A Link between two Fundamental Contributions of Kowalevski

Jishan Hu, Min Yan
Department of Mathematics
Hong Kong University of Science and Technology

November 22, 2004

1 Introduction

Among numerous important mathematical contributions of Kowalevski, the following two are fundamental: The first is her doctoral dissertation, "On the theory of partial differential equations", appeared in 1875 [8]. The main result is what we now call the Cauchy-Kowalevski theorem. The second is "On the problem of the rotation of a solid body about a fixed point", appeared in 1889 [9] and won her the famous Prix Bordin of the French Academy of Sciences. The main result is the discovery of what we now call the Kowalevski top, which completes a program implicit in the works of Euler and Lagrange - to solve in an analytic manner the equations of motion of a rotating rigid body.

It is rather interesting to compare the two contributions from the technical perspective. The first work shows that the formal power series solution to the initial value problem (along a noncharacteristic manifold for PDE) of an analytic differential equation must be convergent (and analytically dependent on the initial data). In particular, Cauchy-Kowalevski theorem justifies the formal power series solutions as being *realistic*. In the second work, Kowalevski determined the integrability of a dynamical system by *formally* testing Laurent series solutions with movable singularity location and enough number (five for a system of six first order equations) of free parameters. The method is what we now call the Painlevé test [2] [11] and is still the most effective way of detecting integrability. Being the formal process, however, the convergence (i.e., the realism) of the series is not considered in the test.

In this note, we will explain that the following two are equivalent for a differential equation

1. The equation passes the Painlevé test in the most strict sense (see the end of Section 2 for the precise meaning), as the Kowalevski top does;

2. There is a (usually meromorphic) change of dependent variables that converts the equation to an analytic system of equations, and converts the Laurent series in the Painlevé test to power series.

In particular, if we can apply the Cauchy-Kowalevski theorem to the power series solution, then we conclude the convergence of the Laurent series in the Painlevé test.

An example of such changes of variables already appeared a century ago in Painlevé's proof [10] that all solutions of of the first Painlevé equation $u'' = 6u^2 + t$ are meromorphic functions on the whole complex plane. However, it did not appeared to have been noticed that the existence of such changes of variables is a general property for equations passing the Painlevé test. This note explained the key ideas for finding the changes of variables. More discussions and examples can be found in our series of papers [3] through [6].

Both the power series and the Laurent series are local by nature. Therefore it is not reasonable to expect a global link between the two. Even though our change of variables does not *directly* tell us anything about the global questions such as whether singularities accumulate, it can be a potentially useful tool for studying the integrable equations, especially on problems related to singularity analysis. Basically, near regular points of a solution, we may use the original equation to study the behavior of the solution. Near movable pole singularities, we may regularize the situation by applying our changes of variables and then study the system of equations for the new variables. Sometimes the two studies can be combined together and provide us a global understanding of the solution. Painlevé had successfully applied the idea in his proof. We also used the idea in a much better proof of the Painlevé property for certain equations [7].

2 Key Idea

We illustrate our key idea through the simple example of Painlevé's first equation:

$$u'' = 6u^2 + t. \tag{1}$$

Consider a (regular) initial value problem

$$u(t_0) = r, \quad u'(t_0) = s. \tag{2}$$

Formally, through a recursive relation, the initial value problem is equivalent to the following power series solution

$$u = r + s(t - t_0) + \left(3r^2 + \frac{t_0}{2}\right)(t - t_0)^2 + \left(2rs + \frac{1}{6}\right)(t - t_0)^3 + \cdots. \tag{3}$$

Cauchy-Kowalevski theorem then concludes that (3) is indeed a convergent series, and u is analytically dependent on t_0, r, s.

Now consider pole-like solutions of (1), with movable location of the pole. By dominant balance, the order of the pole must be 2. Therefore we postulate $u = u_0(t - t_0)^{-2} + u_1(t - t_0)^{-1} + \cdots$ and get a recursive relation for the coefficients. The result is a formal Laurent series solution

$$u = (t - t_0)^{-2} - \frac{t_0}{10}(t - t_0)^2 - \frac{1}{6}(t - t_0)^3 + r(t - t_0)^4 + \frac{t_0^2}{18}(t - t_0)^6 + \cdots \tag{4}$$

with r arbitrary. Note that the existence of Laurent series solution together with one free parameter r (in general, the number is the degree of the equation minus one) in the series imposes a highly non-trivial condition on the shape of the equation. For example, the equation $u'' = 6u^2 + t^2$ has no Laurent series solutions. Moreover, the Chazy's equation $u''' = 2uu'' - 3u'^2$ has two Laurent series solutions and they can never contain free parameters. Through a similar but much more complicated process, Kowalevski discovered her top by studying the conditions derived from the existence of five free parameters in Laurent series solutions in the system of six equations. More discussion can be found on [1] and [2].

Comparing the two solutions (3) and (4), we have the following observations:

1. In addition to t_0, there are two free parameters in (3). Such parameters are exactly the initial data in (regular) initial value problems (2).

The number of such parameters is *always* the order of the equation. Moreover, they *always* appear explicitly as the initial coefficients in (3);

2. In addition to t_0, there is one free parameter in (4). Such parameters can be considered as the "initial data" in some imaginary singular initial value problems. The number of such parameters is *always* at most the order of the equation, and may be fewer. Moreover, they often do not appear as the initial coefficients in (3), but rather buried deep in the series.

In order to convert Laurent series solutions such as (4) to power series solutions such as (3), therefore, we need to bring the free parameters "buried deep" in the Laurent series "to the front". The most naive way to bring r to the front is to truncate at the place where r first appears

$$u = (t - t_0)^{-2} - \frac{t_0}{10}(t - t_0)^2 - \frac{1}{6}(t - t_0)^3 + v(t - t_0)^4, \tag{5}$$

by introducing a new variable

$$v = r + \frac{t_0^2}{18}(t - t_0)^2 + \cdots. \tag{6}$$

However, it is not clear what the transformation between u and v is. It is also not clear what equation and the corresponding initial value problem for v is.

Thus we need to do the truncation in a more clever way, and we also need to get a transformation between the old and the new variables at the end. We consider the equation (1) as a system for the variables u and u'. Then we introduce a new variable θ through *indicial normalization*

$$u = \theta^{-2}, \tag{7}$$

so that θ behaves just like $t - t_0$:

$$\begin{aligned} \theta &= (t - t_0) + \frac{t_0}{20}(t - t_0)^5 + \frac{1}{12}(t - t_0)^6 - \frac{r}{2}(t - t_0)^7 - \frac{t_0^2}{36}(t - t_0)^9 + \cdots \\ &= (t - t_0) + \frac{t}{20}(t - t_0)^5 + \frac{1}{30}(t - t_0)^6 - \frac{r}{2}(t - t_0)^7 - \frac{t^2}{36}(t - t_0)^9 + \cdots \end{aligned} \tag{8}$$

On the other hand, we have the Laurent series for (another variable) u':

$$\begin{aligned} u' &= -3(t-t_0)^{-2} - \frac{t_0}{5}(t-t_0) - \frac{1}{2}(t-t_0)^2 + 4r(t-t_0)^3 + \frac{t_0^2}{3}(t-t_0)^5 + \cdots \\ &= -3(t-t_0)^{-2} - \frac{t}{5}(t-t_0) - \frac{3}{10}(t-t_0)^2 + 4r(t-t_0)^3 + \frac{t^2}{3}(t-t_0)^5 + (9) \end{aligned}$$

From (8), we may rewrite $t-t_0$ as a series in powers of θ. Then we substitute this series into (9) and obtain

$$u' = -2\theta^{-3} - \frac{t}{2}\theta - \frac{1}{2}\theta^2 + 7r\theta^3 + \frac{t^2}{16}\theta^5 + \cdots. \tag{10}$$

Now we may truncate at the first place where the free parameter r appears

$$u' = -2\theta^{-3} - \frac{t}{2}\theta - \frac{1}{2}\theta^2 + \xi\theta^3, \tag{11}$$

by introducing a new variable ξ, which must satisfy

$$\xi = 7r + \frac{t^2}{16}\theta^2 + \cdots = 7r + \frac{t_0^2}{16}(t-t_0)^2 + \cdots. \tag{12}$$

The equations (7) and (11) form a change of variables between (u, u') and (θ, ξ). This converts the original equation (1) into a *regular* system

$$\begin{aligned} \theta' &= 1 + \frac{t}{4}\theta^4 + \frac{1}{4}\theta^5 - \frac{1}{2}\xi\theta^6, \\ \xi' &= \frac{t^2}{8}\theta + \frac{3t}{8}\theta^2 + \left(\frac{1}{4} - t\xi\right)\theta^3 - \frac{5}{4}\xi\theta^4 - \frac{3}{2}\xi^2\theta^5. \end{aligned}$$

Moreover, the formal Laurent series for (u, u') are converted to the formal power series (8) and (12) for (θ, ξ). Note that we may apply Cauchy-Kowalevski theorem to the initial value problem

$$\theta(t_0) = 0, \quad \xi(t_0) = 7r$$

to conclude the convergence of the series (8) and (12). Then we may apply the transformation to see that the Laurent series (4) is also convergent.

We summarize the key steps in the computation above:

1. First introduce a new variable θ through the indicial normalization of some variable, so that θ behaves like (a nonzero multiple of) $t - t_0$.

2. Use θ to substitute $t - t_0$, so that t_0 will not appear in the Laurent series for the other variables. Because the change of variables we try to find should be independent of specific solutions, we should eliminate any trace of t_0, which depends on specific solutions.

3. Introduce more new variables by successively truncating at the free parameters in the θ-Laurent series obtained in the second step. This brings the hidden free parameters to the front - the initial term in the power series for the new variables.

Finally, note that in order to get enough number of new variables (so that we have a change of equal number of variables at the end), the number of free parameters appearing in the Laurent series must be the order of the equation minus one. The condition is exactly what we mean by passing the Painlevé test in the most strict sense.

3 The Algorithm

The change of variables (7) and (11) already appeared in Painlevé's proof [10] that all solutions of (1) are meromorphic functions on the whole complex plane. In fact, the method we used to derive them in the last section closely resembles Painlevé's original. Next we present a more direct and systematic way of deriving such changes of variables, for equations of the form

$$u^{(n)} = f(t, u, u', \cdots, u^{(n-1)}).$$

We will still illustrate our algorithm with the example of the first Painlevé equation (1).

Step 1: Indicial Normalization

This is the same as the Painlevé test. We use the dominant balance to find the order k of the pole. Then we introduce the indicial normalization $u = \theta^{-k}$. For the first Painlevé equation, we have $k = 2$ and $u = \theta^{-2}$.

Step 2: Laurent Series

We compute the Laurent series *in powers of* θ for the derivatives of u. First postulate a θ-power series

$$\theta' = A = a_0 + a_1\theta + a_2\theta^2 + \cdots, \tag{13}$$

where a_* are functions of t. Thinking of A as functions of t and θ, we denote

$$\begin{aligned}
A_t &= a_0' + a_1'\theta + a_2'\theta^2 + \cdots, \\
A_\theta &= a_1 + 2a_2\theta + 3a_3\theta^2 + \cdots, \\
A_{t^2} &= a_0'' + a_1''\theta + a_2''\theta^2 + \cdots, \\
A_{t\theta} &= a_1' + 2a_2'\theta + 3a_2'\theta^2 + \cdots, \\
A_{\theta^2} &= 3a_2 + 6a_3\theta + 12a_4\theta^2 + \cdots, \\
&\vdots
\end{aligned}$$

Then we have

$$\begin{aligned}
u' &= -k\theta^{-k-1}A \\
&= -ka_0\theta^{-k-1} - ka_1\theta^{-k} - ka_2\theta^{-k+1} + \cdots, \\
u'' &= (-k)(-k-1)\theta^{-k-2}A^2 - k\theta^{-k-1}A_t - k\theta^{-k-1}AA_\theta \\
&= k(k+1)a_0^2\theta^{-k-2} + [k(2k+1)a_0a_1 - ka_0']\theta^{-k-1} \\
&\quad +[2k^2a_0a_2 - ka_1' + k^2a_1^2]\theta^{-k} + \cdots, \\
&\vdots \\
u^{(n)} &= (-k)\cdots(-k-n+1)\theta^{-k-n}A^n + \cdots \\
&= (-1)^n k\cdots(k+n-1)a_0^n\theta^{-k-n} + \cdots.
\end{aligned}$$

These are θ-Laurent series, with functions of t, a_*, and derivatives of a_* as coefficients. Moreover, these series actually do not depend on the equation. They are formal series depending only on k.

Now we substitute $u = \theta^{-k}$ and the θ-Laurent series of u', u'', $\cdots$, $u^{(n-1)}$ to the right side of the equation, so that the right side becomes a θ-Laurent series. Compared with the θ-Laurent series of $u^{(n)}$ on the left side, we get a recursive relation. Solving the recursive relation will determine the coefficients a_* and the θ-Laurent series of the derivatives of u.

For the first Painlevé equation, the θ-Laurent series on the right side is simply $6\theta^{-4} + t$. Therefore the recursive relation becomes

$$2(2+1)a_0^2 = 6, \quad 2(4+1)a_0a_1 - 2a_0' = 0, \quad 8a_0a_2 - 2a_1' + 4a_1^2 = 0, \quad \cdots.$$

We fix a solution $a_0 = 1$ of the first equation and then recursively determine the later coefficients:

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = 0, \quad \cdots.$$

As in the Painlevé test, we found that a_6 can be an arbitrary, and the θ-power series for θ' is

$$\theta' = 1 + \frac{t}{4}\theta^4 + \frac{1}{4}\theta^5 + a_6\theta^6 - \frac{t^2}{32}\theta^8 + \cdots.$$

This easily gives rise to the θ-Laurent series (10) for u'.

Step 3: New Variables

We introduce the new variables (in addition to θ introduced in the first step) by truncating the derivatives of u at the places where free coefficients of the θ-power series of θ' appear. Let a_i, a_j, $\cdots$, $i < j < \cdots$, be free coefficients in (13). Then we introduce a new variable ξ by truncating the θ-Laurent series of u' at θ^{-k+i-1}:

$$u' = u_{1,0}\theta^{-k-1} + u_{1,1}\theta^{-k} + \cdots + u_{1,i-1}\theta^{-k+i-2} + \xi\theta^{-k+i-1}.$$

Note that since i is the first index where a free coefficient appears (i.e., i is the smallest *resonance*), the coefficients $u_{1,0}$, $u_{1,2}$, $\cdots$, $u_{1,i-1}$ are functions of t only.

For the first Painlevé equation, we simply truncate (10) at θ^3 to obtain (11), and the whole algorithm is complete.

If $n > 2$, then we need to further introduce a new variable η by truncating the θ-series of u'' at θ^{-k+j-2}. However, we cannot do this immediately because the θ-series of u'' we have at the moment has a_i appearing in the coefficients. We need to update the θ-series of u'' by replacing a_i with the "equivalent" variable ξ. Here we say a_i and ξ are equivalent because we have brought the hidden arbitrary choice a_i to the front, as the initial coefficient of ξ in a power series expansion. Specifically, from

$$\xi = u_{1,i} + u_{1,i+1}\theta + u_{1,i+2}\theta^2 + \cdots, \quad u_{1,i} = -ka_i + \cdots,$$

we may rewrite a_i as a θ-series, with $-k^{-1}\xi$ as the initial term and functions of t, ξ, a_j, $\cdots$, as coefficients of the subsequent terms. Then we substitute the θ-power series for a_i into the θ-Laurent series for u'' and introduce a new variable η by truncating at θ^{-k+j-2}:

$$u'' = u_{2,0}\theta^{-k-2} + u_{2,1}\theta^{-k-1} + \cdots + u_{2,j-1}\theta^{-k+j-3} + \eta\theta^{-k+j-2}.$$

The coefficients $u_{2,0}$, $u_{2,2}$, $\cdots$, $u_{2,i-1}$ are functions of t only, and the coefficients $u_{2,i}$, $u_{2,i+1}$, $\cdots$, $u_{2,j-1}$ are functions of t and ξ only.

The process continues until we introduce the final new variable by truncating the (updated) θ-series of $u^{(n-1)}$.

4 Comments

The first two steps in our algorithm closely resembles the Painlevé test. Basically the only new ingredient is the use of θ instead of $t - t_0$ as the basis of expansion. The extra computation this causes is not much: Whenever taking derivatives, we need to substitute θ' by (13). It is quite easy to set up an equivalence between the coefficients of the θ-power series of θ' in our algorithm and the coefficients of the ordinary Laurent series of u' in the Painlevé test. We emphasis that the algorithm does not require any computational result from the Painlevé test. The first two steps involve about the same amount of computation as the Painlevé test and is equivalent to the Painlevé test.

Because of the equivalence, the free coefficients in our algorithm correspond to the free coefficients in the Painlevé test, and their numbers are the same. Note that in order for the number of new variables (θ and the ones introduced in the third step) to be n, the number of free coefficients must be $n - 1$. In other words, the third step of our algorithm produces a change of variables if and only if the equation passes the Painlevé test (in the sense that there are enough number of non-negative integral resonances).

The new system for θ, ξ, η, $\cdots$, must be regular, due to the fact that we have brought all the free parameters to the front. See [6] for a complete and rigorous proof. Here we only mention that the essence of the proof is the following general fact: Suppose a first order system of differential equations has functions meromorphic in θ and analytic in ξ, η, $\cdots$ on the right side. Suppose the system has formal power series solutions of the form

$$\begin{aligned} \theta &= a(t-t_0) + b(t-t_0)^2 + \cdots \\ \xi &= \xi_0 + \xi_1(t-t_0) + \xi_2(t-t_0)^2 + \cdots \\ \eta &= \eta_0 + \eta_1(t-t_0) + \eta_2(t-t_0)^2 + \cdots \\ &\vdots \end{aligned}$$

where a is some fixed nonzero number and t_0, ξ_0, η_0, $\cdots$, can be arbitrary. Then the functions on the right side are also analytic in θ (near $\theta = 0$).

Our algorithm is also applicable to systems of equations as well as partial differential equations [3] [4]. We have also demonstrated that, for Hamiltonian systems, it is possible construct the change of variables so that it is canonical [5]. All these have been theoretically established under suitable conditions [6].

There are still many open questions. One particular question is to find suitable changes of variables in less than ideal situation, especially in case some resonances are negative. These include the so called lower balances in the Painlevé test.

References

[1] M. J. Ablowitz, P. A. Clarkson, *Solitons, Nonlinear Evolutions and Inverse Scattering*, Cambridge University Press, Cambridge, 1991.

[2] M. Ablowitz, A. Ramani, H. Segur, *A connection between nonlinear evolution equations and ordinary differential equations of P-type I, II*, J. Math. Phys., **21**(1980)715-721, 1006-1015

[3] J. Hu and M. Yan, *Singularity analysis for integrable systems by their mirrors*, Nonlinearity, **12**(1999)1531-1543

[4] J. Hu and M. Yan, *The mirror systems of integrable equations*, Studies in App. Math., **104**(2000)67-90

[5] J. Hu and M. Yan, *Symplectic structure of the Painlevé test*, J. Nonlinear Math. Phys., **8**(2001)145-148

[6] J. Hu and M. Yan, *Analytical aspects of the Painlevé test*, preprint, 2000

[7] J. Hu and M. Yan, *An elementary and direct proof of the Painlevé property for the Painlevé equations I, II and IV*, preprint 2001

[8] S. Kowalevski, *Zur Theorie der partiellen Differentialgleichungen*, J. Reine Angew. Math., **80**(1875)187-207

[9] S. Kowalevski, *Sur le probléme de la rotation d'un corps solide autour d'un point fixé*, Acta Math., **12**(1889)117-232

[10] P. Painlevé, *Mémoires sur les équations différentielles dont l'intégrale générale est uniforme*, Bull. Soc. Math. France, **28**(1900)201-261

[11] J. Weiss, M. Tabor, G. Carnevale, *The Painlevé property for partial differential equations*, J. Math. Phys., **24**(1983)522-526