Mirror Transforms of Hamiltonian Systems

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Abstract

We demonstrate, through various examples of Hamiltonian systems, that symplectic structures have been encoded into the Painlevé test. Each principal balance in the Painlevé test induces a mirror transform that regularizes movable singularities. Moreover, for finite-dimensional Hamiltonian systems, the mirror transforms are canonical.

1 Introduction

In the late 1980s, a connection between the global geometry of flows in the phase space of an autonomous finite-dimensional Hamiltonian system and the Painlevé analysis was revealed by Adler and van Moerbeke [1], Ercolani and Siggia [2] [3]. A key step in such global study is to extend all the flows in the phase space over singularities. The extension was constructed by introducing change of variables at the pole singularities so that the Laurent series solutions are regularized. Adler and van Moerbeke used the Laurent series solutions directly as the change of variables. On the other hand, Ercolani and Siggia's change of variables is more refined and is often canonical. However, their constructions are ad hoc and involve some guess works.

In [4] [5] [6], we showed that for general ODE systems passing the Painlevé test is equivalent to a change of variables at movable pole singularities so that the Laurent series solutions are regularized. Moreover, we provided an algorithm (very similar to the Painlevé test) for finding a nice change of variables, which we call the mirror transform. Similar to Ercolani and Siggia, the mirror transform can be used to extend the phase space so that the flows are all globally defined and analytic.

In this paper, we demonstrate, through several examples, that the mirror transform for Hamiltonian systems can always be made canonical. This replaces the ad hoc construction by Ercolani and Siggia by a systematic and routine process (as easy as the Painlevé test). Our examples include autonomous, non-autonomous, as well as infinite-dimensional Hamiltonian systems.

Technically, underlying our construction of the mirror transform is the LU decomposition of the resonance matrix, up to a certain rearrangement of the dependent variables. We will demonstrate that for Hamiltonian systems, the resonance matrix can always be made into a symplectic one, and there is a pairing property among resonances. These properties make it possible to arrange the mirror transform to be canonical.

The resonance pairing property played an important role in the work of Ercolani and Siggia. In [7], Lochak suggested that symplectic structure and the resonance pairing property are already built into the Painlevé analysis of Hamiltonian systems. However, Lochak did not explain the relation between his argument and the Painlevé test, so that his argument is not directly applicable as a theoretical explanation. We are currently working on a rigorous theoretical explanation of the examples of this paper. The theory will appear in a separate paper.

2 Resonance Matrix

In this section, we fix some terminology for the Painlevé analysis. We also explain the symplectic structure we are going to observe in subsequent examples.

Consider a system of first order ODEs

$$u' = f(t, u), \qquad u = (u_1, \cdots, u_n).$$

A *balance* is a Laurent series solution

$$u_i = c_i(t - t_0)^{-g_i} + u_{i,1}(t - t_0)^{1-g_i} + u_{i,2}(t - t_0)^{2-g_i} + \cdots, \quad i = 1, \cdots, n$$

such that the vector

$$(g_1 c_1, \cdots, g_n c_n) \neq 0, \tag{1}$$

and some free variables (called *resonance parameters*) are involved. For any specific resonance parameter r, the smallest j such that r appears in some $u_{i,j}$ is the *resonance*. The corresponding *resonance vector* is

$$v = \left((t-t_0)^{g_1-j+1} \frac{\partial u_1}{\partial r}, \cdots, (t-t_0)^{g_n-j+1} \frac{\partial u_n}{\partial r} \right)_{t=t_0} = \left(\frac{\partial u_{1,j}}{\partial r}, \cdots, \frac{\partial u_{n,j}}{\partial r} \right).$$

We may also consider t_0 as a resonance parameter. We assign $j = -1$ as the corresponding resonance, and

$$v = \left((t-t_0)^{g_1+1} \frac{\partial u_1}{\partial t_0}, \cdots, (t-t_0)^{g_n+1} \frac{\partial u_n}{\partial t_0} \right)_{t=t_0} = (g_1 c_1, \cdots, g_n c_n)$$

as the corresponding resonance vector. With all the resonance vectors as columns, in the order of increasing resonances, we get a *resonance matrix*.

The balance is *principal* if the number of resonances (including $j = -1$) is n.

The resonances and resonance vectors are often computed as the eigenvalues and eigenvectors of a certain *Kowalevskian matrix*. However, this relation has only been justified in individual cases. There are some cases that the choice of the Kowalevskian matrix becomes a subtle issue. We remark that condition (1) in our definition of a balance allows the possibility that some leading coefficients c_i vanish. Such relaxation is necessary for some of our examples.

In general, we expect no relation between resonances, nor between resonance vectors. For principal balances of Hamiltonian systems, with

$$u = (q_1, \cdots, q_m, p_1, \cdots, p_m),$$

the works of Ercolani and Siggia [2] [3], Lochak [7] suggest that the following is true: The resonances $-1 = j_1 < j_2 \leq \cdots \leq j_{2m}$ should satisfy the following *resonance pairing property*:

$$\begin{aligned} j_1 + j_{2m} &= g_1 + g_{m+1}, \\ j_2 + j_{2m-1} &= g_2 + g_{m+2}, \\ &\vdots \\ j_m + j_{m+1} &= g_m + g_{2m}. \end{aligned}$$

Moreover, it is possible to choose resonance parameters elegantly, so that after rescaling, the resonance vectors satisfy

$$v_i^T J v_j = \begin{cases} 1, & \text{if } i+j = 2m+1,\ 1 \le i \le m; \\ -1, & \text{if } i+j = 2m+1,\ m+1 \le i \le 2m; \\ 0, & \text{if } i+j \ne 2m+1, \end{cases}$$

where

$$J = \begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix}.$$

In other words, the matrix $S = [v_1, v_2, \cdots, v_m, v_{2m}, v_{2m-1}, \cdots, v_{m+1}]$ (obtained after reversing the order of the last half of the resonance vectors) is a symplectic matrix: $S^T J S = J$.

3 Autonomous Hamiltonian Systems

In this section, we demonstrate the symplectic structures encoded into the Painlevé test for autonomous Hamiltonian systems. Moreover, we show how to construct canonical mirror transforms.

3.1 A Hénon-Heiles Hamiltonian system

The Hénon-Heiles system

$$\begin{cases} \dot{q}_1 &= p_1, \\ \dot{q}_2 &= p_2, \\ \dot{p}_1 &= -q_1 - 6q_1q_2, \\ \dot{p}_2 &= -q_2 - 3q_1^2 - 3q_2^2, \end{cases} \tag{2}$$

is given by the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + 3q_1^2q_2 + q_2^3.$$

The system passes the Painlevé test and has two principal balances. One balance is given by

$$q_1 \sim (t-t_0)^{-2} + \frac{1}{6} + r_2 + r_3(t-t_0) + \left(-\frac{7}{120} - r_2 - 3r_2^2\right)(t-t_0)^2$$

$$
\begin{aligned}
& +\left(-\frac{r_3}{3}-2r_2r_3\right)(t-t_0)^3+\frac{r_4}{4}(t-t_0)^4+\cdots, \\
q_2 \quad \sim \quad & -(t-t_0)^{-2}+r_2+r_3(t-t_0)+\left(-\frac{1}{15}-r_2-3r_2^2\right)(t-t_0)^2 \\
& +\left(-\frac{r_3}{3}-2r_2r_3\right)(t-t_0)^3 \\
& +\left(\frac{1}{48}+\frac{11r_2}{24}+3r_2^2+6r_2^3-r_3^2-\frac{r_4}{4}\right)(t-t_0)^4+\cdots, \\
p_1 \quad \sim \quad & -2(t-t_0)^{-3}+r_3+\left(-\frac{7}{60}-2r_2-6r_2^2\right)(t-t_0) \\
& +(-r_3-6r_2r_3)(t-t_0)^2+r_4(t-t_0)^3+\cdots, \\
p_2 \quad \sim \quad & 2(t-t_0)^{-3}+r_3+\left(-\frac{2}{15}-2r_2-6r_2^2\right)(t-t_0) \\
& +(-r_3-6r_2r_3)(t-t_0)^2 \\
& +\left(\frac{1}{12}+\frac{11r_2}{6}+12r_2^2+24r_2^3-4r_3^2-r_4\right)(t-t_0)^3+\cdots.
\end{aligned}
$$

with the resonances -1, 2, 3, 6, the resonance parameters t_0, r_2, r_3, r_4, and the resonance matrix

$$
\begin{pmatrix} 2 & 1 & 1 & 1/4 \\ -2 & 1 & 1 & -1/4 \\ -6 & 0 & 1 & 1 \\ 6 & 0 & 1 & -1 \end{pmatrix}.
$$

The resonances satisfy $(-1)+6=2+3$ and $2+3=2+3$. Moreover, after rescaling the columns and reversing the last two columns, we get a symplectic matrix

$$
\begin{pmatrix} 2 & 1 & 1/28 & 1/2 \\ -2 & 1 & -1/28 & 1/2 \\ -6 & 0 & 1/7 & 1/2 \\ 6 & 0 & -1/7 & 1/2 \end{pmatrix}.
$$

We remark that this example is essentially the example that appeared in [7]. Since the resonance vectors are the eigenvectors of the Kowalevski matrix, the general argument in [7] implies the symplectic structure observed above.

Now we compute the mirror transform. Following the steps in [5], we introduce the indicial normalization $q_1=\theta^{-2}$ and find the following Laurent series in θ:

$$\begin{aligned}
\theta' &\sim 1+\frac{\bar{r}_2}{2}\theta^2-\bar{r}_3\theta^3+\left(-\frac{1}{16}+\frac{7\bar{r}_2^2}{8}\right)\theta^4-\bar{r}_2\bar{r}_3\theta^5-\frac{\bar{r}_4}{2}\theta^6+\cdots,\\
q_2 &\sim -\theta^{-2}-\frac{1}{6}-\frac{2\bar{r}_2}{3}+\bar{r}_3\theta+\left(\frac{1}{24}-\frac{2\bar{r}_2^2}{3}\right)\theta^2+\frac{\bar{r}_2\bar{r}_3}{2}\theta^3+0\theta^4+\cdots,\\
p_1 &\sim -2\theta^{-3}-\bar{r}_2\theta^{-1}+2\bar{r}_3+\left(\frac{1}{8}-\frac{7\bar{r}_2^2}{4}\right)\theta+2\bar{r}_2\bar{r}_3\theta^2+\bar{r}_4\theta^3+\cdots,\\
p_2 &\sim 2\theta^{-3}+\bar{r}_2\theta^{-1}-\bar{r}_3+\left(-\frac{1}{24}+\frac{5\bar{r}_2^2}{12}\right)\theta\\
&\quad+\left(\frac{\bar{r}_2}{24}-\frac{2\bar{r}_2^3}{3}-\bar{r}_3^2-\bar{r}_4\right)\theta^3+\cdots,
\end{aligned}$$

where $\bar{r}_2$, $\bar{r}_3$, and $\bar{r}_4$ are equivalent to the resonance parameters in the principal balance. We truncate the θ-series of q_2 at $\bar{r}_2$ by introducing η_2:

$$q_2=-\theta^{-2}+\eta_2. \tag{3}$$

From the θ-series of η_2, we may express $\bar{r}_2$ as a θ-series, with functions of η_2, $\bar{r}_3$, and $\bar{r}_4$ as coefficients. We substitute this into the θ-series of p_1 and p_2, so that the coefficients become functions of η_2, $\bar{r}_3$, and $\bar{r}_4$. Then we truncate the θ-series of p_2 at $\bar{r}_3$ by introducing η_3:

$$p_2=2\theta^{-3}+\left(-\frac{1}{4}-\frac{3}{2}\eta_2\right)\theta^{-1}+\eta_3. \tag{4}$$

From the θ-series of η_3, we may express $\bar{r}_3$ as a θ-series, with functions of η_2, η_3, and $\bar{r}_4$ as coefficients. We substitute this into the θ-series of p_1, so that the coefficients become functions of η_2, η_3, and $\bar{r}_4$. Then we truncate the θ-series of p_1 at $\bar{r}_4$ by introducing η_4:

$$p_1=-2\theta^{-3}+\left(\frac{1}{4}+\frac{3}{2}\eta_2\right)\theta^{-1}+\eta_3+\left(\frac{1}{32}-\frac{1}{8}\eta_2-\frac{3}{8}\eta_2^2\right)\theta+\eta_4\theta^3. \tag{5}$$

As pointed out in [6], the transform $(q_1,q_2,p_1,p_2)\leftrightarrow(\theta,\eta_2,\eta_3,\eta_4)$ given by $q_1=\theta^{-2}$, (3), (4), (5) converts system (2) into a regular system and converts the Laurent series solution into a power series solution.

Now we study how the transform behaves with regard to the symplectic structure. An easy computation shows that

$$dq_1\wedge dp_1+dq_2\wedge dp_2=-2d\theta\wedge d\eta_4+d\eta_2\wedge d\eta_3.$$

Therefore, if we denote $Q_1 = \theta$, $Q_2 = \eta_2$, $P_1 = -2\eta_4$, $P_2 = \eta_3$, then the following mirror transform

$$\begin{aligned} q_1 &= Q_1^{-2}, \\ q_2 &= -Q_1^{-2} + Q_2, \\ p_1 &= -2Q_1^{-3} + \left(\frac{1}{4} + \frac{3Q_2}{2}\right) Q_1^{-1} + P_2 \\ &\quad + \left(\frac{1}{32} - \frac{Q_2}{8} - \frac{3Q_2^2}{8}\right) Q_1 - \frac{P_1}{2} Q_1^3, \\ p_2 &= 2Q_1^{-3} + \left(-\frac{1}{4} - \frac{3Q_2}{2}\right) Q_1^{-1} + P_2, \end{aligned}$$

preserves the symplectic form. This implies that the mirror transform converts the Hamiltonian system (2) into a Hamiltonian system

$$\begin{aligned} \dot{Q}_1 &= 1 + \left(-\frac{1}{8} - \frac{3Q_2}{4}\right) Q_1^2 - \frac{P_2}{2} Q_1^3 + \left(-\frac{1}{64} + \frac{Q_2}{16} + \frac{3Q_2^2}{16}\right) Q_1^4 + \frac{P_1}{4} Q_1^6, \\ \dot{Q}_2 &= 2P_2 + \left(\frac{1}{32} - \frac{Q_2}{8} - \frac{3Q_2^2}{8}\right) Q_1 - \frac{P_1}{2} Q_1^3, \\ \dot{P}_1 &= -\frac{P_2}{32} + \frac{P_2 Q_2}{8} + \frac{3P_2 Q_2^2}{8} \\ &\quad + \left(-\frac{1}{1024} + \frac{P_1}{4} + \frac{Q_2}{128} + \frac{3P_1 Q_2}{2} + \frac{Q_2^2}{128} - \frac{3Q_2^3}{32} - \frac{9Q_2^4}{64}\right) Q_1 \\ &\quad + \frac{3P_1 P_2}{2} Q_1^2 + \left(\frac{P_1}{16} - \frac{P_1 Q_2}{4} - \frac{3P_1 Q_2^2}{4}\right) Q_1^3 - \frac{3P_1^2}{4} Q_1^5, \\ \dot{P}_2 &= -\frac{1}{64} - \frac{7Q_2}{16} - \frac{21Q_2^2}{16} + \left(\frac{P_2}{8} + \frac{3P_2 Q_2}{4}\right) Q_1 \\ &\quad + \left(\frac{1}{256} + \frac{3P_1}{4} + \frac{Q_2}{128} - \frac{9Q_2^2}{64} - \frac{9Q_2^3}{32}\right) Q_1^2 + \left(-\frac{P_1}{16} - \frac{3P_1 Q_2}{8}\right) Q_1^4, \end{aligned}$$

with the Hamiltonian

$$\begin{aligned} & \bar{H}(Q_1, Q_2, P_1, P_2) \\ = \; & H(q_1, q_2, p_1, p_2) \\ = \; & \frac{1}{128} + \frac{Q_2}{64} + \frac{7Q_2^2}{32} + \frac{7Q_2^3}{16} + P_1 + P_2^2 \end{aligned}$$

$$+\left(\frac{P_2}{32}-\frac{P_2Q_2}{8}-\frac{3P_2Q_2^2}{8}\right)Q_1$$
$$+\left(\frac{1}{2048}-\frac{Q_2}{256}-\frac{Q_2^2}{256}+\frac{3Q_2^3}{64}+\frac{9Q_2^4}{128}-\frac{P_1}{8}-\frac{3P_1Q_2}{4}\right)Q_1^2$$
$$-\frac{P_1P_2}{2}Q_1^3+\left(-\frac{P_1}{64}+\frac{P_1Q_2}{16}+\frac{3P_1Q_2^2}{16}\right)Q_1^4+\frac{P_1^2}{8}Q_1^6.$$

We note that this is a regular function (in fact a polynomial).

3.2 Another Hénon-Heiles Hamiltonian system

The last example is an ideal one, for which the usual notions of the Painlevé analysis work perfectly. Now we turn to another Hénon-Heiles system

$$\left\{\begin{array}{lcl}\dot{q}_1 & = & p_1,\\ \dot{q}_2 & = & p_2,\\ \dot{p}_1 & = & -q_1-2q_1q_2,\\ \dot{p}_2 & = & -q_2-q_1^2-6q_2^2,\end{array}\right. \tag{6}$$

which is given by the Hamiltonian

$$H=\frac{1}{2}(p_1^2+p_2^2+q_1^2+q_2^2)+q_1^2q_2+2q_2^3.$$

The system passes the Painlevé test and has one principal balance

$$\begin{aligned}
q_1 &\sim -r_2(t-t_0)^{-1}+\left(-\frac{5r_2}{12}-\frac{r_2^3}{12}\right)(t-t_0)+\frac{r_3}{2}(t-t_0)^2\\
&\quad+\left(\frac{61r_2}{720}+\frac{11r_2^3}{144}+\frac{r_2^5}{72}\right)(t-t_0)^3+\left(-\frac{r_3}{24}-\frac{r_2^2r_3}{24}\right)(t-t_0)^4\\
&\quad+\left(-\frac{r_2}{243}-\frac{13r_2^3}{25920}+\frac{r_2^5}{2592}+\frac{r_2^7}{15552}+\frac{r_2r_4}{36}\right)(t-t_0)^5+\cdots,\\
q_2 &\sim -(t-t_0)^{-2}-\frac{1}{12}+\frac{r_2^2}{12}+\left(-\frac{1}{240}+\frac{r_2^2}{12}+\frac{r_2^4}{48}\right)(t-t_0)^2\\
&\quad-\frac{r_2r_3}{6}(t-t_0)^3+\frac{r_4}{4}(t-t_0)^4+\cdots,\\
p_1 &\sim r_2(t-t_0)^{-2}-\frac{5r_2}{12}-\frac{r_2^3}{12}+r_3(t-t_0)
\end{aligned}$$

$$+\left(\frac{61r_2}{240}+\frac{11r_2^3}{48}+\frac{r_2^5}{24}\right)(t-t_0)^2+\left(-\frac{r_3}{6}-\frac{r_2^2r_3}{6}\right)(t-t_0)^3$$
$$+\left(-\frac{5r_2}{243}-\frac{13r_2^3}{5184}+\frac{5r_2^5}{2592}+\frac{5r_2^7}{15552}+\frac{5r_2r_4}{36}\right)(t-t_0)^4+\cdots,$$
$$p_2 \sim 2(t-t_0)^{-3}+\left(-\frac{1}{120}+\frac{r_2^2}{6}+\frac{r_2^4}{24}\right)(t-t_0)-\frac{r_2r_3}{2}(t-t_0)^2$$
$$+r_4(t-t_0)^3+\cdots.$$

If we take the natural leading exponents 1, 2, 2, 3, then the resonances would be -1, 0, 3, 6 for the resonance parameters t_0, r_2, r_3, r_4. Although we have the resonance pairing property $-1+6=2+3$, $0+3=1+2$, the resonance matrix fails to become symplectic despite rescaling and reversing.

The failure of the "naive" symplectic property has been noticed by Ercolani and Siggia [3]. They pointed out that one has to be very careful in applying the usual notions of the Painlevé analysis to certain Hamiltonian systems. The main point is that one has to allow some leading coefficients to vanish in order for some examples to fit the usual notions.

We may salvage the situation by adding $0(t-t_0)^{-2}$ and $0(t-t_0)^{-3}$ to q_1 and p_1, making their leading exponents 2 and 3. This changes the resonances to -1, 1, 4, 6, which still satisfies the resonance pairing property $-1+6=2+3$, $1+4=2+3$. Moreover, the resonance matrix becomes

$$\begin{pmatrix} 0 & -1 & 1/2 & 0 \\ -2 & 0 & 0 & 1/4 \\ 0 & 1 & 1 & 0 \\ 6 & 0 & 0 & 1 \end{pmatrix}.$$

After rescaling and reversing the last two columns, we get a symplectic matrix

$$\begin{pmatrix} 0 & -1 & 0 & -1/3 \\ -2 & 0 & -1/14 & 0 \\ 0 & 1 & 0 & -2/3 \\ 6 & 0 & -2/7 & 0 \end{pmatrix}.$$

Now we compute the mirror transform. Following the steps in [5], we introduce the indicial normalization $q_2=\theta^{-2}$ and find the Laurent series in

θ:

$$\begin{aligned}
\theta' &\sim i + \left(\frac{i}{8} - \frac{i\bar{r}_2^2}{8}\right)\theta^2 + \left(-\frac{i}{128} + \frac{13i\bar{r}_2^2}{64} + \frac{7i\bar{r}_2^4}{128}\right)\theta^4 \\
&\quad + \frac{i\bar{r}_2\bar{r}_3}{2}\theta^5 - \frac{\bar{r}_4}{2}\theta^6 + \cdots, \\
q_1 &\sim i\bar{r}_2\theta^{-1} + \left(-\frac{3i\bar{r}_2}{8} - \frac{i\bar{r}_2^3}{8}\right)\theta - \frac{i\bar{r}_3}{2}\theta^2 + \left(-\frac{9i\bar{r}_2}{128} - \frac{3i\bar{r}_2^3}{64} - \frac{i\bar{r}_2^5}{128}\right)\theta^3 \\
&\quad + \left(\frac{11i\bar{r}_2}{2304} + \frac{5i\bar{r}_2^3}{384} + \frac{7i\bar{r}_2^5}{2304} - \frac{\bar{r}_2\bar{r}_4}{18}\right)\theta^5 + \cdots, \\
p_1 &\sim \bar{r}_2\theta^{-2} + \frac{\bar{r}_2}{2} + \bar{r}_3\theta + \left(\frac{\bar{r}_2}{4} + \frac{5\bar{r}_2^3}{16} + \frac{\bar{r}_2^5}{16}\right)\theta^2 + \left(\frac{\bar{r}_3}{8} + \frac{3\bar{r}_2^2\bar{r}_3}{8}\right)\theta^3 \\
&\quad + \left(-\frac{\bar{r}_2}{2304} + \frac{\bar{r}_2^3}{768} + \frac{37\bar{r}_2^5}{2304} + \frac{\bar{r}_2^7}{256} + \frac{2i}{9}\bar{r}_2\bar{r}_4\right)\theta^4 + \cdots, \\
p_2 &\sim -2i\theta^{-3} + \left(-\frac{i}{4} + \frac{i\bar{r}_2^2}{4}\right)\theta^{-1} + \left(\frac{i}{64} - \frac{13i\bar{r}_2^2}{32} - \frac{7i\bar{r}_2^4}{64}\right)\theta \\
&\quad - i\bar{r}_2\bar{r}_3\theta^2 + \bar{r}_4\theta^3 + \cdots.
\end{aligned}$$

As in the previous example, by truncating q_1, p_1, p_2 at $\bar{r}_2$, $\bar{r}_3$, $\bar{r}_4$, respectively, we get the following mirror transform

$$\begin{aligned}
q_1 &= Q_2Q_1^{-1}, \\
q_2 &= Q_1^{-2}, \\
p_1 &= -iQ_2Q_1^{-2} - \frac{7iQ_2}{8} + \frac{iQ_2^3}{8} + P_2Q_1, \\
p_2 &= -2iQ_1^{-3} + \left(-\frac{i}{4} - \frac{iQ_2^2}{4}\right)Q_1^{-1} \\
&\quad + \left(\frac{i}{64} + \frac{7iQ_2^2}{32} - \frac{3iQ_2^4}{64}\right)Q_1 - \frac{P_2Q_2}{2}Q_1^2 - \frac{P_1}{2}Q_1^3.
\end{aligned}$$

It is easy to check that the mirror transform is canonical, so that it converts the Hamiltonian system (6) into a Hamiltonian system

$$\dot{Q}_1 = i + \left(\frac{i}{8} + \frac{iQ_2^2}{8}\right)Q_1^2 + \left(-\frac{i}{128} - \frac{7iQ_2^2}{64} + \frac{3iQ_2^4}{128}\right)Q_1^4$$

$$
\begin{aligned}
& +\frac{P_2Q_2}{4}Q_1^5+\frac{P_1}{4}Q_1^6, \\
\dot{Q}_2 &= \left(-\frac{3iQ_2}{4}+\frac{iQ_2^3}{4}\right)Q_1+P_2Q_1^2+\left(-\frac{iQ_2}{128}-\frac{7iQ_2^3}{64}+\frac{3iQ_2^5}{128}\right)Q_1^3 \\
& +\frac{P_2Q_2^2}{4}Q_1^4+\frac{P_1Q_2}{4}Q_1^5, \\
\dot{P}_1 &= \frac{3iP_2Q_2}{4}-\frac{iP_2Q_2^3}{4} \\
& +\left(\frac{1}{4096}-P_2^2-\frac{iP_1}{4}+\frac{7Q_2^2}{1024}-\frac{iP_1Q_2^2}{4}+\frac{95Q_2^4}{2048}-\frac{21Q_2^6}{1024}+\frac{9Q_2^8}{4096}\right)Q_1 \\
& +\left(\frac{3iP_2Q_2}{128}+\frac{21iP_2Q_2^3}{64}-\frac{9iP_2Q_2^5}{128}\right)Q_1^2 \\
& +\left(\frac{iP_1}{32}-\frac{P_2^2Q_2^2}{2}+\frac{7iP_1Q_2^2}{16}-\frac{3iP_1Q_2^4}{32}\right)Q_1^3-\frac{5P_1P_2Q_2}{4}Q_1^4-\frac{3P_1^2}{4}Q_1^5, \\
\dot{P}_2 &= \frac{83Q_2}{128}-\frac{39Q_2^3}{64}+\frac{15Q_2^5}{128}+\left(\frac{3iP_2}{4}-\frac{3iP_2Q_2^2}{4}\right)Q_1 \\
& +\left(\frac{7Q_2}{1024}-\frac{iP_1Q_2}{4}+\frac{95Q_2^3}{1024}-\frac{63Q_2^5}{1024}+\frac{9Q_2^7}{1024}\right)Q_1^2 \\
& +\left(\frac{iP_2}{128}+\frac{21iP_2Q_2^2}{64}-\frac{15iP_2Q_2^4}{128}\right)Q_1^3 \\
& +\left(-\frac{P_2^2Q_2}{4}+\frac{7iP_1Q_2}{32}-\frac{3iP_1Q_2^3}{32}\right)Q_1^4-\frac{P_1P_2}{4}Q_1^5,
\end{aligned}
$$

with the Hamiltonian

$$
\begin{aligned}
& \bar{H}(Q_1,Q_2,P_1,P_2) \\
= & \frac{1}{256}+iP_1-\frac{83Q_2^2}{256}+\frac{39Q_2^4}{256}-\frac{5Q_2^6}{256}+\left(-\frac{3iP_2Q_2}{4}+\frac{iP_2Q_2^3}{4}\right)Q_1 \\
& +\left(-\frac{1}{8192}+\frac{P_2^2}{2}+\frac{iP_1}{8}-\frac{7Q_2^2}{2048}+\frac{iP_1Q_2^2}{8}-\frac{95Q_2^4}{4096}+\frac{21Q_2^6}{2048}-\frac{9Q_2^8}{8192}\right)Q_1^2 \\
& +\left(-\frac{iP_2Q_2}{128}-\frac{7iP_2Q_2^3}{64}+\frac{3iP_2Q_2^5}{128}\right)Q_1^3
\end{aligned}
$$

$$+\left(-\frac{iP_1}{128}+\frac{P_2^2Q_2^2}{8}-\frac{7iP_1Q_2^2}{64}+\frac{3iP_1Q_2^4}{128}\right)Q_1^4+\frac{P_1P_2Q_2}{4}Q_1^5+\frac{P_1^2}{8}Q_1^6,$$

again obtained by substituting the mirror transform into the original Hamiltonian.

3.3 A 3-freedom spherical symmetric Hamiltonian system

Consider a 3-freedom spherical symmetric Hamiltonian

$$H=\frac{1}{2}(p_1^2+p_2^2+p_3^2)+(q_1^2+q_2^2+q_3^2)^2. \tag{7}$$

The Hamiltonian system passes the Painlevé test and has two principal balances. One balance is characterized by the leading behavior

$$\begin{array}{lll} q_1\sim\rho_1(t-t_0)^{-1}, & q_2\sim\rho_2(t-t_0)^{-1}, & q_3\sim\rho_3(t-t_0)^{-1}, \\ p_1\sim-\rho_1(t-t_0)^{-2}, & p_2\sim-\rho_2(t-t_0)^{-2}, & p_3\sim-\rho_3(t-t_0)^{-2}, \end{array}$$

with the leading coefficients satisfying $2\rho_1^2+2\rho_2^2+2\rho_3^2+1=0$. The resonances are -1, 0, 0, 3, 3, 4, satisfying $(-1)+4=1+2$, $0+3=1+2$, $0+3=1+2$.

The special feature of this example is that the resonances appear in the leading coefficients. We will see that this makes the discussion of the resonance matrix and the construction of the mirror transform somewhat more complicated. Still we see the symplectic structure in the Painlevé analysis and that the mirror transform is canonical.

For the double resonance 3, the resonance parameters form a vector subspace of dimension 2. In case $\rho_1\neq 0$, the vectors $(-\rho_2,\rho_1,0,-2\rho_2,2\rho_1,0)$, $(-\rho_3,0,\rho_1,-2\rho_3,0,2\rho_1)$ form a basis of this subspace, so that any vector can be written as a linear combination of the two vectors, with the resonance parameters r_4 and r_5 as coefficients. Under such an arrangement, the Laurent series solutions are of the form

$$\begin{aligned} q_1 &\sim \rho_1(t-t_0)^{-1}-(\rho_2r_4+\rho_3r_5)(t-t_0)^2+\rho_1r_6(t-t_0)^3+\cdots, \\ q_2 &\sim \rho_2(t-t_0)^{-1}+\rho_1r_4(t-t_0)^2+\rho_2r_6(t-t_0)^3+\cdots, \\ q_3 &\sim \rho_3(t-t_0)^{-1}+\rho_1r_5(t-t_0)^2+\rho_3r_6(t-t_0)^3+\cdots, \\ p_1 &\sim -\rho_1(t-t_0)^{-2}-2(\rho_2r_4+\rho_3r_5)(t-t_0)+3\rho_1r_6(t-t_0)^2+\cdots, \\ p_2 &\sim -\rho_2(t-t_0)^{-2}+2\rho_1r_4(t-t_0)+3\rho_2r_6(t-t_0)^2+\cdots, \\ p_3 &\sim -\rho_3(t-t_0)^{-2}+2\rho_1r_5(t-t_0)+3\rho_3r_6(t-t_0)^2+\cdots, \end{aligned}$$

where $2\rho_1^2+2\rho_2^2+2\rho_3^2+1=0$, $\rho_1\neq 0$, and r_4, r_5, r_6 are arbitrary.

For the double resonance 0, the resonance vectors are of the form

$$(a,b,c,-a,-b,-c), \qquad \rho_1 a+\rho_2 b+\rho_3 c=0,$$

where the condition is the tangent space of the surface $2\rho_1^2+2\rho_2^2+2\rho_3^2+1=0$. Under the condition $\rho_1\neq 0$, we may choose $(\rho_1\rho_3,\rho_2\rho_3,\rho_3^2+1/2)$ and $(\rho_1\rho_2,\rho_2^2+1/2,\rho_2\rho_3)$ as a basis of the tangent space. With this arrangement, the resonance matrix becomes

$$\begin{pmatrix} \rho_1 & \rho_1\rho_3 & \rho_1\rho_2 & -\rho_2 & -\rho_3 & \rho_1 \\ \rho_2 & \rho_2\rho_3 & \rho_2^2+1/2 & \rho_1 & 0 & \rho_2 \\ \rho_3 & \rho_3^2+1/2 & \rho_2\rho_3 & 0 & \rho_1 & \rho_3 \\ -2\rho_1 & -\rho_1\rho_3 & -\rho_1\rho_2 & -2\rho_2 & -2\rho_3 & 3\rho_1 \\ -2\rho_2 & -\rho_2\rho_3 & -\rho_2^2-1/2 & 2\rho_1 & 0 & 3\rho_2 \\ -2\rho_3 & -\rho_3^2-1/2 & -\rho_2\rho_3 & 0 & 2\rho_1 & 3\rho_3 \end{pmatrix}.$$

After rescaling and reversing the last three columns, we get a symplectic matrix

$$\begin{pmatrix} \rho_1 & \rho_1\rho_3 & \rho_1\rho_2 & -2\rho_1/5 & -2\rho_1^{-1}\rho_3/3 & -2\rho_1^{-1}\rho_2/3 \\ \rho_2 & \rho_2\rho_3 & \rho_2^2+1/2 & -2\rho_2/5 & 0 & 2/3 \\ \rho_3 & \rho_3^2+1/2 & \rho_2\rho_3 & -2\rho_3/5 & 2/3 & 0 \\ -2\rho_1 & -\rho_1\rho_3 & -\rho_1\rho_2 & -6\rho_1/5 & -4\rho_1^{-1}\rho_3/3 & -4\rho_1^{-1}\rho_2/3 \\ -2\rho_2 & -\rho_2\rho_3 & -\rho_2^2-1/2 & -6\rho_2/5 & 0 & 4/3 \\ -2\rho_3 & -\rho_3^2-1/2 & -\rho_2\rho_3 & -6\rho_3/5 & 4/3 & 0 \end{pmatrix}.$$

As for the mirror transform, the assumption $\rho_1\neq 0$ allows us to introduce the indicial normalization $q_1=\theta^{-1}$. We expand θ', q_2, q_3, p_1, p_2 and p_3 in powers of θ, and then introduce new variables at the resonances of these expansions. This leads to the following mirror transform:

$$\begin{aligned} q_1 &= Q_1^{-1}, \\ q_2 &= Q_2Q_1^{-1}, \\ q_3 &= Q_3Q_1^{-1}, \\ p_1 &= -\sqrt{2}i\sqrt{1+Q_2^2+Q_3^2}Q_1^{-2}+\left(-P_2Q_2-P_3Q_3\right)Q_1-P_1Q_1^2, \\ p_2 &= -\sqrt{2}i\sqrt{1+Q_2^2+Q_3^2}Q_2Q_1^{-2}+P_2Q_1, \\ p_3 &= -\sqrt{2}i\sqrt{1+Q_2^2+Q_3^2}Q_3Q_1^{-2}+P_3Q_1. \end{aligned}$$

The mirror transform is easily verified to be canonical. Moreover, the Hamiltonian for $(Q_1, Q_2, Q_3, P_1, P_2, P_3)$ is given by substituting the mirror transform into the original Hamiltonian (7),

$$\begin{aligned}\bar{H} &= \sqrt{2}i\sqrt{1+Q_2^2+Q_3^2}P_1 \\ &\quad +\frac{1}{2}(P_2^2+P_3^2+P_2^2Q_2^2+P_3^2Q_3^2+2P_2P_3Q_2Q_3)Q_1^2 \\ &\quad +(Q_2P_2+Q_3P_3)P_1Q_1^3+\frac{1}{2}P_1^2Q_1^4,\end{aligned}$$

which is symplectic near the singularity of the original equation (where $1+Q_2^2+Q_3^2$ is close to $\rho_1^{-2}(\rho_1^2+\rho_2^2+\rho_3^2) = -\rho_1^{-2}/2 \neq 0$).

4 Non-autonomous Hamiltonian Systems

In this section, we examine non-autonomous Hamiltonian systems. We will see that everything works out just as in the autonomous case, with only one exception: the new Hamiltonian function is the "regular part" of the original Hamiltonian function after the mirror transform.

4.1 The first Painlevé equation

We rewrite the first Painlevé equation as a non-autonomous Hamiltonian system

$$\begin{cases} \dot{q} = p, \\ \dot{p} = 6q^2+t, \end{cases} \tag{8}$$

with the Hamiltonian

$$H(q,p) = \frac{1}{2}p^2 - 2q^3 - tq.$$

The system has one principal balance:

$$\begin{aligned} q &\sim (t-t_0)^{-2} - \frac{t_0}{10}(t-t_0)^2 - \frac{1}{6}(t-t_0)^3 + r_2(t-t_0)^4 + \cdots, \\ p &\sim -2(t-t_0)^{-3} - \frac{t_0}{5}(t-t_0) - \frac{1}{2}(t-t_0)^2 + 4r_2(t-t_0)^3 + \cdots, \end{aligned}$$

where r_2 is a resonance parameter with resonance 6. The resonance pairing property holds: $(-1) + 6 = 2 + 3$. The resonance matrix is

$$\begin{pmatrix} 2 & 1 \\ -6 & 4 \end{pmatrix}.$$

Since the 2×2 matrix is non-degenerate, a rescaling of the second column can always make it into a symplectic matrix.

Introducing the indicial normalization $q = \theta^{-2}$, we can find the expansion of p in θ:

$$p \sim -2\theta^{-3} - \frac{t}{2}\theta - \frac{1}{2}\theta^2 + r_2\theta^3 + \cdots.$$

From this we deduce the mirror transform

$$\begin{aligned} q &= Q^{-2}, \\ p &= -2Q^{-3} - \frac{t}{2}Q - \frac{1}{2}Q^2 - \frac{1}{2}PQ^3. \end{aligned}$$

It is easy to verify that the transform is canonical and converts (8) into a regular system:

$$\begin{cases} \dot{Q} = 1 + \frac{t}{4}Q^4 + \frac{1}{4}Q^5 + \frac{P}{4}Q^6, \\ \dot{P} = -\frac{t^2}{4}Q - \frac{3t}{4}Q^2 - \left(\frac{1}{2} + tP\right)Q^3 - \frac{5P}{4}Q^4 - \frac{3P^2}{4}Q^5. \end{cases}$$

The system is a Hamiltonian system given by

$$\bar{H}(Q,P) = P + \frac{t^2}{8}Q^2 + \frac{t}{4}Q^3 + \left(\frac{1}{8} + \frac{tP}{4}\right)Q^4 + \frac{P}{4}Q^5 + \frac{P^2}{8}Q^6.$$

Note that the mirror transform converts the original Hamiltonian into

$$\begin{aligned} H &= \frac{1}{2}p^2 - 2q^3 - tq \\ &= Q^{-1} + P + \frac{t^2}{8}Q^2 + \frac{t}{4}Q^3 + \left(\frac{1}{8} + \frac{tP}{4}\right)Q^4 + \frac{P}{4}Q^5 + \frac{P^2}{8}Q^6 \\ &= Q^{-1} + \bar{H}(Q,P). \end{aligned}$$

Therefore, the new Hamiltonian function is the "regular part" of the original Hamiltonian function under the mirror transform.

4.2 A 2-freedom Hamiltonian system

The last example may be easily dismissed as too special because of its low degree of freedom. In this section, we consider the following non-autonomous Hamiltonian

$$H(q_1, q_2, p_1, p_2) = p_1 q_2 + p_2(2q_1^3 + tq_1 + \alpha),$$

where α is some constant. The corresponding Hamiltonian system is

$$\left\{\begin{array}{rcl} \dot{q}_1 &=& q_2, \\ \dot{q}_2 &=& 2q_1^3 + tq_1 + \alpha, \\ \dot{p}_1 &=& -(6q_1^2 + t)p_2, \\ \dot{p}_2 &=& -p_1. \end{array}\right. \tag{9}$$

The system has two principal balances. For the balance with $q_1 \sim (t - t_0)^{-1}$, the resonance matrix demonstrates again the symplectic structure observed before. The detailed calculation is omitted here. It is easy to verify that the mirror transform

$$\begin{array}{rcl} q_1 &=& Q_1^{-1}, \\ q_2 &=& -Q_1^{-2} - \dfrac{t}{2} - \dfrac{1}{2}(1 + 2\alpha)Q_1 + Q_2 Q_1^2, \\ p_1 &=& 2P_2 Q_1^{-3} - \dfrac{1}{2}(1 + 2\alpha)P_2 + 2P_2 Q_2 Q_1 - P_1 Q_1^2, \\ p_2 &=& P_2 Q_1^{-2}, \end{array}$$

is canonical. Moreover, the mirror system is still a Hamiltonian system given by a polynomial Hamiltonian function

$$\begin{array}{rcl} \bar{H} &=& P_1 + \dfrac{1}{4}(1 + 2\alpha)tP_2 + \dfrac{1}{4}\left[(1 + 2\alpha)^2 - 4tQ_2\right] P_2 Q_1 \\ && + \dfrac{1}{2}\left[tP_1 - 3(1 + 2\alpha)P_2 Q_2\right] Q_1^2 + \dfrac{1}{2}\left[(1 + 2\alpha)P_1 + 4P_2 Q_2^2\right] Q_1^3 \\ && - P_1 Q_2 Q_1^4. \end{array}$$

Compared with the function

$$H = p_1 q_2 + p_2(2q_1^3 + tq_1 + \alpha)$$

$$
\begin{aligned}
= \quad & -\frac{1}{2}P_2Q_1^{-2} + P_1 + \frac{1}{4}(1+2\alpha)tP_2 + \frac{1}{4}((1+2\alpha)^2 - 4tQ_2)P_2Q_1 \\
& +\frac{1}{2}\left[tP_1 - 3(1+2\alpha)P_2Q_2\right]Q_1^2 + \frac{1}{2}\left[(1+2\alpha)P_1 + 4P_2Q_2^2\right]Q_1^3 \\
& -P_1Q_2Q_1^4,
\end{aligned}
$$

obtained by applying the mirror transform to the original Hamiltonian, we see again that the new Hamiltonian is the "regular part" of the original Hamiltonian under the mirror transform.

5 Infinite-dimensional Hamiltonian Systems

In this section, we demonstrate the symplectic structure of resonance matrices and compute the mirror transform for infinite-dimensional Hamiltonian systems. Although the mirror transform does not appear to be canonical anymore, the mirror system is still a Hamiltonian system.

We recall that for any functional

$$
H[u] = \int_{-\infty}^{+\infty} h(x, u, u_t, \cdots)dt
$$

of functions $u = (u_1, \cdots, u_n)$ in (x, t) and of their derivatives in t, the functional derivatives are computed by

$$
\frac{\delta H}{\delta u_i} = \sum_{k=0}^{\infty}(-1)^k \frac{\partial^k}{\partial t^k}\left(\frac{\partial h}{\partial (u_i)_{t^k}}\right).
$$

5.1 Burgers' equation

Burgers' equation

$$
u_t + uu_x + u_{xx} = 0
$$

is equivalent to the first two equations in the following space-evolution Hamiltonian system

$$\begin{cases} q_{1x} &= \dfrac{\delta H}{\delta p_1} &= q_2, \\ q_{2x} &= \dfrac{\delta H}{\delta p_2} &= -q_{1t} - q_1 q_2, \\ p_{1x} &= -\dfrac{\delta H}{\delta q_1} &= p_2 q_2 - p_{2t}, \\ p_{2x} &= -\dfrac{\delta H}{\delta q_2} &= -p_1 + p_2 q_1, \end{cases} \tag{10}$$

given by the Hamiltonian functional

$$H = \int_{-\infty}^{\infty} [p_1 q_2 - p_2 q_1 q_2 - p_2 (q_1)_t] dt. \tag{11}$$

System (10) passes the Painlevé test and has a principal balance given by

$$\begin{aligned} q_1 &\sim 2\phi^{-1} + \psi' + r_3\phi - \frac{\psi''}{4}\phi^2 + \cdots, \\ q_2 &\sim -2\phi^{-2} + r_3 - \frac{\psi''}{2}\phi + \cdots, \\ p_1 &\sim 2r_2\phi^{-1} + r_2\psi' + r_2 r_3\phi + \left(-r_4 - \frac{r_2'\psi'}{2} - \frac{r_2\psi''}{4}\right)\phi^2 + \cdots, \\ p_2 &\sim r_2 - \frac{r_2'}{2}\phi^2 + r_4\phi^3 + \cdots, \end{aligned}$$

where $\phi(x,t) = x - \psi(t)$, and ψ, r_2, r_3, r_4 are arbitrary functions of t. The resonances -1, 0, 2, 3 satisfy the resonance pairing property $(-1)+3 = 1+1$, $0+2 = 2+0$. Moreover, the resonance matrix is

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ -4 & 0 & 1 & 0 \\ 2r_2(t) & 2 & r_2(t) & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

After rescaling and reversing the last two columns, we get a symplectic matrix

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ -4 & 0 & 0 & 1 \\ 2r_2(t) & -2/3 & 1/6 & r_2(t) \\ 0 & -1/3 & -1/6 & 0 \end{pmatrix}.$$

Introducing the indicial normalization $q_1 = \theta^{-1}$, we find the following expansions

$$\begin{aligned}
\theta_x &\sim \frac{1}{2} + 2\theta_t\theta - \bar{r}_3\theta^2 + (4\bar{r}_3\theta_t - 8\theta_{tt})\,\theta^3 \\
&\quad + \left(2\bar{r}_3' - 48\bar{r}_3\theta_t^2 + 96\theta_t\theta_{tt}\right)\theta^4 + \cdots, \\
q_2 &\sim -\frac{1}{2}\theta^{-2} - 2\theta_t\theta^{-1} + \bar{r}_3 + (-4\bar{r}_3\theta_t + 8\theta_{tt})\,\theta \\
&\quad + \left(-2\bar{r}_3' + 48\bar{r}_3\theta_t^2 - 96\theta_t\theta_{tt}\right)\theta^2 + \cdots, \\
p_1 &\sim \bar{r}_2\theta^{-1} + \left(-\frac{\bar{r}_4}{2} + 8\bar{r}_2'\theta_t\right)\theta^2 + \left(2\bar{r}_2'' - 48\bar{r}_2'\theta_t^2\right)\theta^3 + \cdots, \\
p_2 &\sim \bar{r}_2 - 2\bar{r}_2'\theta^2 + \bar{r}_4\theta^3 + \left(-4\bar{r}_3\bar{r}_2' - 2\bar{r}_2'' - 6\bar{r}_4\theta_t + 48\bar{r}_2'\theta_t^2\right)\theta^4 + \cdots,
\end{aligned}$$

where $\bar{r}_2$, $\bar{r}_3$ and $\bar{r}_4$ are arbitrary functions of t. By introducing the new variables successively at the resonances, we get the mirror transform

$$\begin{aligned}
q_1 &= Q_1^{-1}, \\
q_2 &= -\frac{1}{2}Q_1^{-2} - 2Q_{1t}Q_1^{-1} + Q_2, \\
p_2 &= P_2, \\
p_1 &= P_2Q_1^{-1} + 2P_{2t}Q_1 - P_1Q_1^2.
\end{aligned}$$

The transform converts the system (10) into

$$\left\{\begin{aligned}
Q_{1x} &= \frac{1}{2} + 2Q_{1t}Q_1 - Q_2Q_1^2, \\
Q_{2x} &= -2Q_2Q_{1t} + 4Q_{1tt} - 2Q_{2t}Q_1, \\
P_{1x} &= -2P_{2t}Q_2 - 4P_{2tt} + (2P_1Q_2 + 2P_{1t})Q_1, \\
P_{2x} &= -2P_{2t}Q_1 + P_1Q_1^2.
\end{aligned}\right. \tag{12}$$

This is a Hamiltonian system. In fact, its Hamiltonian functional can be obtained as follows. Applying the mirror transform to the original Hamiltonian functional (11), we have

$$\begin{aligned}
H &= \int_{-\infty}^{\infty} [p_1q_2 - p_2(q_1)_t - p_2q_1q_2]\,dt \\
&= \int_{-\infty}^{\infty} \Bigg[P_2Q_{1t}Q_1^{-2} - P_{2t}Q_1^{-1} \\
&\quad + \left(\frac{P_1}{2} - 4P_{2t}Q_{1t}\right) + (2P_{2t}Q_2 + 2P_1Q_{1t})\,Q_1 - P_1Q_2Q_1^2\Bigg]dt.
\end{aligned}$$

Then it is easy to verify that the "regular part" of H

$$\bar{H} = \int_{-\infty}^{\infty} \left[\left(\frac{P_1}{2} - 4P_{2t}Q_{1t} \right) + (2P_{2t}Q_2 + 2P_1Q_{1t})\, Q_1 - P_1Q_2Q_1^2 \right] dt$$

is a Hamiltonian functional for (12).

We are however puzzled by the fact that the mirror transform does not appear to be canonical. We also note that the right side of the mirror system contains derivatives in t of an order greater than 1, i.e., the mirror system is not a Kowalevskian system.

5.2 The Korteweg de Vries equation

The KdV equation

$$u_t + uu_x + u_{xxx} = 0$$

is equivalent to the first three equations in the following space-evolution Hamiltonian system

$$\begin{cases} q_{1x} = q_2, & p_{1x} = p_3q_2 - (p_3)_t, \\ q_{2x} = q_3, & p_{2x} = -p_1 + p_3q_1, \\ q_{3x} = -q_1q_2 - (q_1)_t, & p_{3x} = -p_2, \end{cases}$$

given by the Hamiltonian functional

$$H = \int_{-\infty}^{+\infty} [p_1q_2 + p_2q_3 - p_3q_1q_2 - p_3(q_1)_t]\, dt. \qquad (13)$$

The system passes the Painlevé test and has a principal balance given by

$$\begin{aligned} q_1 &\sim -12\phi^{-2} + \psi' + \frac{r_4}{2}\phi^2 + \frac{\psi''}{6}\phi^3 + \frac{r_5}{4}\phi^4 - \frac{r_4'}{48}\phi^5 + \cdots, \\ q_2 &\sim 24\phi^{-3} + r_4\phi + \frac{\psi''}{2}\phi^2 + r_5\phi^3 - \frac{5r_4'}{48}\phi^4 + \cdots, \\ q_3 &\sim -72\phi^{-4} + r_4 + \psi''\phi + 3r_5\phi^2 - \frac{5r_4'}{12}\phi^3 + \cdots, \\ p_1 &\sim -6r_2\phi^{-4} + (-12r_3 + r_2\psi')\, \phi^{-2} - r_2'\phi^{-1} + \left(\frac{11r_2r_4}{12} + r_3\psi' \right) \\ &\quad + \left(\frac{r_2'\psi'}{12} - \frac{r_3'}{5} + \frac{7r_2\psi''}{30} \right) \phi + \left(\frac{r_3r_4}{2} + \frac{r_2r_5}{4} - \frac{r_2r_4\psi'}{24} \right) \phi^2 \end{aligned}$$

$$
\begin{aligned}
&+\left(8r_6+\frac{r_2'r_4}{24}+\frac{r_3'\psi'}{30}-\frac{r_2r_4'}{48}+\frac{r_3\psi''}{6}-\frac{r_2\psi'\psi''}{90}\right)\phi^3+\cdots,\\
p_2 &\sim 2r_2\phi^{-3}-\frac{r_2'}{12}+\frac{r_2r_4}{12}\phi+\left(-\frac{r_3'}{10}+\frac{r_2\psi''}{30}\right)\phi^2\\
&+\left(\frac{r_2r_5}{12}-\frac{r_2''}{72}\right)\phi^3-5r_6\phi^4+\cdots,\\
p_3 &\sim r_2\phi^{-2}+r_3+\frac{r_2'}{12}\phi-\frac{r_2r_4}{24}\phi^2+\left(\frac{r_3'}{30}-\frac{r_2\psi''}{90}\right)\phi^3\\
&+\left(-\frac{r_2r_5}{48}+\frac{r_2''}{288}\right)\phi^4+r_6\phi^5+\cdots,
\end{aligned}
$$

where $\phi(x,t)=x-\psi(t)$, and ψ, r_2, $\cdots$, r_6 are arbitrary functions of t. The resonances -1, 0, 2, 4, 6, 7 satisfy the resonance pairing property $(-1)+7=2+4$, $0+6=3+3$, $2+4=4+2$. The resonance matrix is

$$
\begin{pmatrix}
-24 & 0 & 0 & 1/2 & 1/4 & 0\\
72 & 0 & 0 & 1 & 1 & 0\\
-288 & 0 & 0 & 1 & 3 & 0\\
-24r_2(t) & -6 & -12 & 11r_2(t)/12 & r_2(t)/4 & 8\\
6r_2(t) & 2 & 0 & r_2(t)/12 & r_2(t)/12 & -5\\
2r_2(t) & 1 & 1 & -r_2(t)/24 & -r_2(t)/48 & 1
\end{pmatrix}.
$$

After rescaling and reversing the last three columns, we get a symplectic matrix

$$
\begin{pmatrix}
-24 & 0 & 0 & 0 & 1/4 & 1/2\\
72 & 0 & 0 & 0 & 1 & 1\\
-288 & 0 & 0 & 0 & 3 & 1\\
-24r_2(t) & 12/7 & -12/5 & -1/105 & r_2(t)/4 & 11r_2(t)/12\\
6r_2(t) & -4/7 & 0 & 1/168 & r_2(t)/12 & r_2(t)/12\\
2r_2(t) & -2/7 & 1/5 & -1/840 & -r_2(t)/48 & -r_2(t)/24
\end{pmatrix}.
$$

By a standard procedure, we introduce the indicial normalization $q_1=\theta^{-2}$, expand θ_x, q_2, q_3, p_1, p_2, p_3 in powers of θ, and truncate successively at the resonances to get the mirror transform

$$
q_1 = Q_1^{-2},
$$

$$
\begin{aligned}
q_2 &= -\frac{i}{\sqrt{3}}Q_1^{-3} - 3Q_{1t}Q_1^{-1} + Q_2Q_1, \\
q_3 &= -\frac{1}{2}Q_1^{-4} + 2\sqrt{3}iQ_{1t}Q_1^{-2} + \left(-\frac{iQ_2}{\sqrt{3}} - \frac{9Q_{1t}^2}{2}\right) \\
&\quad +9Q_{1tt}Q_1 + Q_3Q_1^2, \\
p_1 &= \frac{P_3}{2}Q_1^{-4} + \left(\frac{\sqrt{3}iP_2}{2} + \frac{\sqrt{3}iP_3Q_{1t}}{2}\right)Q_1^{-2} - \frac{\sqrt{3}iP_{3t}}{2}Q_1^{-1} \\
&\quad + \left(\frac{iP_3Q_2}{2\sqrt{3}} - \frac{3P_2Q_{1t}}{2}\right) + \left(\frac{3P_{2t}}{2} - \frac{9P_{3t}Q_{1t}}{2} + \frac{9P_3Q_{1tt}}{2}\right)Q_1 \\
&\quad + \left(\frac{P_2Q_2}{2} + P_3Q_3 + \frac{9P_{3tt}}{2}\right)Q_1^2 - \frac{P_1}{2}Q_1^3, \\
p_2 &= \frac{iP_3}{\sqrt{3}}Q_1^{-3} + P_2Q_1^{-1}, \\
p_3 &= P_3Q_1^{-2}.
\end{aligned}
$$

Again, the new variables $(Q_1, Q_2, Q_3, P_1, P_2, P_3)$ satisfy a Hamiltonian system, with the Hamiltonian $\bar{H}$ being the "regular part" of the original Hamiltonian (13) under the mirror transform

$$
\begin{aligned}
H &= \int_{-\infty}^{+\infty} \left[p_1q_2 + p_2q_3 - p_3q_1q_2 - p_3(q_1)_t\right] dt \\
&= \int_{-\infty}^{+\infty} \Bigg[2P_3Q_{1t}Q_1^{-5} - \frac{P_{3t}}{2}Q_1^{-4} + \left(\sqrt{3}iP_2Q_{1t} - 3\sqrt{3}iP_3Q_{1t}^2\right)Q_1^{-3} \\
&\quad + \left(-\frac{\sqrt{3}iP_{2t}}{2} + 3\sqrt{3}iP_{3t}Q_{1t} + \frac{3\sqrt{3}iP_3Q_{1tt}}{2}\right)Q_1^{-2} - \frac{3\sqrt{3}iP_{3tt}}{2}Q_1^{-1} \\
&\quad + \Bigg(\frac{iP_1}{2\sqrt{3}} - \frac{\sqrt{3}iP_{3t}Q_2}{2} - \frac{9P_{2t}Q_{1t}}{2} + \frac{27P_{3t}Q_{1t}^2}{2} + 9P_2Q_{1tt} \\
&\quad - \frac{27P_3Q_{1t}Q_{1tt}}{2}\Bigg) \\
&\quad + \left(\frac{iP_3Q_2^2}{2\sqrt{3}} + P_2Q_3 - 3P_2Q_2Q_{1t} - 3P_3Q_3Q_{1t} - \frac{27P_{3tt}Q_{1t}}{2}\right)Q_1 \\
&\quad + \left(\frac{3P_{2t}Q_2}{2} + \frac{3P_1Q_{1t}}{2} - \frac{9P_{3t}Q_2Q_{1t}}{2} + \frac{9P_3Q_2Q_{1tt}}{2}\right)Q_1^2
\end{aligned}
$$

$$+\left(\frac{P_2Q_2^2}{2}+P_3Q_2Q_3+\frac{9P_{3tt}Q_2}{2}\right)Q_1^3-\frac{P_1Q_2}{2}Q_1^4\Bigg]dt$$

$$= \int_{-\infty}^{+\infty}\Bigg[2P_3Q_{1t}Q_1^{-5}-\frac{P_{3t}}{2}Q_1^{-4}+\left(\sqrt{3}iP_2Q_{1t}-3\sqrt{3}iP_3Q_{1t}^2\right)Q_1^{-3}$$

$$+\left(-\frac{\sqrt{3}iP_{2t}}{2}+3\sqrt{3}iP_{3t}Q_{1t}+\frac{3\sqrt{3}iP_3Q_{1tt}}{2}\right)Q_1^{-2}-\frac{3\sqrt{3}iP_{3tt}}{2}Q_1^{-1}\Bigg]dt$$

$$+\bar{H}[Q_1,Q_2,Q_3,P_1,P_2,P_3].$$

Similar to Burgers' equation, the mirror transform does not appear to be canonical. The mirror system is also not Kowalevskian.

5.3 The nonlinear Schrödinger equation

The (defocusing) nonlinear Schrödinger equation

$$\begin{cases} iu_t+u_{xx}-2u^2v &= 0,\\ -iv_t+v_{xx}-2uv^2 &= 0,\end{cases} \tag{14}$$

is equivalent to the following space-evolution Hamiltonian system

$$\begin{cases} q_{1x} = q_2, & p_{1x} = -2q_1p_2^2-ip_{2t},\\ q_{2x} = 2q_1^2p_2-iq_{1t}, & p_{2x} = -p_1,\end{cases} \tag{15}$$

given by the Hamiltonian functional

$$H=\int_{-\infty}^{+\infty}\left[p_1q_2+p_2^2q_1^2-\frac{i}{2}p_2q_{1t}+\frac{i}{2}p_{2t}q_1\right]dt. \tag{16}$$

The system has a principal balance

$$\begin{aligned} q_1 &\sim r_2\phi^{-1}+\frac{ir_2\psi'}{2}+\left(-\frac{r_2\psi'^2}{12}+\frac{ir_2'}{6}\right)\phi+\left(-r_2^2r_3-\frac{r_2\psi''}{4}\right)\phi^2\\ &\quad+\left(r_2^2r_4-\frac{r_2'^2r_2^{-1}}{12}-\frac{ir_2\psi'\psi''}{6}+\frac{r_2''}{12}\right)\phi^3+\cdots,\\ q_2 &\sim -r_2\phi^{-2}+\left(-\frac{r_2\psi'^2}{12}+\frac{ir_2'}{6}\right)+\left(-2r_2^2r_3-\frac{r_2\psi''}{2}\right)\phi\\ &\quad+\left(3r_2^2r_4-\frac{r_2'^2r_2^{-1}}{4}-\frac{ir_2\psi'\psi''}{2}+\frac{r_2''}{4}\right)\phi^2+\cdots,\end{aligned}$$

$$
\begin{aligned}
p_1 &\sim r_2^{-1}\phi^{-2} + \left(\frac{\psi'^2 r_2^{-1}}{12} - \frac{i r_2' r_2^{-2}}{6}\right) - 2r_3\phi - 3r_4\phi^2 + \cdots,\\
p_2 &\sim r_2^{-1}\phi^{-1} - \frac{i\psi' r_2^{-1}}{2} + \left(-\frac{\psi'^2 r_2^{-1}}{12} + \frac{i r_2' r_2^{-2}}{6}\right)\phi + r_3\phi^2 + r_4\phi^3 + \cdots,
\end{aligned}
$$

where $\phi(x,t) = x - \psi(t)$, and ψ, r_2, r_3, r_4 are arbitrary functions of t with $r_2(t) \neq 0$. The resonances -1, 0, 3, 4 satisfy the resonance pairing property $(-1) + 4 = 1 + 2$, $0 + 3 = 2 + 1$. The resonance matrix is

$$
\begin{pmatrix}
r_2(t) & 1 & -r_2^2(t) & r_2^2(t)\\
-2r_2(t) & -1 & -2r_2^2(t) & 3r_2^2(t)\\
2r_2^{-1}(t) & -r_2^{-2}(t) & -2 & -3\\
r_2^{-1}(t) & -r_2^{-2}(t) & 1 & 1
\end{pmatrix}.
$$

After rescaling and reversing the last two columns, we get a symplectic matrix

$$
\begin{pmatrix}
r_2(t) & 1 & -r_2(t)/10 & r_2^2(t)/6\\
-2r_2(t) & -1 & -3r_2(t)/10 & r_2^2(t)/3\\
2r_2^{-1}(t) & -r_2^{-2}(t) & 3r_2^{-1}(t)/10 & 1/3\\
r_2^{-1}(t) & -r_2^{-2}(t) & -r_2^{-1}(t)/10 & -1/6
\end{pmatrix}.
$$

By the usual process, we find the mirror transform

$$
\left\{
\begin{aligned}
q_1 &= Q_1^{-1},\\
q_2 &= Q_2 Q_1^{-2},\\
p_1 &= -Q_2^3 Q_1^{-2} + 2iQ_2 Q_{1t} Q_1^{-1} - iQ_{2t} - 2P_2 Q_2 Q_1 - P_1 Q_1^2,\\
p_2 &= Q_2^2 Q_1^{-1} - iQ_{1t} + P_2 Q_1^2.
\end{aligned}
\right.
\tag{17}
$$

The new variables (Q_1, Q_2, P_1, P_2) satisfy a regular differential system

$$
\left\{
\begin{aligned}
Q_{1x} &= -Q_2,\\
Q_{2x} &= -iQ_{1t} + 2P_2 Q_1^2,\\
P_{1x} &= -iP_{2t} - 2P_2^2 Q_1,\\
P_{2x} &= P_1.
\end{aligned}
\right.
\tag{18}
$$

We can easily see that the system (18) is again a Hamiltonian system (in fact not much different from (15)), with the Hamiltonian functional given by

$$
\bar{H} = \int_{-\infty}^{+\infty} \Big[- P_1 Q_2 + P_2^2 Q_1^2 - \frac{i}{2} P_2 Q_{1t} + \frac{i}{2} Q_1 P_{2t} \Big] dt.
$$

This functional is again the "regular part" of the original Hamiltonian (16) under mirror transform (17)

$$\begin{aligned} H &= \int_{-\infty}^{+\infty} \Bigg[-\frac{Q_{1t}^2}{2} Q_1^{-2} + \frac{Q_{1tt}}{2} Q_1^{-1} \\ &\qquad -P_1 Q_2 + P_2^2 Q_1^2 + \frac{i}{2} Q_1 P_{2t} - \frac{i}{2} P_2 Q_{1t} \Bigg] dt \\ &= \int_{-\infty}^{+\infty} \left[-\frac{Q_{1t}^2}{2} Q_1^{-2} + \frac{Q_{1tt}}{2} Q_1^{-1} \right] dt + \bar{H}. \end{aligned}$$

The mirror transform for the nonlinear Schrödinger equation is canonical, in contrast to the last two examples. Mirror system (18) is also Kowalevskian.

By eliminating two "intermediate" variables Q_2 and P_1 in (18), we have

$$\left\{ \begin{aligned} iQ_{1t} - Q_{1xx} - 2Q_1^2 P_2 &= 0, \\ -iP_{2t} - P_{2xx} - 2Q_1 P_2^2 &= 0. \end{aligned} \right.$$

This is a focusing nonlinear Schrödinger equation. The transform from the defocusing case to the focusing case is given by

$$\left\{ \begin{aligned} u &= Q_1^{-1}, \\ v &= Q_{1x}^2 Q_1^{-1} - iQ_{1t} + P_2 Q_1^2. \end{aligned} \right.$$

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