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Quasi-triangular structures on Hopf algebras with positive bases

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Abstract. A basis B of a finite dimensional Hopf algebra H is said to be positive if all the structure constants of H relative to B are non-negative. A quasi triangular structure $R \in H \otimes H$ is said to be positive with respect to B if it has non-negative coefficients in the basis $B \otimes B$ of $H \otimes H$. In our earlier work, we showed that finite dimensional Hopf algebras with positive bases are in one-to-one correspondence with group factorizations $G = G_+G_-$. In this paper, we show that positive quasi-triangular structures on such Hopf algebras are given by a pair of homomorphisms $\xi, \eta : G_+ \to G_-$ satisfying some compatibility conditions. Further properties of such structures are also discussed.

1. Introduction

Consider a finite dimensional Hopf algebra H over $\mathbb{C}$ with a basis B such that all the structure constants with respect to this basis are non-negative. In [**LYZ1**] we proved that any such Hopf algebra is isomorphic to the bicrossproduct Hopf algebra $H(G; G_+, G_-)$ coming from a factorization $G = G_+G_-$ of a finite group G. The construction of $H(G; G_+, G_-)$ has already appeared in [**Mj**] [**T**] and will be recalled in Section 2. We also showed that such Hopf algebras are exactly the linearizations of Hopf algebras in the category of sets with correspondences as morphisms.

In this paper, we further study quasi-triangular structures $R \in H \otimes H$ that are positive in the sense that the coefficients of R in the basis $B \otimes B$ of $H \otimes H$ are non-negative. In Theorem 2.3, we show that for $H = H(G; G_+, G_-)$ and $B = G$, such structures are in one-to-one correspondence with pairs of group homomorphisms $\xi, \eta : G_+ \to G_-$ such that

$$uv = ({}^{\xi(u)}v)(u^{\eta(v)}), \quad \xi({}^{x}u)x^{\,u} = x\xi(u), \quad \eta({}^{x}u)x^{\,u} = x\eta(u),$$

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for all $u, v \in G_+$ and $x \in G_-$. Such data are further interpreted in Theorems 3.1 and 3.2 as certain extra structures on the group factorization $G = G_+G_-$.

We also show, in Theorem 4.3, that positive quasi-triangular Hopf algebras are quasi-equivalent to certain normal forms. When restricted to triangular structures, this implies that positive triangular Hopf algebras are twistings of group algebras. See Corollary 5.2. The result recovers a construction by Etingof and Gelaki [**EG**].

Similar to our theory of Hopf algebras with positive bases, positive quasi-triangular structures also have set-theoretical interpretations as bisections of some groupoids. In particular, they are related to solutions of groupoid-theoretical Yang-Baxter equation introduced in [**WX**]. This sets up the foundation of our study of set-theoretical solutions of Yang-Baxter equation in [**LYZ2**].

The Hopf algebra $H(G; G_+, G_-)$ has been studied in [**BGM**] under a different context. Despite what the title and the abstract may suggest, only the standard quasi-triangular structure of the Drinfel'd double of $H(G; G_+, G_-)$ is studied in [**BGM**]. Since the Drinfel'd double of a Hopf algebra with a positive basis still has a positive basis, and the canonical quasi-triangular structure is easily seen to be positive, the results of [**BGM**] concerning braiding fits nicely into our general theory.

Finally, we would like to point out that our main results do not seem to apply to general bialgebras with positive bases.

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2. Positive quasi-triangular structures

A *factorization* $G = G_+G_-$ of a group G consists of two subgroups G_+ and G_- such that any $g \in G$ can be written as $g = g_+g_-$ for unique $g_+ \in G_+$ and $g_- \in G_-$. The group factorization is also called *matched pair* in [**Mj**] [**T**]. We will denote $(g_+)^{-1} \in G_+$ and $(g_-)^{-1} \in G_-$ simply by g_+^{-1} and g_-^{-1}.

By considering the inverse map in a group factorization, we see that for every $g \in G$, there are unique $g_+, \bar{g}_+ \in G_+$ and $g_-, \bar{g}_- \in G_-$ such that

$$g = g_+g_- = \bar{g}_-\bar{g}_+.$$

This induces the following actions of G_+ and G_- on each other (from left and from right)

$$\begin{aligned}
G_- \times G_+ &\to G_+, \qquad (\bar{g}_-, \bar{g}_+) \mapsto g_+ = {}^{\bar{g}_-}\bar{g}_+,\\
G_- \times G_+ &\to G_-, \qquad (\bar{g}_-, \bar{g}_+) \mapsto g_- = \bar{g}_-^{\bar{g}_+},\\
G_+ \times G_- &\to G_+, \qquad (g_+, g_-) \mapsto \bar{g}_+ = g_+^{g_-},\\
G_+ \times G_- &\to G_-, \qquad (g_+, g_-) \mapsto \bar{g}_- = {}^{g_+}g_-.
\end{aligned}$$

By definition, we have

$$g_+g_- = \left({}^{g_+}g_-\right)\left(g_+^{g_-}\right), \quad g_-g_+ = \left({}^{g_-}g_+\right)\left(g_-^{g_+}\right). \tag{2.1}$$

Moreover, the actions have the following properties

$$\begin{cases} {}^{g_+}(g_-h_-) = {}^{g_+}g_- \; {}^{(g_+^{g_-})}h_-, & (h_+g_+)^{g_-} = h_+^{({}^{g_+}g_-)}\, g_+^{g_-},\\ {}^{g_-}(g_+h_+) = {}^{g_-}g_+ \; {}^{(g_-^{g_+})}h_+, & (h_-g_-)^{g_+} = h_-^{({}^{g_-}g_+)}\, g_-^{g_+}. \end{cases} \tag{2.2}$$

$$\left\{ \begin{array}{ll} (g_+{}^{g_-})^{-1} = {}^{(g_-^{-1})}g_+^{-1}, & ({}^{g_-}g_+)^{-1} = g_+^{-1\,(g_-^{-1})}, \\ (g_-{}^{g_+})^{-1} = {}^{(g_+^{-1})}g_-^{-1}, & ({}^{g_+}g_-)^{-1} = g_-^{-1\,(g_+^{-1})}. \end{array} \right. \tag{2.3}$$

A Hopf algebra $H(G; G_+, G_-)$ has been constructed from a unique factorization $G = G_+G_-$ of a finite group. See [**Mj**] [**T**]. More precisely, $H(G; G_+, G_-)$ is the vector space spanned $\mathbb{C}G$ with the following Hopf algebra structure

$$\left\{ \begin{array}{ll} \text{multiplication:} & \{g\}\{h\} = \delta_{g_+^{g_-}, h_+}\{gh_-\} \\ \text{unit:} & 1 = \sum_{g_+ \in G_+}\{g_+\} \\ \text{co-multiplication:} & \Delta\{g\} = \sum_{h_+ \in G_+}\{g_+h_+^{-1}({}^{h_+}g_-)\} \otimes \{h_+g_-\} \\ \text{co-unit:} & \epsilon\{g\} = \delta_{g_+,e} \\ \text{antipode:} & S\{g\} = \{g^{-1}\} \end{array} \right.$$

where we use $\{g\}$ to denote the group element $g \in G$ considered as an element of $H(G; G_+, G_-)$. We remark that the algebra structure on $H(G; G_+, G_-)$ is that of the cross-product of the group algebra $\mathbb{C}G_-$ of G_- and the function algebra $\mathbb{C}(G_+)$ of G_+ with respect to the above right action of G_- on G_+. The coalgebra structure can be similarly described in terms of the left action of G_+ on G_-. The Hopf algebra $H(G; G_+, G_-)$ has G as the obvious positive basis. In [**LYZ1**], we proved the following classification theorem (the rescaling by positive numbers is necessary because it preserves positive bases).

Theorem 2.1. *Given any finite dimensional Hopf algebra H over $\mathbb{C}$ with a positive basis B, we can always rescale B by some positive numbers, so that (H, B) is isomorphic to $(H(G; G_+, G_-), G)$ for a unique group G and a unique group factorization $G = G_+G_-$.*

Recall that a *quasi-triangular structure* on a Hopf algebra H is an invertible element $R \in H \otimes H$ such that

$$\tau\Delta(a) = R\Delta(a)R^{-1}, \qquad \text{for all } a \in H \tag{2.4}$$

$$(\Delta \otimes id)R = R_{13}R_{23}, \qquad (id \otimes \Delta)R = R_{13}R_{12} \tag{2.5}$$

where $\tau(a \otimes b) = b \otimes a$. We also recall that the conditions imply $(\varepsilon \otimes id)R = (id \otimes \varepsilon)R = 1$ and $(S \otimes id)R = (id \otimes S)R = R^{-1}$.

Definition 2.2. Let H be a finite dimensional Hopf algebra over $\mathbb{C}$ with a positive basis B. An element $R \in H \otimes H$ is said to be *positive* if it is a linear combination of the basis elements $\{b_1\} \otimes \{b_2\}$, $b_1, b_2 \in B$, with non-negative coefficients. A quasi-triangular structure on H is said to be positive if it is given by a positive element.

By Theorem 2.1, we may restrict our attention to $H = H(G; G_+, G_-)$ and $B = G$ in the classification of positive quasi-triangular structures.

Theorem 2.3. *Let $G = G_+G_-$ be a finite group factorization. Let $\xi, \eta : G_+ \to G_-$ be two group homomorphisms such that*

$$uv = ({}^{\xi(u)}v)(u^{\eta(v)}), \tag{2.6}$$

$$\xi({}^{x}u)x^{u} = x\xi(u), \tag{2.7}$$

$$\eta({}^{x}u)x^{u} = x\eta(u), \tag{2.8}$$

for all $u, v \in G_+$, and $x \in G_-$. Then

$$R = \sum_{u,v \in G_+} \{u\,(\eta(v)^u)^{-1}\} \otimes \{v\xi(u)\}$$

is a positive quasi-triangular structure on $H(G; G_+, G_-)$. Conversely, every positive quasi-triangular structure on $H(G; G_+, G_-)$ is given by the construction above.

The theorem will be proved in Section 7. In the subsequent discussion and the proof of the theorem, we need the following technical result.

Lemma 2.4. *The conditions* (2.6), (2.7), (2.8) *imply*

$$\xi(u)^v = \xi(u^{\eta(v)}), \tag{2.9}$$

$$^u\eta(v) = \eta(^{\xi(u)}v). \tag{2.10}$$

Moreover, each of (2.6) *through* (2.10) *is equivalent to the corresponding condition below*

$$uv = (^{\eta(u)}v)(u^{\xi(v)}), \tag{2.11}$$

$$^u x\xi(u^x) = \xi(u)x, \tag{2.12}$$

$$^u x\eta(u^x) = \eta(u)x, \tag{2.13}$$

$$^v\xi(u) = \xi(^{\eta(v)}u), \tag{2.14}$$

$$\eta(v)^u = \eta(v^{\xi(u)}). \tag{2.15}$$

Proof. Given conditions (2.6) and (2.7), the following computation

$$\xi(u)\xi(v) \overset{(2.6)}{=} \xi(^{\xi(u)}v)\xi(u^{\eta(v)}) \overset{(2.7)}{=} \xi(u)\xi(v)(\xi(u)^v)^{-1}\xi(u^{\eta(v)})$$

shows that (2.9) is also true. Similarly, (2.6) and (2.8) imply (2.10).

If (2.6) holds, then we have

$$v^{-1}u^{-1} \overset{(2.6)}{=} \left(^{\xi(v)^{-1}}(u^{-1})\right)\left((v^{-1})^{\eta(u)^{-1}}\right) \overset{(2.3)}{=} (u^{\xi(v)})^{-1}(^{\eta(u)}v)^{-1}.$$

By taking inverse of both sides, we get (2.11). Using (2.3) in the similar way, we may prove that (2.11) implies (2.6). The other equivalences can also be proved similarly.

□

3. The meaning of positive quasi-triangular structures

The statement of Theorem 2.3 appears to be rather technical. In this section, we provide two interpretations of these technicalities as additional structures on the group factorization.

Theorem 3.1. *Let $G = G_+G_-$ be a finite group factorization. Let $\xi, \eta : G_+ \to G_-$ be two group homomorphisms, and denote*

$$G'_+ = \{u\xi(u^{-1}) : u \in G_+\}, \qquad G''_+ = \{\eta(u^{-1})u : u \in G_+\},$$

$$F(u\xi(u^{-1})) = \eta(u)u^{-1} : \quad G'_+ \to G''_+.$$

Then the conditions (2.6), (2.7), (2.8) *in Theorem* 2.3 *are equivalent to*

(a) *Both G'_+ and G''_+ are normal subgroups of G;*

(b) *F is a group isomorphism.*

We note that the map F in the theorem is well-defined, by the uniqueness of the factorization.

PROOF. From

$$u\xi(u^{-1})v\xi(v^{-1}) \overset{(2.1)}{=} u({}^{\xi(u^{-1})}v)(\xi(u^{-1})^{v})\xi(v^{-1}), \tag{3.1}$$

$$\eta(u)u^{-1}\eta(v)v^{-1} \overset{(2.1)}{=} \eta(u)({}^{u^{-1}}\eta(v))((u^{-1})^{\eta(v)})v^{-1}, \tag{3.2}$$

we see that the condition (b) means $(u({}^{\xi(u^{-1})}v))^{-1} = ((u^{-1})^{\eta(v)})v^{-1}$. This is exactly (2.6).

The fact that G'_+ is normal means that for any $u \in G_+$ and $x \in G_-$, we can find $v \in G_+$ such that $xu\xi(u^{-1}) = v\xi(v^{-1})x$. By the uniqueness of factorization, this means

$${}^{x}u = v, \qquad (x^{u})\xi(u^{-1}) = \xi(v^{-1})x.$$

This is clearly equivalent to (2.7). Similarly, the fact that G''_+ is normal is equivalent to (2.8).

It remains to show that (2.6), (2.7), (2.8) imply G'_+ and G''_+ are indeed subgroups. By (3.1) and the definition of G'_+, G'_+ is a subgroup if and only if $\xi(u({}^{\xi(u^{-1})}v))^{-1} = (\xi(u^{-1})^{v})\xi(v^{-1})$. This equality is verified as follows

$$\begin{aligned}\xi\left(u({}^{\xi(u^{-1})}v)\right)^{-1} &\overset{(2.6)}{=} \xi\left(v((u^{-1})^{\eta(v)})^{-1}\right)^{-1} \\ &= \xi((u^{-1})^{\eta(v)})\xi(v^{-1}) \overset{(2.9)}{=} (\xi(u^{-1})^{v})\xi(v^{-1}),\end{aligned}$$

where we used the fact that (2.6) and (2.7) imply (2.9) (see Lemma 2.4). Thus we conclude that G'_+ is indeed a subgroup. Similarly, G''_+ is also a subgroup.

□

Next we give an alternative description for the data $(G = G_+G_-, \xi, \eta)$. Let G_- be a group acting on another group A as automorphisms, with the action denoted by $(x, a) \mapsto x \cdot a : G_- \times A \to A$. Then we have the semi-direct product group $G = A \rtimes G_-$, with the group structure given by

$$(ax)(by) = a(x \cdot b)xy, \qquad a, b \in A, \quad x, y \in G_-.$$

A map $\zeta : A \to G_-$ is called a *shift* if

$$\zeta(a)\zeta(b) = \zeta(a(\zeta(a) \cdot b)). \tag{3.3}$$

The condition is equivalent to the fact that $\{a\zeta(a) : a \in A\}$ is a subgroup of G. Moreover, if ζ is bijective, then ζ is a shift if and only if $\zeta^{-1} : G_- \to A$ is a 1-cocycle of G_- with coefficients in A.

THEOREM 3.2. *There is a one-to-one correspondence between*

1. *triples $(G = G_+G_-, \xi, \eta)$ satisfying the conditions of Theorem* 3.1*;*
2. *triples $(G = A \rtimes G_-, \zeta, F)$, where $\zeta : A \to G_-$ is a shift and F is an automorphism of G, such that $F(x) = x$ for any $x \in G_-$ and $F(a)a \in G_-$ for any $a \in A$.*

Specifically, the correspondence is the following. Given $(G = G_+G_-, \xi, \eta)$, we define

$$
\begin{aligned}
&A = G'_+ = \{u\xi(u^{-1}) : u \in G_+\},\\
&F(u\xi(u^{-1})x) = \eta(u)u^{-1}x, \quad \text{for } u \in G_+ \text{ and } x \in G_- \\
&\zeta(u\xi(u^{-1})) = \xi(u), \quad \text{for } u \in G_+
\end{aligned} \tag{3.4}
$$

Moreover, since A is a normal subgroup, conjugations by elements in G_- give an action of G_- on A as automorphisms. Conversely, given $(G = A \rtimes G_-, \zeta, F)$, we define

$$
\begin{aligned}
&G_+ = \{a\zeta(a) : a \in A\}\\
&\xi = P|_{G_+}\\
&\eta = P \circ F^{-1}|_{G_+}
\end{aligned} \tag{3.5}
$$

where P is the natural homomorphism $G = A \rtimes G_- \to G_-$, $ax \mapsto x$.

Proof of Theorem 3.2. First we show that if $(G = G_+G_-, \xi, \eta)$ satisfies Theorem 3.1, then the construction (3.4) is as described in Theorem 3.2.

For $a = u\xi(u^{-1})$, we have $a\zeta(a) = u$. Therefore the subset $\{a\zeta(a) : a \in A\} = G_+$ is a subgroup of G. This implies ζ is a shift.

By its very definition, F is a homomorphism if and only if $F : G'_+ \to G''_+$ is an equivariant map with respect to the G_--actions defined by conjugations. For $x \in G_-$ and $a = u\xi(u^{-1}) \in G'_+$, the action of x on a is

$$x \cdot a = xu\xi(u^{-1})x^{-1} = v\xi(v^{-1}), \qquad v = (xu)_+ = {}^x u,$$

and the action of x on $F(a)$ is

$$x \cdot F(a) = x\eta(u)u^{-1}x^{-1} = \eta(w)w^{-1}, \qquad w = (xu)_+ = {}^x u.$$

We conclude from this that $v = w$ and $F(x \cdot a) = x \cdot F(a)$.

Finally, we have $F(x) = x$ for any $x \in G_-$ from the definition. Moreover, for any $a = u\xi(u^{-1}) \in A$, we have

$$F(a)a = \eta(u)u^{-1}u\xi(u^{-1}) = \eta(u)\xi(u^{-1}) \in G_-.$$

Now we turn to the construction (3.5).

First of all, since ζ is a shift, we know G_+ is a subgroup of G. Moreover, for any $a \in A$ and $x \in G_-$, the decomposition $ax = (a\zeta(a))(\zeta(a)^{-1}x)$ gives the factorization $G = G_+G_-$.

Since P and F^{-1} are homomorphisms, ξ and η are also homomorphisms.

We express an element in G as ux for unique $u \in G_+$ and $x \in G_-$. The element is in A if and only if it is in the kernel of P. Since $P(ux) = P(u)P(x) = \xi(u)x$, we see that A consists of elements of the form $u\xi(u^{-1})$, $u \in G_+$. In other words, we have $A = G'_+$, which in particular implies G'_+ is a normal subgroup. Similarly, by considering those elements xu in the kernel of $P \circ F^{-1}$, we conclude that $G''_+ = F(A)$. Since F is an automorphism, G''_+ is also a normal subgroup.

Since $F(G'_+) = G''_+$, for any $u \in G_+$, we can find $v \in G_+$ such that $F(u\xi(u^{-1})) = \eta(v)v^{-1}$. Then by condition (b), we have $\eta(v^{-1})vu\xi(u^{-1}) = F(u\xi(u^{-1}))u\xi(u^{-1}) \in G_-$. This implies $uv \in G_-$. On the other hand, $u, v \in G_+$ implies $uv \in G_+$. Therefore by the uniqueness of the factorization, we have $uv = e$. Consequently, the formula $F(u\xi(u^{-1})) = \eta(u)u^{-1}$ holds.

∎

We finish the section with an example.

Any group factorization $G = G_+G_-$ induces another group factorization $\tilde{G} = G \times G = \tilde{G}_+\tilde{G}_-$, with

$$\tilde{G}_+ = \{(g_+, g_-) : g_+ \in G_+, g_- \in G_-\}, \qquad \tilde{G}_- = \{(g, g) : g \in G\}.$$

The Hopf algebra induced by this group factorization is in fact the Drinfel'd double of $H(G; G_+, G_-)$ (see [**LYZ1**]).

Consider homomorphisms

$$\begin{cases} \xi(g_+, g_-) = (g_-, g_-) \\ \eta(g_+, g_-) = (g_+, g_+) \end{cases} : \qquad \tilde{G}_+ \to \tilde{G}_-.$$

The induced subgroups $\tilde{G}'_+ = G \times \{e\}$ and $\tilde{G}''_+ = \{e\} \times G$ (as in Theorem 3.1) are clearly normal. It is also easy to see that the map $F : \tilde{G}'_+ \to \tilde{G}''_+$ is given by $F(a, e) = (e, a)$, which is clearly a group isomorphism. The quasi-triangular structure induced by these data is in fact the standard one on the Drinfel'd double of $H(G; G_+, G_-)$.

To find the alternative description, we use the identification

$$\tilde{G}'_+ \cong G : (a, e) \leftrightarrow a; \qquad \tilde{G}_- \cong G : (g, g) \leftrightarrow g. \tag{3.6}$$

Then the equality $(g, g)(a, e)(g, g)^{-1} = (gag^{-1}, e)$ implies that G acts on $A = G$ by conjugations. Since $(a, e) = u\xi(u^{-1})$ for $u = (a_+, a_-^{-1})$, the 1-cycle is

$$\zeta(a) = a_-^{-1}$$

after the identification (3.6). Moreover, since $(e, a) = (a^{-1}, e)(a, a)$ with respect to the group factorization $\tilde{G} = \tilde{G}'_+\tilde{G}_-$, the automorphism on $G \rtimes_{\mathrm{conj}} G$ is

$$F(a \rtimes g) = F(a \rtimes e)F(e \rtimes g) = (a^{-1} \rtimes e)(e \rtimes a)(e \rtimes g) = a^{-1} \rtimes ag.$$

4. Comparing positive quasi-triangular structures

Let $G = G_+G_-$ and $G = G'_+G_-$ be two factorizations of a finite group G. Then we have two Hopf algebra structures $H(G; G_+, G_-)$ and $H(G; G'_+, G_-)$ on $\mathbb{C}G$. In this section, we will show that the two Hopf algebra structures are quasi-isomorphic. We recall that a quasi-isomorphism between Hopf algebras H and H' consists of an algebra isomorphism $\phi : H' \to H$ and an invertible element $T \in H \otimes H$ satisfying

$$(\phi \otimes \phi)\Delta(a) = T(\Delta\phi(a))T^{-1}, \qquad a \in H' \tag{4.1}$$

and

$$(T \otimes 1)(\Delta \otimes id)T = (1 \otimes T)(id \otimes \Delta)T. \tag{4.2}$$

Moreover, if R is a quasi-triangular structure on H, then

$$R' = (\phi \otimes \phi)^{-1}((\tau T)RT^{-1}) \tag{4.3}$$

is a quasi-triangular structure on H'.

To avoid confusion about two structures on the same space $\mathbb{C}G$, we consider $G = G_+G_-$ as the "standard" factorization, and $G = G'_+G_-$ as the "shifting" of the standard factorization. All the notations in Section 2 refer to operations relative to the factorization $G = G_+G_-$ and the Hopf algebra structure $H(G; G_+, G_-)$. For any $g \in G$, we use $\{g\}$ and $\{g\}'$ to denote g considered as an element in $H(G; G_+, G_-)$ and in $H(G; G'_+, G_-)$, respectively.

The uniqueness of the two factorizations give rise to a "shifting map" $\sigma : G_+ \to G_-$ such that

$$G'_+ = \{\sigma(u)u : u \in G_+\}. \tag{4.4}$$

The fact that G'_+ is a subgroup implies that for any $u, v \in G_+$,

$$\sigma((u^{\sigma(v)})v) = \sigma(u)({}^u\sigma(v)). \tag{4.5}$$

By solving

$$\sigma(u)ux = y\sigma(v)v, \qquad u, v \in G_+, \quad x, y \in G_-,$$

for v and y, we have

$$v = u^x, \quad y = \sigma(u)({}^u x)\sigma(v)^{-1} = \sigma(u)({}^u x)\sigma(u^x)^{-1}.$$

Therefore the left action of G'_+ on G_- and the right action of G_- on G'_+ are given by

$$G'_+ \times G_- \to G'_+, \qquad (\sigma(u)u, x) \mapsto \sigma(u^x)u^x, \tag{4.6}$$

$$G'_+ \times G_- \to G_-, \qquad (\sigma(u)u, x) \mapsto \sigma(u)({}^u x)\sigma(u^x)^{-1}. \tag{4.7}$$

PROPOSITION 4.1. *Denote*

$$\phi\{\sigma(u)ux\}' = \{ux\} : \quad H(G; G'_+, G_-) \to H(G; G_+, G_-), \tag{4.8}$$

and

$$T = \sum_{u,v \in G_+} \{u\sigma(v)\} \otimes \{v\}. \tag{4.9}$$

Then (ϕ, T) is a quasi-isomorphism of Hopf algebras.

PROOF. First, from the action (4.6), we have

$$\{\sigma(u)ux\}'\{\sigma(v)vy\}' = \delta_{u^x,v}\{\sigma(u)uxy\}'.$$

This implies that ϕ preserves the multiplication. It is also easy to see that ϕ preserves the unit. Therefore ϕ is an isomorphism of algebras.

Next, it is easy to verify that

$$T^{-1} = \sum_{u,v \in G_+} \{u\sigma(v)^{-1}\} \otimes \{v\}.$$

Then for any $g_+ \in G_+$ and $g_- \in G_-$, we have

$$\begin{aligned} & T(\Delta\phi\{\sigma(g_+)g_+g_-\}')T^{-1} \\ = & \sum_{h_+ \in G_+} \left\{ \left((g_+h_+)^{\sigma(h_+)^{-1}} \right) \sigma(h_+)({}^{h_+}g_-)\sigma(h_+^{g_-})^{-1} \right\} \otimes \{h_+g_-\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \Delta\{\sigma(g_+)g_+g_-\}' \\ \overset{(4.7)}{=} & \sum_{h_+ \in G_+} \{\sigma(g_+)g_+(\sigma(h_+)h_+)^{-1}\sigma(h_+)({}^{h_+}g_-)\sigma(h_+^{g_-})^{-1}\}' \otimes \{\sigma(h_+)h_+g_-\}'. \end{aligned}$$

Since G'_+ is a subgroup, we have

$$\sigma(g_+)g_+(\sigma(h_+)h_+)^{-1} = \sigma(w)w$$

for some $w \in G_+$. By considering the G_+-components in the factorization $G = G_-G_+$, we find $w = (g_+h_+^{-1})^{\sigma(h_+)^{-1}}$. Therefore

$$\sigma(g_+)g_+(\sigma(h_+)h_+)^{-1} = \sigma\left((g_+h_+^{-1})^{\sigma(h_+)^{-1}}\right)(g_+h_+^{-1})^{\sigma(h_+)^{-1}},$$

and

$$\begin{aligned}&(\phi \otimes \phi)\Delta\{\sigma(g_+)g_+g_-\}' \\ =\; &\sum_{h_+ \in G_+} \left\{\left((g_+h_+^{-1})^{\sigma(h_+)^{-1}}\right)\sigma(h_+)({}^{h_+}g_-)\sigma(h_+^{g_-})^{-1}\right\} \otimes \{h_+g_-\}.\end{aligned}$$

This completes the verification of (4.1).

Finally, it is easy to compute the following

$$\begin{aligned}(T \otimes 1)(\Delta \otimes id)T &= \sum_{u,v,w \in G_+} \{u\sigma(v)({}^{v}\sigma(w))\} \otimes \{v\sigma(w)\} \otimes \{w\}, \\ (1 \otimes T)(id \otimes \Delta)T &= \sum_{u,v,w \in G_+} \{u\sigma(v^{\sigma(w)}w)\} \otimes \{v\sigma(w)\} \otimes \{w\}.\end{aligned}$$

By (4.5), we conclude that (4.2) holds.

□

Now we apply the proposition to the special case in Theorems 2.3 and 3.1. Note that with $\sigma(u) = \xi(u^{-1})$, the group G'_+ in (4.4) is the same as the one given in Theorem 3.1 (this can be seen by taking inverse). In particular, the quasi-triangular structure R on $H(G; G_+, G_-)$ can be transformed to into the quasi-triangular structure (4.3) on $H(G; G'_+, G_-)$. An easy computation gives

$$(\tau T)RT^{-1} = \sum_{u,v \in G_+} \{u\eta(v)^{-1}\xi(v)\} \otimes \{v\},$$

so that

$$R' = \sum_{u,v \in G_+} \{\xi(u^{-1})u\eta(v)^{-1}\xi(v)\}' \otimes \{\xi(v^{-1})v\}'. \tag{4.10}$$

By Theorem 2.3, R' is given by homomorphisms $\xi', \eta' : G'_+ \to G_-$. From the second component in (4.10), we have $\xi'(\xi(v^{-1})v) = e$. By (2.15), the triviality of ξ' implies that the first component in (4.10) is of the form $\{\xi(u^{-1})u\eta'(\xi(v^{-1})v)^{-1}\}'$. Therefore we conclude that

$$\xi'(\xi(u^{-1})u) = e, \qquad \eta'(\xi(v^{-1})v) = \xi(v)^{-1}\eta(v). \tag{4.11}$$

Definition 4.2. Let R be a positive quasi-triangular structure on $H(G; G_+, G_-)$ given by homomorphisms $\xi, \eta : G_+ \to G_-$ as in Theorem 2.3. We call R *normal* if $\xi(u) = e$ for all $u \in G_+$. We call the pair $(H(G; G_+, G_-), R)$ normal if R is normal.

Thus for the special case in Theorem 2.3, Lemma 4.1 implies the following.

Theorem 4.3. *Every pair (H, R), where H is a finite dimensional Hopf algebra with a positive basis and R is a positive quasi-triangular structure in this basis, is quasi-isomorphic to a normal one.*

5. Positive triangular structures

A quasi-triangular structure R is *triangular* if it further satisfies $(\tau R)R = 1 \otimes 1$. For the positive quasi-triangular structure R given by Theorem 2.3, we have

$$(\tau R)R = \sum \{v\xi(u) \left(\eta(\bar{v})^{\bar{u}}\right)^{-1}\} \otimes \{u \left(\eta(v)^{u}\right)^{-1} \xi(\bar{u})\},$$

where the summation is over all $u, v, \bar{u}, \bar{v} \in G_+$ satisfying

$$v^{\xi(u)} = \bar{u}, \qquad {}^{\eta(v)}u = \bar{v}. \tag{5.1}$$

Since $1 \otimes 1 = \sum_{u,v \in G_+} \{u\} \otimes \{v\}$, we see that R is triangular if and only if (5.1) implies

$$\xi(u) = \eta(\bar{v})^{\bar{u}}, \qquad \eta(v)^{u} = \xi(\bar{u}). \tag{5.2}$$

Note that the first equality in (5.1) implies $\eta(v)^u = \eta(v^{\xi(u)}) = \eta(\bar{u})$. Therefore under the assumption (5.1), the second equality of (5.2) is equivalent to $\xi = \eta$. Furthermore, the property $\xi = \eta$ and (5.1) imply

$$\eta(\bar{v})\bar{u} = \eta({}^{\eta(v)}u)\bar{u} = \xi({}^{\eta(v)}u)\bar{u} \overset{(2.14)}{=} ({}^{v}\xi(u))(v^{\xi(u)}) = v\xi(u).$$

In particular, the first equality of (5.2) also holds. Thus we conclude that R is triangular if and only if $\xi = \eta$.

Theorem 5.1. *There is a one-to-one correspondence between the following data*

1. *a finite dimensional positive triangular Hopf algebra;*
2. *a finite group factorization $G = G_+G_-$, and a homomorphism $\xi : G_+ \to G_-$ satisfying $uv = ({}^{\xi(u)}v)(u^{\xi(v)})$ and $\xi({}^{x}u)x^{u} = x\xi(u)$;*
3. *a finite group factorization $G = G_+G_-$, and a homomorphism $\xi : G_+ \to G_-$ such that $A = \{u\xi(u^{-1}) : u \in G_+\}$ is an abelian normal subgroup;*
4. *a finite group G_-, a finite abelian group A acted upon by G_- as automorphisms, and a shift $\zeta : G_- \to A$.*

Proof. We already know that the first item is equivalent to the construction in Theorem 2.3 with $\xi = \eta$. By applying the conditions in Theorem 2.3 to the special case $\xi = \eta$, we see that the first two items are equivalent.

If we apply Theorem 3.1 to the special case $\xi = \eta$, then we need

$$G_+^{'} = \{u\xi(u^{-1}) : \ u \in G_+\}, \qquad G_+^{''} = \{\xi(u^{-1})u : \ u \in G_+\}$$

to be normal subgroups, and

$$F(u\xi(u^{-1})) = \xi(u)u^{-1} : \quad G'_+ \to G''_+$$

to be homomorphic. Since $\xi(u)u^{-1} = (u\xi(u^{-1}))^{-1}$, we have $G_+^{'} = G_+^{''}$ and $F(a) = a^{-1}$. Thus Theorem 3.1 implies the first and the third items are equivalent.

Finally, the fourth item refers to the alternative description in Theorem 3.2. From the discussion above, we see that a positive triangular structure means $A = G'_+$ is abelian, and

$$F(ax) = a^{-1}x, \qquad a \in A, \, x \in G_-.$$

Since these already imply the conditions in the third item are satisfied, we see that there is no further condition on the shift ζ. This proves the equivalence to the fourth item.

$\square$

Now let us apply the theory of Section 4 to normalize positive triangular structures. Since $\xi = \eta$, both ξ' and η' are trivial by (4.11). Another way to see the triviality is to use the fact that if R is triangular, then R' in (4.3) is also triangular. In particular, we have $\xi' = \eta'$. Since ξ' is trivial, so is η'. Anyway, we find $R' = 1' \otimes 1'$, and the triangular Hopf algebra $(H(G; G_+, G_-), R)$ is isomorphic to the twisting of the triangular Hopf algebra $(H(G; A, G_-), 1' \otimes 1')$, with the twist given by

$$\begin{aligned} T' &= (\phi \otimes \phi)^{-1}(T) \\ &= \sum_{u,v \in G_+} \{\xi(u^{-1})u\xi(v^{-1})\}' \otimes \{\xi(v^{-1})v\}' \\ &= \sum_{a,b \in A} \{a\zeta(b^{-1})\}' \otimes \{b\}'. \end{aligned}$$

Finally, we note that since $R' = 1' \otimes 1'$, $H(G; A, G_-)$ is cocommutative. Since any cocommutative Hopf algebra over an algebraically closed field is a group algebra (see [**S**], for example), we have conclude the following.

Corollary 5.2. *Any finite dimensional positive triangular Hopf algebra is the twisting of a group algebra.*

An explicit formula for exhibiting $H(G; A, G_-)$ as a group algebra can be found in Section 4 of [**EG**].

6. Positive quasi-triangular structures and bisections

In [**LYZ1**], we have shown that the positivity condition on a Hopf algebra implies that the Hopf algebra is essentially set-theoretical. In this section, we explain that a positive quasi-triangular structure on such a Hopf algebra is also set-theoretical.

Recall [**Mk**] that a *groupoid* over a set B (called *base space*) is a set Γ (called *total space*) together with

1. two surjections $\alpha, \beta : \Gamma \to B$
2. a product $\mu : (\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2$ in Γ, defined when $\beta(\gamma_1) = \alpha(\gamma_2)$
3. an identity map $e : b \mapsto e_b, B \to \Gamma$

such that the usual axioms similar to those for groups are satisfied. If Γ is finite, then we have an algebra structure on the vector space $\mathbb{C}\Gamma$

$$\{\gamma_1\}\{\gamma_2\} = \begin{cases} \{\gamma_1\gamma_2\} & \text{if } \beta(\gamma_1) = \alpha(\gamma_2) \\ 0 & \text{if } \beta(\gamma_1) \neq \alpha(\gamma_2) \end{cases}, \qquad e = \sum_{b \in B} e_b,$$

called the *groupoid algebra* of Γ.

For example, given a unique factorization $G = G_+G_-$, we have the following groupoid Γ_+ with G as the total space and with G_+ as the base space

1. α_+: $g \mapsto g_+$, $G \to G_+$, and β_+: $g \mapsto \bar{g}_+$, $G \to G_+$
2. $\mu_+ : (g, h) \mapsto gh_-$ when $\bar{g}_+ = h_+$
3. $e_+ : g_+ \mapsto g_+$, $G_+ \to G$

The corresponding groupoid algebra is the algebra structure of $H(G; G_+, G_-)$.

An element $a = \sum_{\gamma \in \Gamma} r(\gamma)\{\gamma\}$ of the groupoid algebra $\mathbb{C}\Gamma$ is called *positive* if $r(\gamma) \geq 0$. In this case, we have a subset

$$L(a) = \{\gamma : \quad r(\gamma) > 0\}$$

of Γ. If a_1 and a_2 are positive, then a_1a_2 is also positive, and we have

$$L(a_1a_2) = \{\gamma_1\gamma_2 : \quad \gamma_1 \in L(a_1),\ \gamma_2 \in L(a_2),\ \beta(\gamma_1) = \alpha(\gamma_2)\}. \tag{6.1}$$

If we define the product of two subsets $L_1, L_2 \subset \Gamma$ as

$$L_1L_2 = \{\gamma_1\gamma_2 : \quad \gamma_1 \in L_1,\ \gamma_2 \in L_2,\ \beta(\gamma_1) = \alpha(\gamma_2)\}, \tag{6.2}$$

then (6.1) becomes $L(a_1a_2) = L(a_1)L(a_2)$.

The subset

$$E_\Gamma = L\left(\sum_{\gamma\in\Gamma}\{\gamma\}\right) = \{e_b : \quad b \in B\} \subset \Gamma$$

is a unit of the product (6.2). The next result tells us which subsets of Γ are invertible.

Proposition 6.1. *Let $L \subset \Gamma$ be a subset of a groupoid Γ over B. Then the following are equivalent*

1. *There is a subset $K \subset \Gamma$ such that $LK = E_\Gamma$ and $KL = E_\Gamma$;*
2. *The restrictions $\alpha|_L, \beta|_L : L \to B$ are bijections.*

Proof. We first prove that the first statement implies the second.

Since $LK = E_\Gamma$, for any $b \in B$, there are $\gamma_1 \in L$ and $\gamma_2 \in K$ such that $\beta(\gamma_1) = \alpha(\gamma_2)$ and $\gamma_1\gamma_2 = e_b$. In particular, we have $\alpha(\gamma_1) = \alpha(\gamma_1\gamma_2) = b$ and $\beta(\gamma_2) = \beta(\gamma_1\gamma_2) = b$. Thus $\alpha|_L$ and $\beta|_K$ are surjective.

Now suppose we have $\gamma_1, \gamma_1' \in L$ such that $\alpha(\gamma_1) = \alpha(\gamma_1') = a$. By the surjectivity of $\beta|_K$, we can find $\gamma_2 \in K$ such that $\beta(\gamma_2) = a$. Since $KL = E_\Gamma$, we conclude that $\gamma_2\gamma_1 = e_b$ and $\gamma_2\gamma_1' = e_{b'}$ for some $b, b' \in B$. Then

$$\beta(\gamma_1) = \beta(\gamma_2\gamma_1) = b = \alpha(\gamma_2\gamma_1) = \alpha(\gamma_2) = \alpha(\gamma_2\gamma_1') = b' = \beta(\gamma_2\gamma_1') = \beta(\gamma_1').$$

In particular, the products $\gamma_1\gamma_2$ and $\gamma_1'\gamma_2$ make sense. Since $\gamma_1\gamma_2, \gamma_1'\gamma_2 \in LK = E_\Gamma$, we see that $\gamma_1\gamma_2 = \gamma_1'\gamma_2 = e_a$. Combining this with $\gamma_2\gamma_1 = e_b = e_{b'} = \gamma_2\gamma_1'$, we conclude that $\gamma_1 = \gamma_2^{-1} = \gamma_1'$. This proves the injectivity of $\alpha|_L$.

The bijectivity of $\beta|_L : L \to B$ can be proved similarly.

Conversely, given the second statement, it is easy to verify that $K = \{\gamma^{-1} : \gamma \in L\}$ satisfies the first statement.

$\square$

A subset $L \subset \Gamma$ of a groupoid satisfying the equivalent conditions of the proposition above is called a *bisection*. With product (6.2), the collection $\mathcal{U}(\Gamma)$ of all bisections of a groupoid Γ form a group.

Now we specialize the theory above to the positive quasi-triangular structure R in Theorem 2.3. Denote

$$\mathcal{R} = L(R) = \left\{\left(u\left(\eta(v)^u\right)^{-1},\ v\xi(u)\right) : \quad u, v \in G_+\right\}. \tag{6.3}$$

Since the equality $R^{-1} = (S \otimes id)R$ implies that R^{-1} is also positive, we see that $\mathcal{R}$ has $L(R^{-1})$ as inverse and is a bisection of $\Gamma_+ \times \Gamma_+$. Moreover, we have

$$L(R_{12}) = \mathcal{R} \times \{e\} = \mathcal{R}_{12}, \qquad \text{etc.}$$

Applying L to the Yang-Baxter equation satisfied by R, we see that $\mathcal{R}$ also satisfies the following groupoid-theoretical Yang-Baxter equation introduced in [**WX**]

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \tag{6.4}$$

which is an equality inside the group $\mathcal{U}(\Gamma_+ \times \Gamma_+ \times \Gamma_+)$ of bisections.

To get set-theoretical solutions of the Yang-Baxter equation from (6.4), we recall that the quasi-triangular structure R induces a solution of the Yang-Baxter equation on any H-module. The set-theoretical analogue of modules is sets acted upon by groupoids.

Let Γ be a groupoid over B. A (left) Γ-*set* consists of a set X, a map $J : X \to B$, and an action

$$(\gamma, x) \mapsto \gamma x \in X, \qquad \text{for } \gamma \in \Gamma,\ x \in X, \text{ satisfying } \beta(\gamma) = J(x).$$

such that $J(\gamma x) = \alpha(\gamma)$ and the usual conditions similar to those for group actions are satisfied. The vector space $\mathbb{C}X$ has an obvious module structure over the groupoid algebra.

For any $L \in \mathcal{U}(\Gamma)$ and $x \in X$, the equation

$$Lx = \gamma x, \qquad \text{for the unique } \gamma \in L \text{ satisfying } \beta(\gamma) = J(x)$$

defines a (left) action of the group $\mathcal{U}(\Gamma)$ of bisections on the set X. Now if $\mathcal{R} \in \mathcal{U}(\Gamma \times \Gamma)$ satisfies the groupoid-theoretical Yang-Baxter equation (6.4), then the map $\mathcal{R}_X : X \times X \to X \times X$ induced by $\mathcal{R}$ is a set-theoretical solution of the Yang-Baxter equation over X.

Now we compute the set-theoretical solution of the Yang-Baxter equation induced by the action of the bisection (6.3) on the simplest Γ_+-set, the unit Γ_+-set $id : G_+ \to G_+$. In this case, $\mathcal{R}_{G_+}$ is given by the following diagram

$$\begin{tikzcd}
 & \left(u\,(\eta(v)^u)^{-1},\, v\xi(u)\right) \arrow[dl, "\beta_+\times\beta_+"'] \arrow[dr, "\alpha_+\times\alpha_+"] & \\
(u^{(\eta(v)^u)^{-1}}, v^{\xi(u)}) \arrow[rr, "\mathcal{R}_{G_+}"] & & (u,v)
\end{tikzcd}$$

By $u^{(\eta(v)^u)^{-1}} = {}^{\eta(v)}u$, we have

$$\mathcal{R}_{G_+}^{-1}(u,v) = ({}^{\eta(v)}u, v^{\xi(u)}).$$

Solving the equation, we get the set-theoretical solution

$$\mathcal{R}_{G_+}(u,v) = (u^{\eta(v)}, {}^{\xi(u)}v) \tag{6.5}$$

of the Yang-Baxter equation over G_+.

Direct computation shows that if ξ and η are two group homomorphisms satisfying (2.6), then (6.5) is already a set-theoretical solution of the Yang-Baxter equation. Moreover, in order for (2.6) and (6.5) to make sense, we do not even need to know anything about G_-. The only data we need are actions $(u,v) \mapsto {}^{\xi(u)}v$ and $(u,v) \mapsto u^{\eta(v)}$ of G_+ on itself. This is the basic data for the construction in [**LYZ2**].

We would like to end the section by mentioning that a comprehensive theory can be established for the Yang-Baxter equation on groupoids. Indeed, we can formulate the definitions of Hopf groupoids and quasi-triangular structures on them. We can further show that any group factorization of a group induces a Hopf groupoid, and that all of its quasi-triangular structures are given by the bisections of the form (6.3). Moreover, we can introduce the notion of quasi-isomorphisms of Hopf groupoids and establish the set-theoretical analogue of Theorem 4.3.

7. Proof of the classification theorem

We prove Theorem 2.3 in this section. We start with the condition (2.5).

LEMMA 7.1. *Suppose $\xi, \eta : G_+ \to G_-$ are two group homomorphisms satisfying* (2.14), (2.15). *Suppose $r : G_+ \times G_+ \to \mathbf{R}^{>0}$ is a function such that for any $u, v, w \in G_+$,*

$$r(uw, v) = r(u, v) r(w, v^{\xi(u)}), \tag{7.1}$$

$$r(u, wv) = r(u, v) r({}^{\eta(v)}u, w). \tag{7.2}$$

Then

$$R = \sum_{u,v \in G_+} r(u, v) \{u\,(\eta(v)^u)^{-1}\} \otimes \{v \xi(u)\} \tag{7.3}$$

satisfies (2.5). *Conversely, if R is invertible, positive, satisfies* (2.5), *and R^{-1} is also positive, then R is given by the construction above.*

PROOF. It is straightforward to verify that, when all conditions are satisfied, R indeed satisfies (2.5). In what follows, we prove the converse.

Note that the tensor algebra $H(G; G_+, G_-) \otimes H(G; G_+, G_-)$ is the groupoid algebra of the product groupoid $\Gamma_+ \times \Gamma_+$. Thus we may apply the operation L introduced in Section 6 to the positive elements R and R^{-1}. In particular, the equality $RR^{-1} = 1 \otimes 1 = R^{-1}R$ implies that $L(R)$ is invertible with respect to the product (6.2). By Proposition 6.1, the restriction of $\alpha_{\Gamma_+ \times \Gamma_+}(g, g) = (g_+, g_+)$ on $L(R)$ is bijective. This implies that

$$R = \sum_{u,v \in G_+} r(u, v) \{u \phi(u, v)\} \otimes \{v \psi(u, v)\}, \tag{7.4}$$

where $\phi, \psi : G_+ \to G_-$ are two maps, and $r : G_+ \times G_+ \to \mathbf{R}^{>0}$ is a positively valued function.

From

$$\begin{aligned}
(\Delta \otimes id)R &= \sum_{u,v,w \in G_+} r(u, v) \{uw^{-1}({}^w\phi(u, v))\} \otimes \{w \phi(u, v)\} \otimes \{v \psi(u, v)\}, \\
R_{13}R_{23} &= \sum_{u,v,w \in G_+} r(u, v) r(w, v^{\psi(u,v)}) \{u \phi(u, v)\} \otimes \{w \phi(w, v^{\psi(u,v)})\} \\
&\qquad \otimes \{v \psi(u, v) \psi(w, v^{\psi(u,v)})\},
\end{aligned}$$

we see that $(\Delta \otimes id)R = R_{13}R_{23}$ means

$$r(uw, v) = r(u, v) r(w, v^{\psi(u,v)}), \tag{7.5}$$

$${}^w\phi(uw, v) = \phi(u, v), \tag{7.6}$$

$$\phi(uw, v) = \phi(w, v^{\psi(u,v)}), \tag{7.7}$$

$$\psi(uw, v) = \psi(u, v) \psi(w, v^{\psi(u,v)}), \tag{7.8}$$

for all $u, v, w \in G_+$. Similarly, $(id \otimes \Delta)R = R_{13}R_{12}$ means

$$r(u, wv) = r(u, v) r(u^{\phi(u,v)}, w), \tag{7.9}$$

$$\phi(u, wv) = \phi(u, v) \phi(u^{\phi(u,v)}, w), \tag{7.10}$$

$${}^v\psi(u, wv) = \psi(u^{\phi(u,v)}, w), \tag{7.11}$$

$$\psi(u, wv) = \psi(u, v). \tag{7.12}$$

Equation (7.12) implies that

$$\psi(u,v)=\xi(u) \tag{7.13}$$

for a map $\xi : G_+ \to G_-$. Then (7.8) becomes $\xi(uw)=\xi(u)\xi(w)$, so that ξ is a group homomorphism.

Equation (7.6) implies that $\phi(u,v)={}^{u^{-1}}\phi(e,v)$. Therefore we introduce $\eta(v)=\phi(e,v)^{-1} : G_+ \to G_-$ and have

$$\phi(u,v)={}^{u^{-1}}(\eta(v)^{-1}) \overset{(2.3)}{=} (\eta(v)^u)^{-1}. \tag{7.14}$$

Moreover, we have

$$u^{\phi(u,v)}=u^{({}^{u^{-1}}(\eta(v)^{-1}))} \overset{(2.2)}{=} {}^{\eta(v)}u. \tag{7.15}$$

Then by (7.14) and (7.15), equation (7.10) becomes

$$\eta(wv)^u=\eta(w)^{({}^{\eta(v)}u)}\eta(v)^u.$$

Taking $u=e$, we see that η is a group homomorphism. By making use of this fact, the equation above becomes (2.2), which is always satisfied.

By (7.13) and (7.15), equation (7.11) becomes (2.14). By (7.14), equation (7.7) becomes $\eta(v)^{uw}=\eta(v^{\xi(u)})^w$. Applying the right action by w^{-1}, we have (2.15). Finally, by (7.15), equations (7.5) and (7.9) become (7.1) and (7.2).

□

Proof of Theorem 2.3. We first prove that any positive quasi-triangular structure is given by the theorem.

Let R be a positive quasi-triangular structure. Then $R^{-1}=(S\otimes id)R$ implies that R^{-1} is also positive. Then by Lemma 7.1, R is of the form (7.3), and we have properties (2.14) and (2.15). Note that by Lemma 2.4, we also have properties (2.9) and (2.10).

In the product

$$\begin{aligned} R\Delta\{g\} \quad = \quad & \left(\sum_{u,v\in G_+} r(u,v)\{u\,(\eta(v)^u)^{-1}\}\otimes\{v\xi(u)\}\right) \\ & \left(\sum_{h_+\in G_+}\{g_+h_+^{-1}({}^{h_+}g_-)\}\otimes\{h_+g_-\}\right), \end{aligned}$$

we must have

$$h_+=v^{\xi(u)}, \qquad g_+h_+^{-1}=u^{(\eta(v)^u)^{-1}} \overset{(2.15)}{=} u^{\eta(v^{\xi(u)})^{-1}}=u^{\eta(h_+)^{-1}}.$$

Therefore

$$\begin{aligned} u &= (g_+h_+^{-1})^{\eta(h_+)}, \\ v &= h_+^{\xi\left((g_+h_+^{-1})^{\eta(h_+)}\right)^{-1}} \overset{(2.9)}{=} h_+^{\left(\xi(g_+h_+^{-1})^{h_+}\right)^{-1}} \\ &\overset{(2.3)}{=} {}^{h_+^{-1}}h_+^{\left(\xi(g_+h_+^{-1})^{-1}\right)} \overset{(2.2)}{=} \left((h_+^{-1})^{\left(\xi(g_+h_+^{-1})^{-1}\right)}\right)^{-1} \overset{(2.3)}{=} {}^{\xi(g_+h_+^{-1})}h_+, \\ \eta(v)^u &\overset{(2.15)}{=} \eta(v^{\xi(u)})=\eta(h_+), \\ \xi(u) &= \xi((g_+h_+^{-1})^{\eta(h_+)}) \overset{(2.9)}{=} \xi(g_+h_+^{-1})^{h_+}, \end{aligned}$$

and

$$
\begin{aligned}
R\Delta\{g\} &= \sum_{h_+\in G_+} r((g_+h_+^{-1})^{\eta(h_+)}, {}^{\xi(g_+h_+^{-1})}h_+) \\
&\quad \left\{((g_+h_+^{-1})^{\eta(h_+)})\eta(h_+)^{-1}({}^{h_+}g_-)\right\} \\
&\quad \otimes\left\{({}^{\xi(g_+h_+^{-1})}h_+)(\xi(g_+h_+^{-1})^{h_+})g_-\right\}.
\end{aligned} \tag{7.16}
$$

This should be equal to

$$
\begin{aligned}
\tau\Delta\{g\}R &= \sum_{h_+\in G_+} r(h_+{}^{g_-}, (g_+h_+^{-1})^{({}^{h_+}g_-)}) \\
&\quad \left\{h_+g_-\left(\eta((g_+h_+^{-1})^{({}^{h_+}g_-)})^{(h_+{}^{g_-})}\right)^{-1}\right\} \\
&\quad \otimes\left\{g_+h_+^{-1}({}^{h_+}g_-)\xi(h_+{}^{g_-})\right\}.
\end{aligned} \tag{7.17}
$$

Observe that in the G_+-parts of each term in $\tau\Delta\{g\}R$, we have $(g_+h_+^{-1})h_+ = g_+$. Therefore by $\tau\Delta\{g\}R = R\Delta\{g\}$ and $r>0$, we have

$$\left({}^{\xi(g_+h_+^{-1})}h_+\right)\left((g_+h_+^{-1})^{\eta(h_+)}\right) = g_+.$$

This is exactly (2.6).

We now compare $\tau\Delta\{g\}R$ and $R\Delta\{g\}$ term by term. In order to avoid confusion, we change the index h_+ in $\tau\Delta\{g\}R$ to $\bar{h}_+$.

The equality $\tau\Delta\{g\}R = R\Delta\{g\}$ and $r>0$ suggests us to consider the map

$$h_+ \mapsto \bar{h}_+ = (g_+h_+^{-1})^{\eta(h_+)}. \tag{7.18}$$

By using (2.6), (2.9), (2.10), we can show that the map has the following inverse

$$\bar{h}_+ \mapsto h_+ = (g_+\bar{h}_+^{-1})^{\xi(\bar{h}_+)}. \tag{7.19}$$

This means that the term in $\tau\Delta\{g\}R$ indexed by $\bar{h}_+$ and the term in $R\Delta\{g\}$ indexed by h_+ must be equal. By comparing the coefficients, the G_+-components, and the G_--components of the corresponding terms, we have

$$r(\bar{h}_+{}^{g_-}, (g_+\bar{h}_+^{-1})^{({}^{\bar{h}_+}g_-)}) = r((g_+h_+^{-1})^{\eta(h_+)}, {}^{\xi(g_+h_+^{-1})}h_+), \tag{7.20}$$

$$g_-\left(\eta\left((g_+\bar{h}_+^{-1})^{({}^{\bar{h}_+}g_-)}\right)^{(\bar{h}_+{}^{g_-})}\right)^{-1} = \eta(h_+)^{-1}({}^{h_+}g_-), \tag{7.21}$$

$$g_+\bar{h}_+^{-1} = {}^{\xi(g_+h_+^{-1})}h_+, \tag{7.22}$$

$$({}^{\bar{h}_+}g_-)\xi(\bar{h}_+{}^{g_-}) = (\xi(g_+h_+^{-1})^{h_+})g_-. \tag{7.23}$$

As pointed out earlier, (7.18) and (7.22) implies (2.6).

Let $h_+ = e$. Then from (7.18) we have $\bar{h}_+ = g_+$, so that (7.23) becomes

$$({}^{g_+}g_-)\xi(g_+{}^{g_-}) = \xi(g_+)g_-.$$

This is (2.12), which is equivalent to (2.7) by Lemma 2.4.

We have from (2.9) and (7.18) that

$$\xi(\bar{h}_+) = \xi((g_+h_+^{-1})^{\eta(h_+)}) = \xi(g_+h_+^{-1})^{h_+},$$

so that

$$\begin{aligned} h_+{}^{\xi(\bar h_+)^{-1}} &= h_+{}^{\left(\xi(g_+h_+^{-1})^{h_+}\right)^{-1}} \overset{(2.3)}{=} {}^{h_+^{-1}}h_+{}^{\left(\xi(g_+h_+^{-1})^{-1}\right)} \overset{(2.2)}{=} \left((h_+^{-1})^{\xi(g_+h_+^{-1})^{-1}}\right)^{-1} \\ &\overset{(2.3)}{=} {}^{\xi(g_+h_+^{-1})}h_+ \overset{(2.6)}{=} (g_+h_+^{-1})h_+\left((g_+h_+^{-1})^{\eta(h_+)}\right)^{-1} \overset{(7.18)}{=} g_+\bar h_+^{-1}. \end{aligned}$$

Thus $(g_+\bar h_+^{-1})^{\xi(\bar h_+)} = h_+$, and in (7.21) we have

$$\begin{aligned} \eta\left((g_+\bar h_+^{-1})^{({}^{\bar h_+}g_-)}\right)^{(\bar h_+{}^{g_-})} &\overset{(2.15)}{=} \eta\left((g_+\bar h_+^{-1})^{({}^{\bar h_+}g_-)\xi(\bar h_+{}^{g_-})}\right) \\ &\overset{(2.12)}{=} \eta\left((g_+\bar h_+^{-1})^{\xi(\bar h_+)g_-}\right) = \eta(h_+{}^{g_-}). \end{aligned}$$

Therefore (7.21) becomes

$$\eta(h_+)g_- = ({}^{h_+}g_-)\eta\left((g_+\bar h_+^{-1})^{({}^{\bar h_+}g_-)}\right)^{(\bar h_+{}^{g_-})} = ({}^{h_+}g_-)\eta(h_+{}^{g_-}).$$

This is (2.13), which is equivalent to (2.8) by Lemma 2.4.

Finally, substituting (7.18) and (7.22) into (7.20) gives

$$r(\bar h_+{}^{g_-}, (g_+\bar h_+^{-1})^{({}^{\bar h_+}g_-)}) = r(\bar h_+, g_+\bar h_+^{-1}).$$

In other words, we have

$$r(u,v) = r(u^x, v^{({}^ux)}). \tag{7.24}$$

We claim that (7.1) and this imply r is constant 1.

The one-to-one correspondence

$$u \mapsto u\xi(u^{-1}) : \quad G_+ \to G'_+$$

translates the group structure (3.1) on G'_+ to a group structure

$$u \star v = u({}^{\xi(u^{-1})}v) \tag{7.25}$$

on G_+. We have

$$r(u\star w, v) = r(u({}^{\xi(u^{-1})}w), v) \overset{(7.1)}{=} r(u,v)r({}^{\xi(u^{-1})}w, v^{\xi(u)}).$$

Let $x = {}^{w^{-1}}\xi(u)$. Then

$${}^wx = \xi(u), \qquad w^x = w^{({}^{w^{-1}}\xi(u))} \overset{(2.2)}{=} \left((w^{-1})^{\xi(u)}\right)^{-1} \overset{(2.3)}{=} {}^{\xi(u^{-1})}w.$$

Therefore

$$r(u\star w, v) = r(u,v)r(w^x, v^{({}^wx)}) \overset{(7.24)}{=} r(u,v)r(w,v).$$

Thus for fixed v, $r(?, v) : (G_+, \star) \to \mathbf{R}^{>0}$ is a group homomorphism. Since G_+ is finite, we conclude that $r = 1$.

This completes the proof of that R is given as in the theorem.

Conversely, suppose ξ, η are homomorphisms satisfying the conditions in the theorem. Then the homomorphisms also have the properties in Lemma 2.4. Then by Lemma 7.1, (2.5) is satisfied. Moreover, we may use (2.6), (2.9), (2.10) to show that (7.18) and (7.19) are inverse to each other, just as what we have done in the proof of the other direction. This implies that (7.18) and (7.19) give a one-to-one correspondence between the terms in $\tau\Delta\{g\}R$ and $R\Delta\{g\}$. Thus in order to show $\tau\Delta\{g\}R = R\Delta\{g\}$, it remains to verify (7.20) through (7.23). The detailed computation is almost the same as what we have done in the first part of the proof, except for (7.23). The following computation verifies (7.23)

$$({}^{\bar{h}_+}g_-)\xi(\bar{h}_+{}^{g_-}) \overset{(2.12)}{=} \xi(\bar{h}_+)g_- \overset{(7.18)}{=} \xi((g_+h_+^{-1})^{\eta(h_+)})g_- \overset{(2.9)}{=} (\xi(g_+h_+^{-1})^{h_+})g_-.$$

This completes the proof of the converse.

□

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