Equivariant Periodicity for Abelian Group Actions

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Abstract

For a manifold M, the structure set $S(M, \text{rel } \partial)$ is the collection of manifolds homotopy equivalent to M relative to the boundary. Siebenmann [11] showed that in topological category, the structure set is 4-periodic: $S(M, \text{rel } \partial) \cong S(M \times D^4, \text{rel } \partial)$ up to a copy of $\mathbf{Z}$. The periodicity has been extended in [27] to topological manifolds with homotopically stratified group actions by odd order groups, with D^4 replaced by the unit ball of any 4-fold permutation representation. In this paper, we extend such equivariant periodicity to the case that the group is compact abelian, and D^4 is replaced by the unit ball of twice of any complex representation.

1 Introduction

One of the most fundamental phenomena in the homotopy classification theory of topological manifolds is Siebenmann's periodicity theorem [11]: There is a 1-1 correspondence between the manifolds homotopy equivalent (relative to the boundary) to a manifold M and the same thing for $M \times D^4$. (This is actually not entirely correct in the context of manifolds [17]. But the deviation is small, and the theorem as stated in [11] is true if one replaces manifolds by ANR-homology manifolds [3]). The object of this paper is to generalize this to manifolds with group actions.

An equivariant generalization is given in [25]: for arbitrary stratified spaces there is a "Siebenmann type periodicity" for crossing with D^4, and an equivariant theorem follows by consideration of the quotient. However, the most interesting and natural equivariant generalization involves consideration of DV, the unit disk of an orthogonal representation V, in place of D^4. As a matter of fact, Siebenmann periodicity is a cousin of Bott periodicity, which has such an equivariant generalization. For odd order groups a class of "periodicity representations" is given in [27]. Equivariant products are rather complicated from a purely stratified point of view, and the operation does not have a natural meaning for general stratified spaces, so that one hopes that deeper elements of the theory of group actions should follow from such periodicity theorems.

Indeed, the equivariant periodicity theorem seems to play a more useful role than the nonequivariant one. One reason for this is the following: The geometric topology of G-manifolds seems to be best analyzed in a category that only involves *isovariant* maps. These are maps which not only map fixed sets of subgroups to one another, but also sends complements of such sets to each other. This is a difficult notion to work with (constant maps are equivariant but not isovariant, for instance). Browder has shown that assuming a *large gap hypothesis*, equivariant homotopy equivalences are homotopic to isovariant ones (the gap hypothesis is, in any case, an important one in transformation groups). Using a periodicity theorem, one can cross with a suitably large representation (meaning with large enough gaps) to achieve the desired gap hypothesis, without losing information. Then one can do geometry and homotopy theory in a more congenial environment.

Successful applications of this idea (and, indeed, of the results of this paper) have already been executed: In [25] where these results are used in disproving the equivariant topological rigidity conjecture (for equivariantly aspherical manifolds), and in [5] they are applied to the problem of the variation of the homotopy type of the fixed point set of a group action within a given equivariant homotopy type (the replacement problem). Further applications of the ideas presented here will appear in [26] and [6] where decomposition theorems will be proven for equivariant surgery groups and structure sets, and to functoriality of equivariant surgery theory.

In this paper, we are mainly concerned with the actions by abelian groups. For actions of odd order groups see [8] [16] [27].

Denote by $S_G(M, \text{rel } \partial M)$ the space of G-*isovariant* homotopy structures of M relative to the boundary ∂M (the 0-th homotopy is the homeomorphism classes of G-manifolds isovariantly homotopy equivalent to M, which are already homeomorphic on the boundaries).

Theorem 1 *Let $V = \mathbf{C}^2$ be twice of the natural representation of S^1. Suppose that M is a homotopically stratified S^1-manifold with codimension ≥ 3 gap and nontrivial S^1-action. Then there is a periodicity equivalence*

$$S_{S^1}(M, \text{rel } \partial M) \simeq S_{S^1}(M \times DV, \text{rel } \partial(M \times DV)).$$

By virtually the same proof, we also see that the periodicity is "inductive".

Theorem 2 *Let $\kappa : G \to S^1$ be a character of a compact Lie group G. Let $V = \mathbf{C}^2$ be twice of the G-representation induced from κ. Suppose that M is a homotopically stratified G-manifold with codimension ≥ 3 gap and nontrivial G-action. Then there is a periodicity equivalence*

$$S_G(M, \text{rel } \partial M) \simeq S_G(M \times DV, \text{rel } \partial(M \times DV)).$$

Since any complex representation of a compact abelian Lie group G is a direct sum of characters, we have the following result by repeatedly applying the above Theorem.

Theorem 3 *Suppose that G is a compact abelian Lie group, W is a complex G-representation and $V = W \oplus W$. Suppose that M is a homotopically stratified G-manifold such that M has codimension ≥ 3 gap, and $M \times V$ and M have the same isotropy everywhere. Then there is periodicity equivalence*

$$S_G(M, \text{rel } \partial M) \simeq S_G(M \times DV, \text{rel } \partial(M \times DV)).$$

This isotropy condition was defined in [27]: Any point in M has an arbitrarily small neighborhood U such that the sets of isotropy groups of $U \times V$ and of U are identical. The condition essentially means that $M \times V$ and M have the same isovariant fixed point structures (or the same posets, in the terminology of [9]). In case every subgroup of G appears as an isotropy subgroup of V, the condition means that M has *strongly saturated orbit structure* as defined in [8].

With certain applications in mind, we would also like the equivariant periodicity to be natural.

Theorem 4 *The periodicity is compatible with the restriction to fixed points of subgroups and, provided the subgroup has finite index, the restriction to the action of subgroups.*

We expect the finite index condition to be unnecessary. However, the proof in that case seems to involve some delicate points.

We note that in general, one cannot much improve these results. Indeed the class of "periodicity representations" is precisely the representations that are twice a complex representation for the case of the torus group. However, an important conjectural extension of our result is suggested by the following (see [28] for further evidence).

Conjecture: Twice any complex representation of any compact Lie group is a periodicity representation.

The keys to proving periodicity theorems on structure sets are a surgery theory that has a suitable "local-global" form (see [25]) and an appropriate "periodicity theorem" for L-groups. Indeed the result of [27] follows from a core result on L-groups that is the same as the key result in [8]. Till this paper, no periodicity theorems were known for even order groups, even $\mathbf{Z}_2$, let alone for compact Lie groups. As explained in [8] (see proposition 3.7 on page 96), the difficulty encountered is that there does not seem to be an equivariant variant of $\mathbf{CP}^2$ to cross with for the even order case. The trouble is that one needs a manifold whose equivariant signature is the one dimensional trivial representation. In addition, one needs the fixed point set of every subgroup to be connected and simply connected. These do not seem to exist. A similar issue arises in the work of [13] on decomposition theorems for equivariant surgery groups.

In fact, in [16] equivariant transversality was shown to follow from a topological version of equivariant Bott periodicity (i.e., from the construction of a K-theoretic Thom classes for topological bundles). However, equivariant transversality fails for $\mathbf{Z}_2$, which might suggest that periodicity does as well (See [22] for an explanantion of how to prove equivariant Bott periodicity using the signature operator instead of the Dirac operator. That proof fails for $\mathbf{Z}_2$ exactly for the same computational reason that produces nonlinear similiarities for even order cyclic groups of order > 4). We avoid this difficulty by making use of the complex structure of the representations, so that our periodicities of topological structure sets are not topologically invariant! We hope to return to this issue in a future paper on Thom isomorphism for structure sets of equivariant "bundles", where such problems are much more serious. This defect is, in some ways, an advantage, in that in the equivariant case there are a number of distinct periodicities which puts a useful algebraic structure on structure sets (unequivariantly, there are only two, which differ by a sign). Again, this will be dealt with elsewhere.

The way we get around the lack of "periodicity G-manifolds" (which are supposed play a similar role equivariantly to that of $\mathbf{CP}^2$ in the classical periodicity) is to make use of certain G-spaces that are not manifolds (or even pseudomanifolds). The idea is to consider stratified spaces whose singularities are themselves boundaries of other stratified spaces with some special "π-π structure". This π-π structure ensures that the singularities are not "too serious" in a certain algebraic sense, and the stratified spaces can be used with success in manifold theory.

The advantage of using such spaces can be understood via consideration of the important work of [7]. To define a purely free manifold theoretic product from arbitrary

G-manifolds, one would want to cobord to a free manifold. The Conner-Floyd approach is to make the singular set into an appropriate boundary, and to insist that the "normal bundle data" bound as well. This bundle theory actually is dominant in the size of equivariant bordism theory. Our contribution is to show that it can be ignored for surgery theoretic purposes.

Philosophically, the reason one can do this goes back to Atiyah's analysis [1] of the lack of multiplicativity of the signature (which is the main contribution of manifold cobordism theory from the point of view of surgery theory; see [21]). The idea is to find something that bounds the boundary of a tubular neighborhood of the singular set and use this to replace the normal sphere (considered by Conner and Floyd) around the "coboundary" of the fixed sets. If a signature were multiplicative, and the singular set bounded even a simply connected manifold, the signature of the boundary of the tubular neighborhood of the singular set would vanish. This would be an important first step. However, this does not hold for mapping cylinders of Atiyah's bundles.

Still, Atiyah showed that the deviation from multiplicity has a characteristic class formula, so that if the singular manifold bounded a manifold with the same fundamental group as itself, this deviation term would vanish as well, and we would have the vanishing of the signature. Our exotic products provide a precise chain level construction (performed thanks to some magic in stratified surgery theory) that applies to more complex singularities and to more sophisticated invariants (some of finite order, for instance) than merely the signature. In fact, the result also includes the multiplicativity of higher signatures noticed by Lusztig [15] as well. We note that Lück and Ranicki [14] have also analyzed Atiyah's formula from a surgery theoretic point of view. Indeed, in the manifold case, their result is much more precise than what we accomplish, but we need the added generality of nonmanifold singular sets when we get to noncyclic groups. It is an interesting project to try to combine their formulae with our construction.

The most important problem posed by this work is how to make the "exotic product" idea yet more exotic, by allowing the singular set to bound in a more exotic (less geometric) fashion. Currently, that seems like the most likely route to general nonabelian results.

Our paper is organized as follows. In section two we introduce a particular useful "periodicity space". We will see how crossing with this space leads to the periodicity in the following sections. Section three gives a result for surgery obstruction groups. Section four gives the corresponding result for stable structure sets.

Section five destablizes our periodicity theorem. Unfortunately we have not found a way to axiomatize the proof in section five in a useful way, nor have we a direct approach to proving the periodicity theorem for (the unstable) structure sets in general. If the reader were only interested in the PL locally linear category, destablization would not be necessary, although the periodicity would be marred by (1) the usual Kirby-Siebenmann difficulties and (2) the kind of boundary conditions imposed by Nicas on Siebenmann's periodicity. The reason is that many of the G-manifolds produced by the theory, without a boundary condition, will only be locally simple homotopy linear, not actually locally linear.

Finally, in the last section we discuss the naturalities present under restriction to subgroup or to fixed point set.

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Notations and Conventions

In this paper, we work on *manifold stratified spaces* (or *stratified spaces* for short, at least in this paper) X: X has a filtration $\{X_\alpha\}$ of closed subspaces indexed by a partially ordered set such that $X_\alpha \subset X_\beta$ for $\alpha < \beta$ and the *strata* $X^\alpha = X_\alpha - X_{<\alpha}$ are topological manifolds. We will always assume that up to stratified homotopy, the neighborhood of lower strata in higher strata is the mapping cylinder of a fibration map, i.e., X is *homotopically stratified spaces* in the sense of Quinn [19]. This is the (weaker) homotopical version of *geometrically stratified spaces of* Browder and Quinn [2]. Moreover, we always assume the maps between such stratified spaces to be stratified and *homotopically transverse*, meaning that the induced map on the fibrations are fibrewise homotopy equivalences.

A stratified space X with a stratum-preserving G-action is a *G-stratified space*. The quotient X/G has an induced stratification doubly indexed by the isotropy subgroups of X and the indices of X. This generalizes the induced stratification (indexed by the isotropy subgroups only) on the quotient of a nonstratified X, which may also be considered as having a single stratum. In this paper, we will always assume that the quotient stratified space X/G is homotopically stratified. Under the assumption, we also say that the group action is homotopically stratified.

To simplify notation, we will pretend $\partial M = \emptyset$ throughout this paper. In addition, we will write $M \times (A, \text{rel } B)$ for $(M \times A, \text{rel } \partial M \times A \cup M \times B)$. In particular, by $M \times (DV, \text{rel } SV)$ we mean $(M \times DV, \text{rel } \partial(M \times DV))$.

An equivariant map $f : X \to Y$ is called an *equivariant* π_0*-equivalence* if $f^H : X^H \to Y^H$ is a one-to-one correspondence of connected components for each subgroup H. f is called an *equivariant* π_1*-equivalence* if, in addition to being a one-to-one correspondence of connected components, f^H induces an isomorphism on the fundamental groups of each component.

The notions of equivariant π_0- and π_1-equivalences have isovariant analogues. Instead of considering the restriction of f on connected components of X^H, we will only consider the restrictions of f on the *isovariant components*: connected components of $X_H = X^H - X^{>H}$. If an equivariant map $X \to Y$ induces a one-to-one correspondence on the collections of isovariant components, then the map is called an *isovariant* π_0*-equivalence*. If, in addition, the map also induces an isomorphism on the isovariant components, then it is called an *isovariant* π_1*-equivalence*.

A G-manifold M has *codimension* ≥ 3 *gap* if for any equivariant components $X^H_\alpha \subset X^K_\beta$, we have either $X^H_\alpha = X^K_\beta$ or $\dim X^H_\alpha + 3 \leq \dim X^K_\beta$. If f is an isovariant map between G-manifolds with codimension ≥ 3 gaps, then f is an equivariant π_0-equivalence (π_1-equivalence) if and only if it is an isovariant π_0-equivalence (π_1-equivalence).

We will need generalization of the notion of π_0- and π_1-equivalences to stratified G-spaces. By this we mean the (equivariant or isovariant) π_0- and π_1-equivalences for the restriction of the stratified G-map on each strata.

2 Periodicity Spaces

Products of G-stratified spaces are G-stratified spaces. In this paper, we will find it necessary to take the product of a G-manifold M with certain periodicity space P, which is a geometrically G-stratified space instead of a G-manifold. The manifold M is trivially G-stratified, having only one stratum. Thus the G-stratification on $M \times P$ is given by the G-stratification of P, and the quotient $(M \times P)/G$ has an induced stratification. Since P is geometrically G-stratified, $(M \times P)/G$ will be geometrically (homotopically) stratified if M/G is geometrically (homotopically) stratified.

To construct P, we start with the complex representation $V = \mathbf{C}^2$ where S^1 acts by complex scalar multiplication. We add a trivial representation $\mathbf{C}$ to V and obtain the induced S^1-action on the complex projective space $\mathbf{CP}^2 = \mathbf{CP}(V \oplus \mathbf{C})$. Under the obvious identification (the boundary SV of DV maps onto $\mathbf{CP}(V) = S^2$ via Hopf projection)

$$\mathbf{CP^2} = \mathbf{CP}(V \oplus \mathbf{C}) = DV \cup S^2,$$

the S^1-action is semifree with fixed points

$$(\mathbf{CP}^2)^{S^1} = 0 \coprod S^2.$$

We note that $\mathbf{CP}(V \oplus \mathbf{C})$ is not a periodicity manifold in the sense of [8] or [27] because the fixed point set is not connected.

Since the expected periodicity representation comes from the neighborhood DV of the origin 0, we need to eliminate the contribution from the other component S^2. This is achieved by expanding $\mathbf{CP}^2$ by gluing the obvious nullcobordism D^3 of S^2:

$$P = \mathbf{CP}^2 \cup_{S^2} D^3.$$

This is a manifold geometrically stratified space. By letting S^1 act trivially on D^3, P becomes an S^1-stratified space.

3 Periodicity of S^1-Surgery Obstruction Groups

The periodicity will come from the following operations

$$M \overset{\times P}{\Longrightarrow} M \times P \overset{\text{incl}}{\Longleftarrow} M \times (DV, \text{rel } SV). \tag{1}$$

The operations will be applied to the stable surgery obstructions $L_{S^1}^{-\infty}$ and the Tate cohomology $\hat{H}(\mathbf{Z}_2; Wh^{top,\leq 0})$ of the topological Whitehead torsion. Both are functors over S^1-equivariant homotopically stratified spaces. The operations (1) can also be applied to unstable surgery obstructions L_{S^1}, and the subsequent periodicity results remain true. However, this fact is not directly needed in this paper.

An element of $L_{S^1}^{-\infty}(M)$ is represented by a stable S^1-surgery problem with a reference S^1-map to M. The operation $\times P$ means crossing the problem by the space P, and crossing the reference map by id_P. An element in $L_{S^1}^{-\infty}(M \times (DV, \text{rel } SV))$ is represented by a stable S^1-surgery problem with a reference S^1-map to $M \times DV$. The inclusion operation does not change the surgery problem itself, and only takes a new viewpoint on the reference map. Specifically, we view the reference map as mapping into $M \times (DV - SV)$ part of $M \times P$ (this is a stratified map), so that over $M \times (D^3 \supset S^2)$ there is only the empty problem. The simple geometric description of the operations (2) readily implies that commutativity of all the diagrams in the subsequent proofs.

We first consider the case of free actions.

Lemma 5 *Suppose S^1 acts freely on M. Then* (1) *induces equivalences of surgery obstructions*

$$L_{S^1}^{-\infty}(M) \simeq L_{S^1}^{-\infty}(M \times P) \simeq L_{S^1}^{-\infty}(M \times (DV, \text{rel } SV)).$$

Proof: Consider the diagram

$$\begin{array}{ccc} L_{S^1}^{-\infty}(M \times \mathbf{CP}^2) & \stackrel{\text{incl}}{\longleftarrow} & L_{S^1}^{-\infty}(M \times (DV, \text{rel } SV)) \\ \times \mathbf{CP}^2 \uparrow & \stackrel{\phi}{\nwarrow} & \downarrow \text{incl} \\ L_{S^1}^{-\infty}(M) & \stackrel{\times P}{\longrightarrow} & L_{S^1}^{-\infty}(M \times P) \end{array} \tag{2}$$

where the map ϕ first restricts to the closed stratum $\mathbf{CP}^2 \subset P$ and then forgets the stratification structure $\mathbf{CP}^2 \supset S^2$. The two triangles are commutative by the geometric meaning of the operations.

In [14], Lück and Ranicki showed that $\times\mathbf{CP}^2$ depends only on the S^1-equivariant signature of $\mathbf{CP}^2$. Since S^1 acts homotopically trivially on $\mathbf{CP}^2$, the equivariant signature is in fact the nonequivariant one, which is 1: $\mathbf{Z} \otimes \mathbf{Z} \to \mathbf{Z}$. As a result, the map $\times\mathbf{CP}^2$ is an equivalence.

The horizontal inclusion induces an isomorphism $\pi_1(M \times DV) \cong \pi_1(M \times \mathbf{CP}^2)$. Since S^1 acts freely on the products, we have $\pi_1((M\times DV)/S^1) \cong \pi_1((M\times\mathbf{CP}^2)/S^1)$. Therefore the horizontal inclusion induces an equivalence on the surgery obstructions.

The vertical inclusion fits into a fibration

$$L^{-\infty}_{S^1}(M \times (DV, \text{rel } SV)) \overset{\text{incl}}{\to} L^{-\infty}_{S^1}(M \times P) \overset{\text{rest}}{\to} L^{-\infty}_{S^1}(M \times (D^3 \supset S^2)), \tag{3}$$

where by writing $D^3 \supset S^2$ we mean the stratification structure in D^3. By $\pi - \pi$ theorem, $L^{-\infty}_{S^1}(M \times (D^3 \supset S^2))$ is trivial. Therefore the inclusion is an equivalence.

Combining the above equivalences we proved the equivalence between surgery obstructions.

□

Now we move on to the general case. A small gap condition is needed.

Lemma 6 *Suppose the nonfree part of S^1-action on M has codimension ≥ 3. Then* (1) *induces equivalences*

$$L^{-\infty}_{S^1}(M) \simeq L^{-\infty}_{S^1}(M \times P) \simeq L^{-\infty}_{S^1}(M \times (DV, \text{rel } SV)).$$

Proof: Since S^1 acts semifreely on P, for any $\{1\} \neq H \subset S^1$ we have

$$(M \times P)^H = M^H \times P^{S^1} = M^H \times (0 \coprod D^3).$$

Denote by $M_s = \cup_{g \in S^1} M^g$ the part of M on which S^1 acts nonfreely. Then M_s is a manifold S^1-stratified space.

As in the proof of the previous lemma, the fibration (3) and $\pi - \pi$ theorem implies the inclusion is an equivalence.

To prove that $\times P$ is an equivalence, we compare two fibrations:

$$\begin{array}{ccccc} L^{-\infty}_{S^1}(M - M_s) & \overset{\text{incl}}{\to} & L^{-\infty}_{S^1}(M) & \overset{\text{rest}}{\to} & L^{-\infty}_{S^1}(M_s) \\ \downarrow & & \downarrow \times P & & \downarrow \times P^{S^1} \\ L^{-\infty}_{S^1}(M \times P - M_s \times P^{S^1}) & \overset{\text{incl}}{\to} & L^{-\infty}_{S^1}(M \times P) & \overset{\text{rest}}{\to} & L^{-\infty}_{S^1}(M_s \times P^{S^1}) \end{array} \tag{4}$$

By $\pi - \pi$ theorem, we have

$$\begin{aligned} & L^{-\infty}_{S^1}(M_s \times P^{S^1}) \\ = \ & L^{-\infty}_{S^1}(M_s \times 0) \times L^{-\infty}_{S^1}(M_s \times (D^3 \supset S^2)) \\ \simeq \ & L^{-\infty}_{S^1}(M_s \times 0). \end{aligned}$$

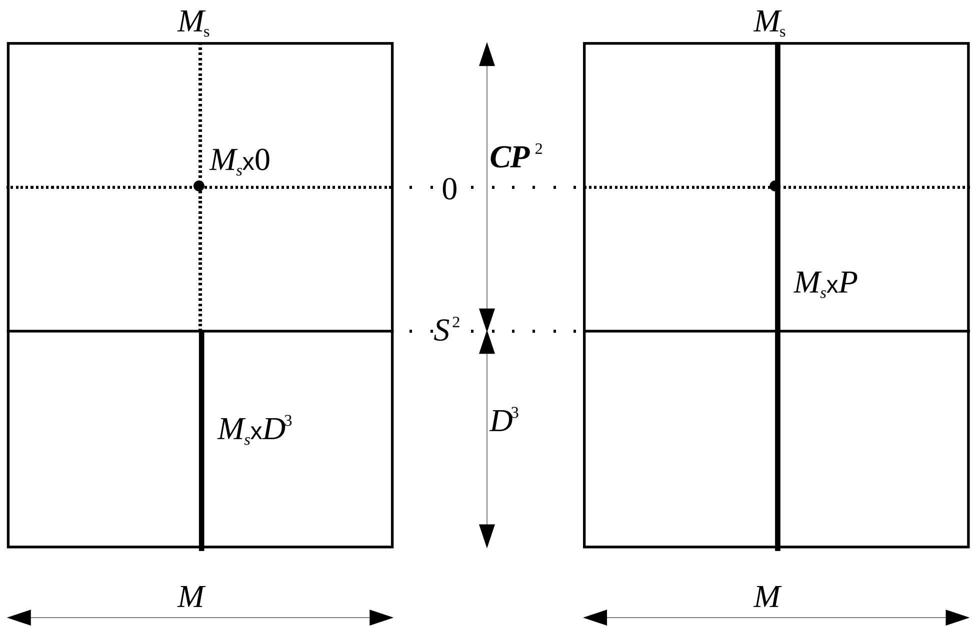
Thus the map on the right of (4) is an equivalence. Therefore in order to show the middle of (4) is an equivalence, it suffices to show that the left is an equivalence. We note that left side is the composition

$$L^{-\infty}_{S^1}(M - M_s) \stackrel{\times P}{\to} L^{-\infty}_{S^1}((M - M_s) \times P) \stackrel{\text{incl}}{\to} L^{-\infty}_{S^1}(M \times P - M_s \times P^{S^1}).$$

The map $\times P$ is an equivalence by the lemma 5. The inclusion can be considered as a gluing

$$M \times P - M_s \times P^{S^1} = [(M - M_s) \times P] \cup_{(M-M_s)\times(P-P^{S^1})} [M \times (P - P^{S^1})].$$

We claim that the gluing neither introduces new S^1-strata, nor changes the connectivity and the fundamental groups of isovariant components inside each G-stratum. This would imply that the inclusion is an isovariant π_1-equivalence, so that it induces an equivalence on the surgery obstructions.



First, both $(M - M_s) \times (P - P^{S^1})$ and $M \times (P - P^{S^1})$ are parts of an S^1-stratum $M\times(DV-SV)$ of $M\times P$. Hence the gluing is merely an extension of the existing S^1-strata, so that no new S^1-strata are introduced. Second, S^1 acts freely on the extended part $M\times(P-P^{S^1})$. Therefore the extension happens only in the free part of $M\times(DV-SV)$, so that no new fixed points are introduced. Finally the assumption that M_s has codimension

≥ 3 implies that the inclusion $(M - M_s) \times (P - P^{S^1}) \to M \times (P - P^{S^1})$ is an isovariant π_1-equivalence. By Van-Kampen's theorem, this implies that the inclusion $(M - M_s) \times P \to M \times P - M_s \times P^{S^1}$ is an S^1-stratified isovariant π_1-equivalence. In particular, the inclusion induces an equivalence on stable surgery obstruction. This completes the proof of the lemma.

□

4 Periodicity of Stable S^1-Surgery Theory

In this section we prove the stable version of theorem (1). We also include a naturality property of the stable periodicity. The property will be needed in deriving unstable periodicity later on.

The operations (1) may be applied to $S_{S^1}^{-\infty}$. Omitting the $\mathbf{R}^i$-factor, an element of $S_{S^1}^{-\infty}(M)$ is represented by a stable isovariant homotopy equivalence $N \to M$. "$\times P$" takes the element to $N \times P \to M \times P$. An element of $S_{S^1}^{-\infty}(M \times (DV, \text{rel } SV))$ is represented by a homotopy equivalence $(W, \partial W) \to M \times (DV, SV)$ such that the restriction $\partial W \to M \times SV$ is a homeomorphism. The inclusion operation simply glues $M \times (D^3 \supset S^2)$ to the homotopy equivalence by making use of the homeomorphism on the boundary.

Unlike surgery obstructions, the operations (1) do not induce equivalences on $S_{S^1}^{-\infty}$. However, our stable periodicity will be compatible with these operations. We summerise the stable version of theorem 1 and the compatibility in the following theorem.

Theorem 7 *Let $V = \mathbf{C}^2$ be twice of the natural representation of S^1. Suppose that M is a homotopically stratified S^1-manifold such that the nonfree part has codimension ≥ 3. Then there is periodicity equivalence and commutative diagram*

$$\begin{array}{ccc} S_{S^1}^{-\infty}(M) & \overset{\text{per}}{\simeq} & S_{S^1}^{-\infty}(M \times (DV, \text{rel } SV)) \\ {\scriptstyle \times P} \searrow & & \swarrow {\scriptstyle \text{incl}} \\ & S_{S^1}^{-\infty}(M \times P) & \end{array}$$

Proof: $S_G^{-\infty}$ may be computed by the following fibration (see Stable Classification Theorem on page 134 of [25]):

$$S_G^{-\infty}(M) \to H(M/G; L_G^{-\infty}(locM)) \to L_G^{-\infty}(M). \tag{5}$$

Lemma 6 says that (1) induces natural equivalences of functors:

$$L^{-\infty}_{S^1}(?) \overset{\times P}{\simeq} L^{-\infty}_{S^1}(? \times P) \overset{\text{incl}}{\simeq} L^{-\infty}_{S^1}(? \times (DV, \text{rel } SV)).$$

Applying homology, we obtain homotopy equivalent assembly maps:

$$\begin{array}{ccc} H(M/S^1; L^{-\infty}_{S^1}(locM)) & \to & L^{-\infty}_{S^1}(M) \\ \simeq\downarrow \times P & & \simeq\downarrow \times P \\ H(M/S^1; L^{-\infty}_{S^1}((locM) \times P)) & \to & L^{-\infty}_{S^1}(M \times P) \\ \simeq\uparrow \text{incl} & & \simeq\uparrow \text{incl} \\ H(M/S^1; L^{-\infty}_{S^1}((locM) \times (DV, \text{rel } SV))) & \to & L^{-\infty}_{S^1}(M \times (DV, \text{rel } SV)) \end{array} \tag{6}$$

By the stable surgery fibration (5), the homotopy fibre of the top map is $S^{-\infty}_{S^1}(M)$. If we can identify the bottom map with the assembly map of $L^{-\infty}_{S^1}$ over $M \times DV$, then the homotopy fibre of the bottom map is $S^{-\infty}_{S^1}(M \times (DV, \text{rel } SV))$ and stable periodicity follows.

By applying the "Fubini equivalence" (proven exactly the same way as the Leray spectral sequence of a map in a generalized homology theory) associated to the stratified system of fibrations $DV/G_x \to (M \times DV)/S^1 \to M/S^1$ (G_x =isotropy group of $x \in M$; this stratified system is entirely analogous to the stratification of the quotient of a smooth G-vector bundle, see [24] for example), we may compare the assembly map of the functor $L^{-\infty}_{S^1}(? \times (DV, \text{rel } SV))$ over M/S^1 and the assembly map of the functor $L^{-\infty}_{S^1}(?)$ over $(M \times DV)/S^1$:

$$\begin{array}{ccc} H(M \times DV/S^1; L^{-\infty}_{S^1}(loc(M \times DV))) & & \\ \simeq\downarrow \text{Fubini} & \searrow & \\ H(M/S^1; H(DV/G_x; L^{-\infty}_{S^1}(loc(M \times DV)))) & & L^{-\infty}_{S^1}(M \times (DV, \text{rel } SV)) \\ \downarrow \alpha & \nearrow & \\ H(M/S^1; L^{-\infty}_{S^1}((locM) \times (DV, \text{rel } SV))) & & \end{array} \tag{7}$$

We note that $loc(M \times DV)$ means the local S^1-structure of the product space $M \times DV$, while $(locM) \times DV$ means the product of the local G-structure of M with the whole G-space DV. Moreover, the map α is the "partial assembly map" obtained in the following way: The assembly map of the functor $L^{-\infty}_{S^1}((locM) \times ?)$ over DV

$$\alpha_0 : H(DV/G_x; L^{-\infty}_{S^1}(loc(M \times DV))) \to L^{-\infty}_{S^1}((locM) \times DV)$$

may be considered as a natural transformation between functors of the variable $locM$. Then α is obtained by applying the homology functor to α_0. The naturality of the assembly map with respect to the Fubini equivalence (see Section 8 of [19], and [27]) shows that the diagram (7) is commutative.

The proof of stable periodicity is thus reduced to showing that α is a homotopy equivalence. This is a consequence of α_0 being a homotopy equivalence. Note that DV/G_x is the cone of the stratified space SV/G_x, with the cone point as an additional stratum. The assembly map over such space is always a homotopy equivalence (see lemma 3.21 on page 1038 of [27]). This proves that α_0 is a homotopy equivalence.

To show the diagram in the lemma is commutative, we consider the following diagram

$$\begin{array}{ccc} H(M/S^1; L_{S^1}^{-\infty}(locM)) & \to & L_{S^1}^{-\infty}(M) \\ \downarrow \times P & & \simeq\downarrow \times P \\ H((M\times P)/S^1; L_{S^1}^{-\infty}(loc(M\times P)) & \to & L_{S^1}^{-\infty}(M\times P) \\ \uparrow \text{incl} & & \simeq\uparrow \text{incl} \\ H((M\times DV)/S^1; L_{S^1}^{-\infty}(loc(M\times DV)) & \to & L_{S^1}^{-\infty}(M\times (DV, \text{rel } SV)) \end{array} \tag{8}$$

The inclusion map on the left side is the usual map in homology theory, The map $\times P$ on the left side has the following geometrical interpretation: As pointed out on page 134 and explained in section 8.3 of [25], the homology $H(M/S^1; L_{S^1}^{-\infty}(locM))$ may be interpreted as the normal invariants (=isovariant surgery problems) over M. By taking the product of a normal invariant with the stratified space P, we have a normal invariant over the stratified space $M\times P$, which belongs to the homology $H((M\times P)/S^1; L_{S^1}^{-\infty}(loc(M\times P))$. A purely algebraic interpretation would involve a canonical controlled stratified visible Sullivan class for P, as an element in the homology $H_*(P/G; VL_{S^1}(locP))$. But this is not needed here.

The fibres of the assembly maps in (8) are the stable structures $S_{S^1}^{-\infty}(M)$, $S_{S^1}^{-\infty}(M\times P)$, $S_{S^1}^{-\infty}(M\times (DV, \text{rel } SV))$, and the induced maps on the stable structures are $\times P$ and inclusion.

The left side of (8) has a natural map to the left side of (6). The map over M is the identity. The map over $M\times DV$ is the Fubini equivalence followed by partial assembly (left side of (6)). The map over $M\times P$ is the similar Fubini equivalence followed by partial assembly (except the partial assembly is over P/G_x, instead of over the cone DV/G_x. So the partial assembly is not a homotopy equivalence). The natural map from (8) to (6) induces natural maps on the homotopy fibres of the assemblies. The induced diagram is the commutative diagram in the theorem.

□

We end this section by remarking that theorem 7 also applies to other abelian groups. In fact, the proofs in the last three sections are still valid if we replace S^1 by G. Perhaps the only thing worth mentioning is that $\mathbf{CP}^2$ still represents an invertible element of the Euler ring of G after localizing at 2. Theorem 3 (the stable version) is obtained by writing W as a direct sum of several characters (one dimensional complex representations) and then repeatedly applying the theorem 2 (the stable version). The same isotropy everywhere condition implies that at each stage, the conditions of the theorem 2 are satisfied.

5 Destablization

The proof of destablization overall follows from the same strategies employed in the analysis of the stable structure set. However, the details seems to be irreducibly more complicated.

The stable and unstable structures are related by generalized Rothenberg fibration (see Destablization Theorem on page 135 of [25])

$$S_G(M) \to S_G^{-\infty}(M) \to \hat{H}(\mathbf{Z}_2; Wh_G^{top,\leq 0}(M)). \tag{9}$$

The same operations (1) used on $S_{S^1}^{-\infty}$ can be compatibly defined on $Wh_G^{top,\leq 0}$, which by theorem 7, are compatible with the periodicity equivalence on $S_{S^1}^{-\infty}$. Therefore the periodicity equivalence is compatible with the operations (1) on $\hat{H}(\mathbf{Z}_2; Wh_{S^1}^{top,\leq 0})$. As a result, if the operations (1) induce equivalences on $\hat{H}(\mathbf{Z}_2; Wh_{S^1}^{top,\leq 0})$, then by (9) we obtain the periodicity equivalence on the unstable structure S_{S^1}.

One can almost repeat the proof for the periodicity on the surgery obstructions, as was done in [28]. However, some technical difficulties (taking Tate cohomology does not commute with truncating involutive spectrum) add more complications to the argument. In this paper, we use a more direct approach.

Our proof will be presented in terms of the isovariant topological Whitehead group $Wh_G^{Top} = \pi_0 Wh_G^{top,\le 0}$ (this corresponds to, and for stratified case, generalizes $Wh_G^{Top,Iso}$ of [23]). Because $\pi_{-k} Wh_G^{top,\le 0} = Wh^{top}_{G,\mathbf{R}^k-\text{bounded}}(\mathbf{R}^k \times X)$, the bounded version of the proof will also show that (1) induce equivalences on all $\hat{H}(\mathbf{Z}_2; \pi_{-k} Wh_G^{top,\le 0})$, $k \ge 0$. By decomposing the Rothenberg fibration (9) into many fibrations

$$S_G^{-k}(M) \to S_G^{-k-1}(M) \to \hat{H}(\mathbf{Z}_2; \pi_{-k} Wh_G^{top,\le 0}(M)),$$

and by making use of the isomorphism $S_G^{-\infty}(M) \cong \lim_{k\to\infty} S_G^{-k}(M)$, we inductively deduce the periodicity on S_G from the periodicity on $S_G^{-\infty}$.

For a G-manifold M with codimension ≥ 3 gap, we may describe the group $Wh_G^{Top}(M)$ as the homeomorphism classes of (equivariant or isovariant, which are the same in the presence of codimension ≥ 3 gap) G-h-cobordisms over M. The upside down operation describes the involution on $Wh_G^{Top}(M)$. This description (including that of the involution) holds as well for manifold G-homotopically stratified spaces. In particular, this enables us to define the maps such as $Wh_{S^1}^{Top}(M \times (DV, \text{rel } SV)) \overset{\text{incl}}{\to} Wh_{S^1}^{Top}(M \times (\mathbf{CP}^2, \text{rel } S^2)) \overset{\text{incl}}{\to} Wh_{S^1}^{Top}(M \times P) \overset{\times P}{\leftarrow} Wh_{S^1}^{Top}(M)$ in the most natural way. These maps are clearly compatible with the operations (1) on $S_{S^1}^{\infty}$.

We will need the following property of Wh_G^{Top}: Suppose X is a homotopically stratified space and $Y \subset X$ is a closed union of strata of X. Then there is a natural exact sequence

$$0 \to Wh_G^{Top}(X, \text{rel } Y) \overset{\text{incl}}{\to} Wh_G^{Top}(X) \overset{\text{rest}}{\to} Wh_G^{Top}(Y) \to 0. \tag{10}$$

Moreover, in case X is a manifold stratified space, the inclusion and restriction maps preserve the involutions.

Suppose M is a homotopically stratified G-manifold with codimension ≥ 3 gap. Then it was shown in [23] that $Wh_G^{Top}(M)$ may be identified with a subgroup $Wh_{G,\rho}^{Top,Equi}(M)$ of the equivariant topological Whitehead group $Wh_G^{Top,Equi}(M)$ (in [23], these are denoted as $Wh_G^{Top,\rho}(M) \subset Wh_G^{Top}(M)$). In fact, for any locally compact G-ANR X, Steinberger defined $Wh_G^{Top,Equi}(X)$ as the equivalence classes of G-ANR strong deformation retracts (Y, X), where the equivalence relation can either be given by G-CE maps or by stable G-homeomorphisms after crossing with the equivariant Hilbert cube. As a consequence of this description, $Wh_G^{Top,Equi}(X)$ is an equivariant homotopy functor ($f : X \to X'$ takes (Y, X) to $(Y \cup_X X', X')$). Moreover, crossing with any G-ANR Z gives rise to a homomorphism $\times Z : Wh_G^{Top,Equi}(X) \to Wh_G^{Top,Equi}(X \times Z)$, which can be further projected down to $Wh_G^{Top,Equi}(X)$ by the G-equivariant map $X \times Z \to X$.

The subgroup $Wh_{G,\rho}^{Top,Equi}(X)$ consists of elements of $Wh_G^{Top,Equi}(X)$ represented by G-ANR strong deformation retracts (Y, X) such that the inclusion $X \to Y$ is an isovariant π_0-equivalence. In particular, if a certain operation does not change this property, then the operation induces a homomorphism on $Wh_{G,\rho}^{Top,Equi}$. Specifically, this observation will be applied to the operations of products with $\mathbf{CP}^2$ and DV, and the operations induced by the maps $M \times \mathbf{CP}^2 \to M$, $M \times DV \to M$, and $M = M \times 0 \to M \times DV$. If M has codimension ≥ 3 gap, then these operations induce maps on $Wh_G^{Top,Equi}$ and restrict to maps on $Wh_{G,\rho}^{Top,Equi}$.

The next lemma reduces the proof that (1) induces equivalences on $\hat{H}(\mathbf{Z}_2; Wh_{S^1}^{Top})$ to an algebraic problem.

Lemma 8 *Suppose M is a homotopically stratified S^1-manifold with codimension ≥ 3 gap. Let $A = Wh_{S^1}^{Top}(M)$ and $*$ be the usual involution on A. Then after localizing at 2, we have*

1. $Wh_{S^1}^{Top}(M \times P) \cong A \oplus A \oplus A$, *with involution given by*

$$(\alpha, \beta, \gamma)^* = (-\alpha^* + \beta^*, \beta^*, \gamma' + \lambda(\beta)),$$

   where $'$ is another (possibly different from $*$) involution on A, and $\lambda : A \to A$ is a homomorphism satisfying $\lambda^2 = 0$;

2. $Wh_{S^1}^{Top}(M \times (DV, \mathrm{rel}\ SV)) \cong A$, *with the inclusion* $Wh_{S^1}^{Top}(M \times (DV, \mathrm{rel}\ SV)) \to Wh_{S^1}^{Top}(M \times P)$ *given by* $a \to (0, 0, a)$;

3. $\times P : Wh_{S^1}^{Top}(M) \to Wh_{S^1}^{Top}(M \times P)$ *is given by* $a \to (a, 2a, a)$.

Proof: First we claim that the inclusion induces an isomorphism

$$Wh_{S^1}^{Top}(M \times (DV, \mathrm{rel}\ SV)) \cong Wh_{S^1}^{Top}(M \times (\mathbf{CP}^2, \mathrm{rel}\ S^2)). \tag{11}$$

This is because the difference between the two topological Whitehead torsions is the possible "leaking" along $M \times S^2$; that is, we have exact sequence

$$H_0((M \times S^2)/S^1; Wh^{PL}(\text{holink})) \to Wh^{Top}_{S^1}(M \times (DV, \text{rel } SV)) \to$$

$$\to Wh^{Top}_{S^1}(M \times (\mathbf{CP}^2, \text{rel } S^2)) \to H_{-1}((M \times S^2)/S^1; Wh^{PL}(\text{holink})).$$

We note that the link of $(M \times S^2)/S^1$ in $(M \times \mathbf{CP}^2)/S^1$ is S^1/G_x, which is either a circle or a point. The fundamental group of the link is then $\mathbf{Z}$ or trivial. In either case, the piecewise linear K-theory $Wh^{PL}(\text{holink})$ is trivial at dimension ≤ 1. Therefore the homologies in the exact sequence vanish, and the inclusion is an equivalence.

By (10), we have the following natural involutive short exact sequence

$$0 \to Wh^{Top}_{S^1}(M \times (\mathbf{CP}^2, \text{rel } S^2)) \overset{\text{incl}}{\to} Wh^{Top}_{S^1}(M \times P) \overset{\text{rest}}{\to} Wh^{Top}_{S^1}(M \times (D^3 \supset S^2)) \to 0$$

The inclusion $Wh^{Top}_{S^1}(M \times (DV, \text{rel } SV)) \to Wh^{Top}_{S^1}(M \times P)$ clearly factors through $Wh^{Top}_{S^1}(M \times (\mathbf{CP}^2, \text{rel } S^2))$. By making use of the isomorphism (11), we see that the top row in the following diagram is exact.

$$\begin{array}{rcccl}
0 \to Wh^{Top}_{S^1}(M \times (DV, \text{rel } SV)) & \overset{\text{incl}}{\to} & Wh^{Top}_{S^1}(M \times P) & \overset{\text{rest}}{\to} & Wh^{Top}_{S^1}(M \times (D^3 \supset S^2)) \to 0 \\
\| & \overset{\text{incl}}{\searrow} & \downarrow \text{rest} & \overset{\times P}{\nwarrow} & \uparrow \times (D^3 \supset S^2) \\
Wh^{Top,Equi}_{S^1,\rho}(M \times (DV, \text{rel } SV)) & & Wh^{Top}_{S^1}(M \times \mathbf{CP}^2) & \overset{\times \mathbf{CP}^2}{\leftarrow} & Wh^{Top}_{S^1}(M) \\
\text{proj} \downarrow & \overset{\text{incl}}{\searrow} & \| & & \| \\
Wh^{Top,Equi}_{S^1,\rho}(M) & \overset{\text{proj}}{\leftarrow} & Wh^{Top,Equi}_{S^1,\rho}(M \times \mathbf{CP}^2) & \overset{\times \mathbf{CP}^2}{\leftarrow} & Wh^{Top,Equi}_{S^1,\rho}(M)
\end{array} \tag{12}$$

In the diagram, the equalities $Wh^{Top}_{S^1} = Wh^{Top,Equi}_{S^1,\rho}$ are applied to M, $M \times (DV, \text{rel } SV)$, and $M \times \mathbf{CP}^2$, which are all S^1-manifolds with codimension ≥ 3 gaps. The commutativity of the diagram follows from the geometric meaning of the maps.

The projection $M \times DV \to M$ and the inclusion $M = M \times 0 \to M \times DV$ are equivariant homotopy inverse to each other. Therefore they induce an isomorphism between $Wh^{Top,Equi}_{S^1}(M \times (DV, \text{rel } SV))$ and $Wh^{Top,Equi}_{S^1}(M)$. Since M has codimension ≥ 3 gap, the two maps restrict to an isomorphism between $Wh^{Top,Equi}_{S^1,\rho}$. Consequently, the vertical projection in (12) is an isomorphism, and the composition

$$Wh^{Top}_{S^1}(M \times P) \overset{\text{rest}}{\to} Wh^{Top}_{S^1}(M \times \mathbf{CP}^2) = Wh^{Top,Equi}_{S^1,\rho}(M \times \mathbf{CP}^2) \overset{\text{proj}}{\to} Wh^{Top,Equi}_{S^1,\rho}(M)$$

$$\overset{\text{proj}}{\cong} Wh_{S^1,\rho}^{Top,Equi}(M \times (DV, \text{rel } SV)) = Wh_{S^1}^{Top}(M \times (DV, \text{rel } SV))$$

is a splitting to the inclusion

$$Wh_{S^1}^{Top}(M \times (DV, \text{rel } SV)) \to Wh_{S^1}^{Top}(M \times P).$$

Thus the splitting induces a decomposition

$$Wh_{S^1}^{Top}(M \times P) \cong Wh_{S^1}^{Top}(M \times (D^3 \supset S^2)) \oplus Wh_{S^1}^{Top}(M \times (DV, \text{rel } SV)), \tag{13}$$

such that the projection to the first summand is involutive. Note that we are asserting nothing about the commutation of the second projection with the involution.

By making use of the collar of $M \times S^2$ in $M \times D^3$, we have a decomposition

$$Wh_{S^1}^{Top}(M \times (D^3 \supset S^2)) = Wh_{S^1}^{Top}(M \times (D^3, \text{rel } S^2)) \oplus Wh_{S^1}^{Top}(M \times S^2). \tag{14}$$

The situation (especially the involution) is then similar to the Whitehead torsion of a manifold with boundary. As above, since $M \times D^3$ and $M \times S^2$ have codimension ≥ 3 gaps, their topological Whitehead torsion groups may be identified with $Wh_{S^1,\rho}^{Top,Equi}$. Since the projections $M \times S^2 \to M$ and $M \times D^3 \to M$ are isovariant π_1-equivalences, the projections induce isomorphisms of both summands with $Wh_{S^1,\rho}^{Top,Equi}(M)$. By identifying $Wh_{S^1,\rho}^{Top,Equi}(M)$ with $Wh_{S^1}^{Top}(M)$, (14) then becomes

$$Wh_{S^1}^{Top}(M \times (D^3 \supset S^2)) \cong Wh_{S^1}^{Top}(M) \oplus Wh_{S^1}^{Top}(M). \tag{15}$$

The isomorphism (the left of (12))

$$Wh_{S^1}^{Top}(M \times (DV, \text{rel } SV)) = Wh_{S^1,\rho}^{Top,Equi}(M \times (DV, \text{rel } SV)) \overset{\text{proj}}{\cong} Wh_{S^1,\rho}^{Top,Equi}(M) = Wh_{S^1}^{Top}(M),$$

may be combined with (13) and (15) to give rise to a decomposition

$$Wh_{S^1}^{Top}(M \times P) \cong Wh_{S^1}^{Top}(M) \oplus Wh_{S^1}^{Top}(M) \oplus Wh_{S^1}^{Top}(M).$$

However, this is not what we want, because the map $\times P : Wh_{S^1}^{Top}(M) \to Wh_{S^1}^{Top}(M \times P)$ will not become $a \to (a, 2a, a)$ under such a decomposition.

What we really want is to show that the composition $\mathrm{proj}\circ(\times\mathbf{CP}^2)$ at the bottom of (12) is an isomorphism after localizing at 2. As a result, we have an isomorphism

$$\begin{aligned} & Wh^{Top}_{S^1}(M\times(DV,\mathrm{rel}\ SV))_{(2)} = Wh^{Top,Equi}_{S^1,\rho}(M\times(DV,\mathrm{rel}\ SV))_{(2)} \\ \overset{\mathrm{proj}}{\cong}\ & Wh^{Top,Equi}_{S^1,\rho}(M)_{(2)} \overset{\mathrm{proj}\circ(\times\mathbf{CP}^2)}{\cong} Wh^{Top,Equi}_{S^1,\rho}(M)_{(2)} = Wh^{Top}_{S^1}(M)_{(2)} \end{aligned} \tag{16}$$

by first following the left and then following the bottom of (12). Then we will combine (13), (15), and (16) to form a decomposition

$$Wh^{Top}_{S^1}(M\times P)_{(2)} \cong Wh^{Top}_{S^1}(M)_{(2)}\oplus Wh^{Top}_{S^1}(M)_{(2)}\oplus Wh^{Top}_{S^1}(M)_{(2)} = A\oplus A\oplus A. \tag{17}$$

The composition $\mathrm{proj}\circ(\times\mathbf{CP}^2)$ may be extended to a natural map of the following exact sequence (see [23]) relating topological and piecewise linear K-theoretical obstructions:

$$Wh^{PL,Equi}_{S^1,\rho}(M)_c \to Wh^{PL,Equi}_{S^1,\rho}(M) \to Wh^{Top,Equi}_{S^1,\rho}(M) \to \tilde{K}^{PL,Equi}_{0,S^1,\rho}(M)_c \to \tilde{K}^{PL,Equi}_{0,S^1,\rho}(M) \tag{18}$$

where the subscript c means controlled K-theory. It was explained in sections 7 and 14 of [12] that, as a categorical nonsense, the effect of $\mathrm{proj}\circ(\times\mathbf{CP}^2)$ on the equivariant piecewise linear Whitehead torsion and finiteness obstructions comes from the module structure on the relevant obstruction groups over the Euler ring of S^1. Since the argument of [12] is a categorical one, the conclusion also applies to controlled equivariant piecewise linear Whitehead torsion and finiteness obstructions. Now the Euler numbers of $\mathbf{CP}^2/S^1$ and $(\mathbf{CP}^2)^{S^1}$ are 1 and 3, which implies that $\mathbf{CP}^2$ represents an invertible element of the Euler ring after localizing at 2. Consequently, the composition $\mathrm{proj}\circ(\times\mathbf{CP}^2)$ is an equivalence on the PL-terms in (18) after localizing at 2. By five lemma, this implies that the composition at the bottom of (12) is an isomorphism after localizing at 2.

To describe the involution in (17), we observe that the projection to the first two factors, being the restriction from $M\times P$ to $M\times(D^3\supset S^2)$), is involutive. As in the case of manifold with boundary, the involution on the two factors is given by $(\alpha,\beta)^* = ((-1)^3\alpha^*+(-1)^2\beta^*,\beta^*) = (-\alpha^*+\beta^*,\beta^*)$.

Although we feel that the isomorphism (16) is likely to be involutive, the proof is not immediately obvious. Since we will not need this fact anyway, we denote by $'$ the involution on A induced from the natural involution on $Wh_{S^1}^{Top}(M \times DV, \mathrm{rel}\ SV)_{(2)}$ via (16). The fact that the inclusion $Wh_{S^1}^{Top}(M \times DV, \mathrm{rel}\ SV) \to Wh_{S^1}^{Top}(M \times P)$ is involutive then implies that $(0, 0, \gamma)^* = (0, 0, \gamma')$.

Thus to complete the description of the involution in (17), it remain to consider the third coordinate of $(\alpha, \beta, 0)^*$. Geometrically, this is the transfer of β along the projection $M \times SV \to M \times S^2$:

$$Wh_{S^1}^{Top}(M \times S^2) \stackrel{\mathrm{trf}}{\to} Wh_{S^1}^{Top}(M \times SV) \stackrel{\mathrm{incl}}{\to} Wh_{S^1}^{Top}(M \times DV, \mathrm{rel}\ SV). \tag{19}$$

When the two ends of (19) are identified with A by projection and (16), this transfer is our homomorphism λ.

To see $\lambda^2 = 0$, we translate (19) to an equivalent map on $Wh_{S^1,\rho}^{Top,Equi}$, which becomes the left side of the following diagram:

$$\begin{array}{rcl}
Wh_{S^1,\rho}^{Top,Equi}(M)_{(2)} & & \\
\mathrm{proj} \uparrow \cong & & \\
Wh_{S^1,\rho}^{Top,Equi}(M \times S^2)_{(2)} & & \\
\mathrm{trf} \downarrow & & \\
Wh_{S^1,\rho}^{Top,Equi}(M \times SV)_{(2)} & \stackrel{\mathrm{incl}_1}{\longleftarrow} & Wh_{S^1,\rho}^{Top,Equi}((M - M_s) \times SV)_{(2)} \\
\mathrm{proj} \downarrow & & \downarrow \mathrm{proj} \\
Wh_{S^1,\rho}^{Top,Equi}(M)_{(2)} & \stackrel{\mathrm{incl}_2}{\longleftarrow} & Wh_{S^1,\rho}^{Top,Equi}(M - M_s)_{(2)} \\
\mathrm{proj}\circ(\times \mathbf{CP}^2) \uparrow \cong & & \cong \uparrow \mathrm{proj}\circ(\times \mathbf{CP}^2) \\
Wh_{S^1,\rho}^{Top,Equi}(M)_{(2)} & \stackrel{\mathrm{incl}_3}{\longleftarrow} & Wh_{S^1,\rho}^{Top,Equi}(M - M_s)_{(2)}
\end{array} \tag{20}$$

Since S^1 acts freely on SV, $M \times SV$ and $(M - M_s) \times SV$ are free S^1-spaces. Therefore $Wh_{S^1,\rho}^{Top,Equi}$ is the classical Whitehead group of the quotient space for these spaces. Since the classical Whitehead group depends only on the fundamental group, and the inclusion $(M - M_s) \times SV \to M \times SV$ is am isomorphism on π_1, we conclude that incl_1 is an isomorphism. It then follows from the commutativity of (20) that the image of λ lies inside $Wh_{S^1,\rho}^{Top,Equi}(M, \mathrm{rel}\ M_s)_{(2)}$. On the other hand, if we start from an element of $Wh_{S^1,\rho}^{Top,Equi}(M)_{(2)}$ that comes from $Wh_{S^1,\rho}^{Top,Equi}(M, \mathrm{rel}\ M_s)_{(2)}$, then the element is nontrivial only over the free part of M. However, the fibre of $(M \times SV)/S^1 \to (M \times S^2)/S^1$ is

S^1 over the free part $((M - M_s) \times S^2)/S^1$. This implies that the transfer is trivial, so that λ vanishes on $Wh^{Top,Equi}_{S^1,\rho}(M, \text{rel } M_s)_{(2)}$. Consequently, $\lambda^2 = 0$.

It remains to show that $\times P$ sends a to $(a, 2a, a)$ under the natural identifications. Since G acts trivially on $(D^3 \supset S^2)$, the commutative diagram (12) shows that the first two coordinates of $\times P$ is simply given by multiplying Euler numbers. This gives rise to $(a, 2a)$. Since the third coordinate of $Wh^{Top}_{S^1}(M \times P)$ is given by the isomorphism (16), the third coordinate of $\times P$ is by the very construction sending a to a.

This completes the proof of the lemma.

□

Since localization at 2 does not change Tate cohomologies, the lemma 8 reduces the destablization of the periodicity to the following algebraic computation.

Lemma 9 *Suppose A is an abelian group, and $*, '$ are two involutions on A. Suppose $\lambda : A \to A$ is a homomorphism, such that*

1. $\lambda^2 = 0$;
2. $(\alpha, \beta, \gamma)^* = (-\alpha^* + \beta^*, \beta^*, \gamma' + \lambda(\beta))$ *is an involution on* $A \oplus A \oplus A$;
3. $\phi(a) = (a, 2a, a), (A, *) \to A \oplus A \oplus A$ *is an involutive homomorphism.*

Then the induced map $\phi_* : \hat{H}(\mathbf{Z}_2; A) \to \hat{H}(\mathbf{Z}_2; A \oplus A \oplus A)$ is an isomorphism. Moreover, the inclusion $\psi(a) = (0, 0, a), (A,') \to A \oplus A \oplus A$ also induces an isomorphism on the Tate cohomology.

Although the lemma is not trivial, the proof is rather straightforward and is therefore omitted here.

Finally, let us note that the remark made at the end of last section also applies to the discussions in this section. Therefore the periodicity theorem 1 also applies to other abelian groups. As a result, theorem 2 is proved.

6 Naturality under the Restriction to Fixed Sets and Subgroups

In this last section we prove the theorem 4.

The naturality in the theorem 4 means the commutativity of the following diagram

$$\begin{array}{ccccc} S_{WH}(M^H) & \xleftarrow{\text{rest}_1} & S_G(M) & \xrightarrow{\text{rest}_2} & S_H(M) \\ \mid\wr\, (V^H, WH) & & \mid\wr\, (V,G) & & \mid\wr\, (V,H) \\ S_{WH}(M^H \times DV^H) & \xleftarrow{\text{rest}_1} & S_G(M \times DV) & \xrightarrow{\text{rest}_2} & S_H(M \times DV) \end{array} \tag{21}$$

where $V = \mathbf{C}^2$ is the representation from $\kappa : G \to S^1$, and the vertical maps are periodicity equivalences corresponding to different groups and representations. The commutativity of (21) will follow from the relation between the two restrictions (to fixed points and to actions by subgroups) in (21) and the whole proof of the theorems 1 and 2.

The restriction of the stable structure to fixed points of subgroups comes as the fibre of two compatible assembly maps

$$\begin{array}{ccc} H(M/G; L_G^{-\infty}(locM)) & \to & L_G^{-\infty}(M) \\ \downarrow \text{rest}_1 & & \downarrow \text{rest}_1 \\ H(M/G; L_{WH}^{-\infty}((locM)^H)) & \to & L_{WH}^{-\infty}(M^H) \\ \| & & \| \\ H(M^H/WH; L_{WH}^{-\infty}(loc(M^H))) & \to & L_{WH}^{-\infty}(M^H) \end{array} \tag{22}$$

The commutativity of the diagram comes from the obvious naturality of restriction for the functor $L^{-\infty}$.

The effect of the restriction to the actions of subgroups on the assembly map is more complicated. First we have the usual natural transformation

$$\begin{array}{ccc} H(M/G; L_G^{-\infty}(loc_G M)) & \to & L_G^{-\infty}(M) \\ \downarrow \text{rest}_2 & & \downarrow \text{rest}_2 \\ H(M/G; L_H^{-\infty}(loc_G M)) & \to & L_H^{-\infty}(M) \end{array} \tag{23}$$

Note that we use $loc_G M$ to denote the local G-equivariant structure. If we restrict the action to the subgroup H, then we have $loc_G M = Gx \times_H loc_H M$. Now we apply the

Fubini equivalence (constructed by induction on orbit type) to the stratified system of fibrations $Gx/H \to M/H \to M/G$ and obtain

$$\begin{array}{ccc} H(M/G; L_H^{-\infty}(loc_G M)) & & \\ \| & & \\ H(M/G; L_H^{-\infty}(Gx \times_H loc_H M)) & & \\ \simeq\uparrow \alpha & \stackrel{\text{assemblies}}{\Longrightarrow} & L_H^{-\infty}(M) \\ H(M/G; H(Gx/H; L_H^{-\infty}(loc_H M))) & & \\ \|_{\text{Fubini}} & & \\ H(M/H; L_H^{-\infty}(loc_H M)) & & \end{array} \tag{24}$$

where $\Longrightarrow$ means the natural assembly maps from the four homologies on the left to the stable surgery obstruction. Moreover, α is the "partial assembly map" obtained by applying the homology to the assembly map (considered as a natural transformation):

$$\alpha_0 : H(Gx/H; L_H^{-\infty}(loc_H M)) \to L_H^{-\infty}(Gx \times_H loc_H M).$$

Since both sides are products of Gx/H copies of $L_H^{-\infty}(loc_H M)$, α_0 is an equivalence. The naturality of the assembly map with respect to the Fubini equivalence (see Section 8 of [19], and [27]) shows that the diagram (24) is commutative.

The fibre of the top of (23) is $S_G^{-\infty}(M)$. The fibre of the bottom of (24) is $S_H^{-\infty}(M)$. Combining the diagrams (23) and (24) together we get a diagram whose induced map on the fibre is the restriction map $S_G^{-\infty} \to S_H^{-\infty}$.

In both restriction cases, the discussion above shows that the naturality problem for the stable structure (i.e., the commutativity of (21) with $S^{-\infty}$ in place of S) is reduced to the naturality problem for the stable surgery obstruction. Upon closer inspection, we see that besides the natural properties of the homology theory described in [20] and [27], each of the commutative squares involved is one of the two types: First, the "$\times Z$" operation (Z is a G-stratified space) is natural with respect to the restrictions:

$$\begin{array}{ccccc} L_{WH}^{-\infty}(M^H) & \leftarrow & L_G^{-\infty}(M) & \rightarrow & L_H^{-\infty}(M) \\ \downarrow \times Z^H & & \downarrow \times Z & & \downarrow \times Z \\ L_{WH}^{-\infty}(M^H \times Z^H) & \leftarrow & L_G^{-\infty}(M \times Z) & \rightarrow & L_H^{-\infty}(M \times Z). \end{array} \tag{25}$$

Second, the inclusion operation is natural with respect to the restrictions (M is a G-transverse subspace of N):

$$\begin{array}{ccccc} L^{-\infty}_{WH}(M^H) & \leftarrow & L^{-\infty}_G(M) & \rightarrow & L^{-\infty}_H(M) \\ \downarrow \text{incl} & & \downarrow \text{incl} & & \downarrow \text{incl} \\ L^{-\infty}_{WH}(N^H) & \leftarrow & L^{-\infty}_G(N) & \rightarrow & L^{-\infty}_H(N). \end{array} \qquad (26)$$

Such naturalities are obvious from the geometrical meaning of the operations. This completes the proof of the stable version of the theorem 4.

The naturality of the destablization process is more direct. This follows from the commutativity of the naturality of the operations (1) with respect to the restrictions on $Wh^{top,\leq 0}$ (i.e., the commutativity of the diagrams (25) and (26) with $Wh^{top,\leq 0}$ in place of $L^{-\infty}$).

In conclusion, we see the periodicity in the theorem 2 is natural with respect to the restriction to fixed points of subgroups and the restriction to the action of subgroups. Since the periodicity in theorem 3 is obtained by repeatedly applying theorem 2, its naturality with respect to the two restrictions is also true.

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