlying swap which has the same cash-flow frequency for both fixed and floating legs. Similar to the situation of swaps on nominal interest rates, it is straightforward to show that the market prevailing inflation swap rate (that nullifies the value of a swap) is

$$S_{m,n}(t) = \frac{\sum_{i=m+1}^{n} \Delta T_i P(t, T_i) f_i^{(I)}(t)}{\sum_{i=m+1}^{n} \Delta T_i P(t, T_i)}.$$
(29)

The above expression can be recast into

$$S_{m,n}(t) + \frac{1}{\Delta T_{m,n}} = \sum_{i=m+1}^{n} \omega_i \mu_i(t),$$
(30)

where

$$w_i(t) = \frac{\Delta T_i P(t, T_i)}{A_{m,n}(t)}, \quad A_{m,n}(t) = \sum_{i=m+1}^n \Delta T_i P(t, T_i),$$

and

$$\frac{1}{\Delta T_{m,n}} = \sum_{i=m+1}^{n} w_i(t) \frac{1}{\Delta T_i}$$

Using brute force, we can derive the dynamics of the displaced swap rate under the forward measure,  $Q_{m,n}$ , which is the martingale measure corresponding to annuity numeraire,  $A_{m,n}(t)$ , such that

$$d\left(S_{m,n}(t) + \frac{1}{\Delta T_{m,n}}\right) = \left(S_{m,n}(t) + \frac{1}{\Delta T_{m,n}}\right)$$
$$\times \sum_{i=m+1}^{n} \left[\alpha_i(t)\gamma_i^{(I)}(t) + (\alpha_i(t) - w_i(t))\Sigma(t, T_i)\right] \cdot d\mathbf{Z}_t^{(m,n)},$$
(31)

where  $\mathbf{Z}_{t}^{(m,n)}$  is a  $Q_{m,n}$ -Brownian motion, and

$$\alpha_i(t) = \frac{\omega_i \mu_i(t)}{\sum_{j=m+1}^n \omega_j \mu_j(t)}$$

By appropriately freezing coefficients of (31), the displaced forward inflation swap rate  $S_{m,n}(t) + \frac{1}{\Delta T_{m,n}}$  becomes a lognormal process, and closed-form pricing of inflation swaptions will then follow. Consider a  $T_m$ -maturity swaption on the YYIIS over the period  $[T_m, T_n]$  and with strike K, we can derive its value as

$$V_t = A_{m,n}(t) \left[ \left( S_{m,n}(t) + \frac{1}{\Delta T_{m,n}} \right) \Phi(d_1^{(m,n)}) - \tilde{K}_{m,n} \Phi(d_2^{(m,n)}) \right], \quad (32)$$

where

$$\begin{split} \tilde{K}_{m,n} &= K + \frac{1}{\Delta T_{m,n}}, \\ d_1^{(m,n)} &= \frac{\ln\left(S_{m,n}(t) + 1/\Delta T_{m,n}\right)/\tilde{K}_{m,n} + \frac{1}{2}\sigma_{m,n}^2(t)(T_m - t)}{\sigma_{m,n}(t)\sqrt{T_m - t}}, \\ d_2^{(m,n)} &= d_1^{(m,n)} - \sigma_{m,n}(t)\sqrt{T_m - t}, \\ \sigma_{m,n}(t) &= \frac{1}{T_m - t} \int_t^{T_m} \left\| \sum_{i=m+1}^n \left[ \alpha_i(t)\gamma_i^{(I)}(s) + (a_i(t) - w_i(t))\Sigma(s, T_i) \right] \right\|^2 ds. \end{split}$$

The swaption formula, (32), implies a hedging strategy for the swaption. At any time t, the hedger should long  $N(d_1^{(m,n)})$  units of the underlying inflation swap for hedging. When n = m + 1, the swaption reduces to a caplet. With the Black's formula, inflation caps, floors and swaptions can be quoted using implied volatilities.

### Model Calibration

A comprehensive calibration of the inflation-rate model (15) means simultaneous determination of volatility vectors for inflation forward rates, based on market data of YYIIS, inflation caps and inflation swaptions. For nonparametric calibration, one can adopt the methodology for the calibration of LIBOR market model developed by Wu (2003).

For demonstration, we have calibrated the two-factor market model to price data of Euro ZCIIS and (part of the) inflation caps as of April 7, 2008<sup>8</sup>,

<sup>&</sup>lt;sup>8</sup>We do not have the data of YYIIS and swaptions.

and observed very nice performance. Figure 2 shows the term structures of inflation forward rates as well as nominal forward rates. Figure 3 shows the local volatility function obtained by calibrating the model to implied cap volatilities of various maturities but a fixed strike K = 2%.



Figure 2 Term structure of the forward nominal rates and forward inflation rates.



Figure 3 Calibrated local volatility surface,  $\gamma_i^{(I)}(t)$ .

### Smile Modeling

With the dynamics of displaced diffusions only, the market model cannot price volatility smiles in cap/floor markets. For that purpose we should extend or modify the current model in ways parallel to the extensions to the LIBOR market model, on which there are rich literatures (see e.g. Brigo and Mercurio (2006) for an introductions of smile models). One quick solution for smiles modeling is to adopt the SABR (Hagen, *et al.*, 2003) dynamics for the expected displaced forward inflation rates,  $\mu_j(t)$ , and consider the following model:

$$\begin{cases} d\mu_j(t) = \mu_j^{\beta_j}(t)v_j(t)dZ_t^j, \\ dv_j(t) = \epsilon_j v_j(t)dW_t^j, \end{cases}$$
(33)

where  $\beta_j$  and  $\epsilon_j$  are constants, and both  $Z_t^j$  and  $W_t^j$  are one-dimensional (correlated) Brownian motions under the  $T_j$ -forward measure. Mecurio and Mereni (2009) proposed and studied the above model for  $\beta_j = 1$ , and demonstrate a quality fitting of implied volatility smiles.

#### Conclusion

Through this paper we have clarified the important notions of inflation forward rates and market models, and established the consistency between the market model and the framework of currency analogy. Our theory will definetely help to improve the pricing efficiency and hedging effectiveness of inflation derivatives.

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# A Proofs of Propositions

*Proof of Proposition 1*: Do the following zero-net transactions.

- 1. At time  $t \geq T_0$ ,
  - (a) Long the forward contract to buy  $\frac{I(T_1)}{I(T_0)}$  dollar value (or  $\frac{I(T_1)}{I(T_0)F_R(t,T_1,T_2)}$ units) of  $T_2$ -maturity real bond deliverable at  $T_1$  for the unit price  $F_R(t, T_1, T_2)$ ;
  - (b) long one unit of  $T_1$ -maturity real bond at the price of  $P_R(t, T_0, T_1)$ ;
  - (c) short  $\frac{P_R(t,T_0,T_1)}{P_R(t,T_0,T_2)}$  unit(s) of  $T_2$ -maturity real bond at the unit price of  $P_R(t,T_0,T_2)$ .
- 2. At time  $T_1$ , exercise the forward contract to buy the  $T_2$ -maturity real bond (that pays  $I(T_2)/I(T_1)$ ) at the unit price  $F_R(t, T_1, T_2)$ , applying all proceeds from the  $T_1$ -maturity real bond.
- 3. At Time  $T_2$ , settle all transactions.

The net profit or loss from the transactions is

$$P\&L = \left(\frac{1}{F_R(t, T_1, T_2)} - \frac{P_R(t, T_0, T_1)}{P_R(t, T_0, T_2)}\right) \frac{I(T_2)}{I(T_0)}.$$
(34)

For the absense of arbitrage, the forward price must be set equal to (11)

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