

Formulas

Trigonometric formulas

$$\sin a \pm \sin b = 2 \sin \frac{1}{2}(a \pm b) \cos \frac{1}{2}(a \mp b)$$

$$\cos a + \cos b = 2 \cos \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b) \quad \cos a - \cos b = 2 \sin \frac{1}{2}(a + b) \sin \frac{1}{2}(b - a)$$

$$\sin a \cos b = \frac{1}{2}(\sin(a - b) + \sin(a + b))$$

$$\sin a \sin b = \frac{1}{2}(\cos(a - b) - \cos(a + b)) \quad \cos a \cos b = \frac{1}{2}(\cos(a - b) + \cos(a + b))$$

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \quad \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

Fourier series

$$\text{Fourier series of } f(x) \text{ defined on } [-L, L]: \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L))$$

$$\text{where } a_n = \frac{1}{L} \int_{-L}^L f(x) \cos(n\pi x/L) dx, \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin(n\pi x/L) dx.$$

$$\text{Fourier cosine series } (x \in [0, L]): \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) \quad \text{where } a_n = \frac{2}{L} \int_0^L f(x) \cos(n\pi x/L) dx.$$

$$\text{Fourier sine series } (x \in [0, L]): \quad \sum_{n=1}^{\infty} b_n \sin(n\pi x/L) \quad \text{where } b_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L) dx.$$

Solution for wave equation with a source on \mathbb{R} $u_{tt} - c^2 u_{xx} = f$

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds + \frac{1}{2c} \iint_{\Delta_{(x,t)}} f$$

where $\phi(x) = u(x, 0)$, $\psi(x) = u_t(x, 0)$, and $\Delta_{(x,t)}$ is the characteristic triangle for (x, t) .

The Kirchhoff-Poisson solution for wave equation $u_{tt} - c^2 \Delta u = 0$

$$\text{In 3D: } u(\vec{x}, t) = \frac{1}{4\pi c^2} \frac{\partial}{\partial t} \left[\frac{1}{t} \iint_{S_t} \phi(\vec{\xi}) d\sigma \right] + \frac{1}{4\pi c^2 t} \iint_{S_t} \psi(\vec{\xi}) d\sigma \quad \text{where } S_t \text{ is a sphere of radius } ct.$$

$$\text{In 2D: } u(x, y, t) = \frac{1}{2\pi c} \frac{\partial}{\partial t} \left[\iint_{D_t} \frac{\phi(\xi, \eta)}{\sqrt{(ct)^2 - (\xi - x)^2 - (\eta - y)^2}} d\xi d\eta \right] \\ + \frac{1}{2\pi c} \iint_{D_t} \frac{\psi(\xi, \eta)}{\sqrt{(ct)^2 - (\xi - x)^2 - (\eta - y)^2}} d\xi d\eta \quad \text{where } D_t \text{ is a disk of radius } ct.$$

Solution for the diffusion equation with a source on \mathbb{R} $u_t - k u_{xx} = f(x, t) \quad (t > 0)$

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t) \phi(y) dy + \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) f(y, s) dy ds \quad (t > 0)$$

where $S(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}$ and $\phi(x) = u(x, 0)$.

$$\text{Definition of the error function: } \quad \text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp$$

$$\Delta \text{ in polar coordinates: } \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

Δ in spherical coordinates:
$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

Poisson's formula for harmonic function in a disk:
$$u(\mathbf{x}) = \frac{1}{2\pi a} \int_{|\mathbf{x}'|=a} \frac{a^2 - |\mathbf{x}|^2}{|\mathbf{x} - \mathbf{x}'|^2} u(\mathbf{x}') ds', \text{ or}$$

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 - r^2}{a^2 - 2ar \cos(\theta - \phi) + r^2} h(\phi) d\phi \text{ where } h(\phi) \text{ is the value of } u \text{ on the circle } r = a.$$

Green's first identity:
$$\iint_{\partial \mathcal{D}} v \frac{\partial u}{\partial n} dS = \iiint_{\mathcal{D}} \nabla v \cdot \nabla u dV + \iiint_{\mathcal{D}} v \Delta u dV$$

Green's second identity:
$$\iiint_{\mathcal{D}} (u \Delta v - v \Delta u) dV = \iint_{\partial \mathcal{D}} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS$$

Representation formula for a harmonic function u

$$u(\mathbf{x}_0) = \iint_{\partial \mathcal{D}} \left[u(\mathbf{x}) \frac{\partial}{\partial n} K(\mathbf{x}, \mathbf{x}_0) - K(\mathbf{x}, \mathbf{x}_0) \frac{\partial u}{\partial n} \right] dS \text{ where } K(\mathbf{x}, \mathbf{x}_0) \equiv -1/(4\pi|\mathbf{x} - \mathbf{x}_0|) \text{ for } \mathbb{R}^3.$$

Solution of the Dirichlet problem for $\Delta u = f$ using Green's function

$$u(\mathbf{x}_0) = \iint_{\partial \mathcal{D}} u(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial n} dS + \iiint_{\mathcal{D}} f(\mathbf{x}) G(\mathbf{x}, \mathbf{x}_0) dV.$$

On the plane, dS and dV are replaced by dl (length element) and dA (area element), respectively.

Bessel's differential equation of order s :
$$\frac{d^2}{dr^2} u + \frac{1}{r} \frac{d}{dr} u + \left(1 - \frac{s^2}{r^2}\right) u = 0$$

Solutions of Bessel's equation

For a non-integer s , the two independent solutions are

$$J_{\pm s}(r) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(\pm s+j+1)} \left(\frac{r}{2}\right)^{\pm s+2j} \text{ where } \Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt \text{ is the } \Gamma \text{ function.}$$

For $s = n$, a non-negative integer, the solutions are

$$J_n(r) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!} \left(\frac{r}{2}\right)^{n+2j}$$

$$Y_n(r) = \frac{2}{\pi} \left(\gamma + \ln \frac{r}{2} \right) J_n(r) - \frac{(1 - \delta_{n0})}{\pi} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!} \left(\frac{2}{r}\right)^{n-2j} - \frac{1}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j (H_j + H_{n+j})}{j!(n+j)!} \left(\frac{r}{2}\right)^{n+2j}$$

Bessel functions of order $s = \pm 1/2$:
$$J_{1/2}(r) = \sqrt{\frac{2}{\pi r}} \sin r \quad J_{-1/2}(r) = \sqrt{\frac{2}{\pi r}} \cos r$$

For $\beta_{sm} a, \beta_{sm'} a$ being roots of the Bessel function J_s

$$\int_0^a J_s(\beta_{sm} r) J_s(\beta_{sm'} r) r dr = \delta_{mm'} \frac{1}{2} a^2 [J'_s(\beta_{sm} a)]^2 = \delta_{mm'} \frac{1}{2} a^2 [J_{s\pm 1}(\beta_{sm} a)]^2$$

Spherical harmonics:
$$Y_l^m(\theta, \phi) = P_l^{|m|}(\cos \theta) \cdot e^{im\phi}$$

$$\int_0^{2\pi} \int_0^{\pi} Y_l^m(\theta, \phi) \overline{Y_{l'}^{m'}(\theta, \phi)} \sin \theta d\theta d\phi = \delta_{ll'} \delta_{mm'} \frac{4\pi}{2l+1} \frac{(l+m)!}{(l-m)!}$$