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Symplectic Structure of the Painlevé Test

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Abstract

In this note, we present a result to show that the symplectic structures have been naturally encoded into the Painlevé test. In fact, for every principal balance, there is a symplectic change of dependent variables near movable poles.

1 Introduction

Recently, we provided an algorithm for finding an appropriate change of variables that regularizes the solutions near movable singularities for differential equations passing the Painlevé test [1, 2]. In [3], we further proved that our algorithm is in fact equivalent to the Painlevé test. An immediate consequence of this is a conceptual proof that demonstrates, for ordinary differential equations, that the formal Laurent series produced from a successful application of the Painlevé test is always convergent.

The main result of this paper is that our algorithm is compatible with Hamiltonian systems. Specifically, we show that, by carrying out our algorithms more carefully, we can make sure that the change of variable used for constructing the mirror system preserves the symplectic structure and converts the original Hamiltonian system to another Hamiltonian system given by a regular Hamiltonian function near movable singularities.

2 The first Painlevé equation

In this section, we use the first Painlevé equation P_1, $u'' = 6u^2 + t$, as an example to demonstrate the symplectic structure of the Painlevé test. We rewrite the equation as a Hamiltonian system

$$\dot{q}_1 = p_1, \qquad \dot{p}_1 = 6q_1^2 + q_2, \qquad \dot{q}_2 = 1, \qquad \dot{p}_2 = q_1, \tag{1}$$

whose Hamiltonian is given by $H = \frac{1}{2}p_1^2 - 2q_1^3 - q_2 q_1 + p_2$. The formal Laurent series solution is given by

$$q_1 \sim t^{-2}\left[1 + 0t + 0t^2 + 0t^3 - \frac{r_1\, t^4}{10} - \frac{t^5}{6} + \frac{r_3\, t^6}{4} + \cdots\right],$$

$$q_2 \sim t^{-4}\left[0 + 0t + 0t^2 + 0t^3 + r_1 t^5 + t^5 + 0t^6 + \cdots\right],$$

$$p_1 \sim t^{-3}\left[-2+0t+0t^2+0t^3-\frac{r_1\,t^4}{5}-\frac{t^5}{2}+r_3\,t^6+\cdots\right],$$

$$p_2 \sim t^{-1}\left[-1+r_2t+0t^2+0t^3-\frac{r_1\,t^4}{30}-\frac{t^5}{24}+\frac{r_3\,t^6}{20}+\cdots\right].$$

From the expansions, we can construct the following table:

resonance	t	r_2	r_1	r_3
j	-1	1	4	6
q_1	2	0	-1/10	1/4
q_2	0	0	1	0
p_1	-6	0	-1/5	1
p_2	-1	1	-1/30	1/20

In the first column vector are the products of the leading exponents to the corresponding leading coefficients; the other vectors are the coefficients of the resonances. After rescaling, the following column vectors in the table form a symplectic basis of $\mathbf{R}^4$:

$$\begin{pmatrix}2\\0\\-6\\-1\end{pmatrix},\quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix},\quad \frac{2}{7}\begin{pmatrix}1/4\\0\\1\\1/20\end{pmatrix},\quad (-1)\begin{pmatrix}-1/10\\1\\-1/5\\-1/30\end{pmatrix}.$$

This nice symplectic structure has an important consequence in the study of Painlevé analysis for Hamiltonian systems. To see this, let us make a transform for the original Hamiltonian system to a system governing the behavior of solutions near pole singularity. By following a standard procedure [2], we first introduce the indicial normalization $q_1=\theta^{-2}$ and have the following expansions in terms of small θ:

$$\begin{cases}\theta' = 1-3r_2\theta^4+\dfrac{1}{2}\theta^5+r_3\theta^6+\cdots,\\ q_2 = -12r_2+\theta+0\theta^2+\cdots,\\ p_1 = -2\theta^{-3}+6r_2\theta-\theta^2-2r_3\theta^3+\cdots,\\ p_2 = -\theta^{-1}+r_1+r_2\theta^3-\dfrac{1}{8}\theta^4-\dfrac{1}{5}r_3\theta^5+\cdots.\end{cases}$$

Then we introduce new variables η_1, η_2, and η_3 by successively truncating the θ-series of q_2, p_2, and p_1 at r_1, r_2, and r_3. The result is

$$\begin{cases}q_2=\eta_1,\\ p_1=-2\theta^{-3}-\dfrac{1}{2}\eta_1\theta-\dfrac{1}{2}\theta^2+\eta_3\theta^3,\\ p_2=-\theta^{-1}+\eta_2.\end{cases} \tag{2}$$

The transform $(q_1,q_2,p_1,p_2)\leftrightarrow(\theta,\eta_1,\eta_2,\eta_3)$ given by $q_1=\theta^{-2}$ and (2) converts the system (1) into a regular system for the new variables. The proof is given in [3].

A simple computation shows that the symplectic form

$$dq_1\wedge dp_1+dq_2\wedge dp_2=-2d\theta\wedge d\eta_3+d\eta_1\wedge d\eta_2.$$

Therefore, if we denote $Q_1 = \theta$, $Q_2 = \eta_1$, $P_1 = -2\eta_3$ and $P_2 = \eta_2$, then the following transform

$$\begin{cases} q_1 = Q_1^{-2}, \\ q_2 = Q_2, \\ p_1 = -2Q_1^{-3} - \frac{1}{2}Q_1Q_2 - \frac{1}{2}Q_1^2 - \frac{1}{2}Q_1^3P_1, \\ p_2 = -Q_1^{-1} + P_2, \end{cases} \tag{3}$$

preserves the symplectic form. Moreover, the transform converts the Hamiltonian system (1) into a Hamiltonian system given by

$$\begin{aligned} H(q_1, p_1, q_2, p_2) &= \bar{H}(Q_1, Q_2, P_1, P_2) \\ &= P_1 + P_2 + \frac{1}{8}Q_1^2Q_2^2 + \frac{1}{4}Q_1^3Q_2 + \frac{1}{8}Q_1^4 + \frac{1}{4}Q_1^4Q_2P_1 + \frac{1}{4}Q_1^5P_1 + \frac{1}{8}Q_1^6P_1^2, \end{aligned}$$

which is again a polynomial.

We remark that the fact that the mirror transform is symplectic is actually a general property for Hamiltonian systems, provided that the systems pass the Painlevé test.

3 Main theorem

In this section, we consider general polynomial Hamiltonian systems

$$u' = J\nabla H, \qquad \nabla H = \left(\frac{\partial H}{\partial u_1}, \cdots, \frac{\partial H}{\partial u_n}\right), \qquad J = \begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix} \tag{4}$$

in which $n = 2m$ and $H(u)$ is a polynomial.

A *balance* for the system is a formal Laurent series solution of the form

$$u_i = c_i(t-t_0)^{-g_i} + u_{1,i}(t-t_0)^{1-g_i} + \cdots + u_{j,i}(t-t_0)^{j-g_i} + \cdots, \quad i = 1, \ldots, n \tag{5}$$

in which the coefficients are analytic functions of t_0, and we have at least one i, such that $c_i \neq 0$ and $g_i < 0$. The integers g_i are called the *leading exponents* of the balance.

Note that the leading exponents in Laurent series are usually understood as the orders of the poles. We allow some (but not all) of the leading terms in (5) to be trivial.

We call a balance to be *principal* if it allows $(n-1)$ free parameters. If we include t_0, this brings the total number of free parameters to n. We say the ODE system (4) *passes the Painlevé test* if all balances are principal.

Definition. *We say the Hamiltonian system (4) is* almost scalar invariant *relative to the leading exponents $g_1, \ldots, g_n$ if $g_i + g_{i+m} = h - 1$ for all $i = 1, \ldots, m$.*

Theorem. *Suppose the Hamiltonian system* (4) *is almost scalar invariant relative to the leading exponents of a principal balance. Then up to exchanging some p_i with q_i and rearranging the order among the pairs (q_1, p_1), …, (q_m, p_m), there is a change of variables* *of the form*

$$\begin{cases} q_1 = Q_1^{-g_1}, \\ q_2 = Q_1^{-g_2}\left(a_{2,0} + a_{2,1}Q_1 + \cdots + a_{2,j_1-1}Q_1^{j_1-1} + Q_2Q_1^{j_1}\right), \\ \quad\vdots \\ q_m = Q_1^{-g_m}\left(a_{m,0} + a_{m,1}Q_1 + \cdots + a_{m,j_{m-1}-1}Q_1^{j_{m-1}-1} + Q_mQ_1^{j_{m-1}}\right), \\ p_1 = Q_1^{-g_{1+m}}\left(b_{1,0} + b_{1,1}Q_1 + \cdots + b_{1,j_{n-1}-1}Q_1^{j_{n-1}-1} - g_1^{-1}P_1Q_1^{j_{n-1}}\right), \\ p_2 = Q_1^{-g_{2+m}}\left(b_{2,0} + b_{2,1}Q_1 + \cdots + b_{2,j_{n-2}-1}Q_1^{j_{n-2}-1} + P_2Q_1^{j_{n-2}}\right), \\ \quad\vdots \\ p_m = Q_1^{-g_n}\left(b_{m,0} + b_{m,1}Q_1 + \cdots + b_{m,j_m-1}Q_1^{j_m-1} + P_mQ_1^{j_m}\right) \end{cases}$$

such that

1. $j_1 \le \cdots \le j_{n-1}$;
2. $a_{i,j}$ *is an analytic function of* Q_k *with* $j_k \le j$;
3. $b_{i,j}$ *is an analytic function of* $Q_2, \ldots, Q_m$, *and* P_k *with* $j_k \le j$;
4. *The symplectic form is preserved*
$$dq_1\wedge dp_1+\cdots+dq_m\wedge dp_m = dQ_1\wedge dP_1+\cdots+dQ_m\wedge dP_m;$$
5. *The Hamiltonian system is converted to another Hamiltonian system with a polynomial of* (Q_*, P_*) *as the Hamiltonian function;*
6. *The principal balance is converted to a solution of the new Hamiltonian system with the initial data equivalent to the resonances.*

The crucial points to prove the main result are that under the above conditions:

1. The Kowalewski exponents appear in pairs;
2. The 2-form $dq_1 \wedge dp_1 + \cdots + dq_m \wedge dp_m$ is regular under the transform.

We remark that the second point was first realized by Ercolani and Siggia [4].

References

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