Saddlepoint approximation methods for pricing derivatives on discrete realized variance

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ABSTRACT

We consider the saddlepoint approximation methods for pricing derivatives whose payoffs depend on the discrete realized variance of the underlying price process of a risky asset. Most of the earlier pricing models of variance products and volatility derivatives use the quadratic variation approximation as the continuous limit of the discrete realized variance. However, the corresponding discretization gap may become significant for short-maturity derivatives. Under Lévy models and stochastic volatility models with jumps, we manage to obtain the saddlepoint approximation formulas for pricing variance products and volatility derivatives using the small time asymptotic approximation of the Laplace transform of the discrete realized variance. As an alternative approach, we also develop the conditional saddlepoint approximation method based on a given simulated stochastic variance path via Monte Carlo simulation. This analytic-simulation approach reduces the dimensionality of the simulation of the discrete variance derivatives; and in some cases, the simulation procedure of the realized variance can be effectively performed using an appropriate exact simulation method. We examine numerical accuracy and reliability of various types of the saddlepoint approximation techniques when applied to pricing derivatives on discrete realized variance under different types of asset price processes. The limitations of the saddlepoint approximation methods in pricing variance products and volatility derivatives are also discussed.

1 Introduction

Given N monitoring dates $0 = t_0 < t_1 < \cdots < t_N = T$, the discrete realized variance $I(0,T;N)$ of the underlying asset price process S_t over the time period $[0,T]$ is defined to be

$$I(0,T;N) = \frac{A}{N}\sum_{k=1}^{N}\left(\ln\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2 = \frac{A}{N}\sum_{k=1}^{N}\left(X_{t_k} - X_{t_{k-1}}\right)^2, \tag{1.1}$$

where $X_t = \ln S_t$ is the log asset price process and A is the annualized factor. It is common to take $A = 252$ for daily monitoring; and there holds $A/N = 1/T$. There are various types of derivatives on discrete realized variance that have been structured and traded in the financial market. The most basic products include the variance swaps, volatility swaps, and options on realized variance or volatility. The third generation variance swap products include the gamma swaps, skewness swaps and conditional variance swaps. Recently, financial institutions offer more exotic forms of volatility exposure, like the timer options (Bernard and Cui, 2011) and volatility target options.

In most of the earlier pricing models of variance products and volatility derivatives, the discrete realized variance defined in Eq. (1.1) is often approximated by the quadratic variation of the log asset price process X_t over $[0,T]$. The quadratic variation, commonly denoted by $[X,X]_T$, can be considered to be the asymptotic limit of the discrete realized variance in probability as $N \to \infty$. In the sequel, we let $I(0,T;\infty) = \frac{1}{T}[X,X]_T$. Suppose one fixes the monitoring frequency of the discrete realized variance to be daily, which means that A is fixed to be 252. As T and N change in relative proportion, we then expect that the quadratic variation approximation for $I(0,T;N)$ is good enough for derivatives on discrete realized variance with long maturity. The quadratic variation approximation has been widely adopted in pricing variance products and volatility derivatives in the literature due to its nice analytic tractability (Carr *et al.*, 2005). For vanilla variance swaps, the quadratic variation approximation is known to work well even for short-maturity derivatives (Sepp, 2008). However, accuracy of the quadratic variation approximation deteriorates for short-maturity derivatives with non-linear payoffs, like options on the realized variance (Bühler, 2006).

Apart from the quadratic variation approximation to the discrete realized variance, Zhu and Lian (2011) manage to obtain closed form pricing formulas for variance swaps. Indeed, analytic tractability can go beyond the vanilla variance swaps to variance swaps with more exotic form of payoff structures. Crosby and Davis (2010) derive analytic pricing formulas for discretely sampled gamma swaps and skewness swaps under time-changed Lévy processes. Their pricing formulas can be decomposed into a term representing the price of the continuously sampled counterpart plus a correction term that can be shown to be $O(N^{-1})$, where N is the number of monitoring instants. Zheng and Kwok (2013) obtain pricing formulas for the corridor variance swaps and conditional variance swaps under the stochastic volatility model with simultaneous jumps. Unfortunately, for options on discrete realized variance, analytic pricing formulas cannot be obtained. As a result, one has to develop various analytic approximation techniques. Sepp (2012) shows that the discrete realized variance under the Heston model can be approximated in distribution by the continuous realized variance plus a correction term which stems from an independent log-normal model. However, Drimus and Farkas (2012) argue that Sepp's approach only works well for near-the-money options since the discretization effect is indeed dependent on the continuous variance under stochastic volatility models. Instead, they prove that conditional on the stochastic variance process, the discrete realized variance is asymptotically normal as the number of monitoring instants goes to infinity. By using the limiting distribution, they are able to derive several pricing formulas of variance options based on a simulated path of the stochastic variance process as well as a set of non-simulation analytic pricing formulas with the use of further asymptotic approximations of the conditional mean and variance. As expected, their approach works well for options with long maturity, under which a sufficiently large value of N in Eq. (1.1) is ensured. For short-maturity options on discrete realized variance, corresponding to a relatively small value of N, the approximation based on the central limit theorem would not perform so well and neither would the performance of their approximation method. Under the Lévy and more general semimartingale dynamics, Keller-Ressel and Muhle-Karbe (2010) investigate the asymptotic discretization gap between the value of an option on discrete realized variance and that on continuous realized variance. In particular, they manage to find the asymptotic distribution of the discretization gap as $T \to 0^+$ for any payoff that satisfies certain continuity and uniform-boundedness conditions.

For general pricing purpose, it may be preferable to have direct approximation formulas for pricing derivatives on discrete realized variance rather than deriving the corresponding continuous counterpart plus a correction term. The derivation of the value of the continuously sampled counterpart may not be an easy task under the general Lévy dynamics. This procedure alone often requires the Fourier inversion calculations, which can be accomplished only if the moment generating function is available in closed form. One feasible approach is to derive the analytic approximation of the Laplace/Fourier transform of the price functions of volatility derivatives with general payoff structures. The analytic approximation of the relevant Laplace/Fourier integral can be effectively achieved via the saddlepoint approximation method by estimating the principal contribution of the integrand in the transform integral under an appropriately chosen contour of integration in the complex domain.

In this paper, we derive various saddlepoint approximation formulas for pricing options and volatility swaps on discrete realized variance under Lévy models and stochastic volatility models. We manage to derive the small time asymptotic approximation of the Laplace transform of discrete realized variance and obtain the saddlepoint approximation pricing formulas for options on discrete realized variance. Also, thanks to the conditional independency of the log asset price returns given a realization of the stochastic variance path, we develop the conditional saddlepoint approximation method for pricing options on discrete realized variance and volatility swaps. Using this analytic-simulation approach, the dimensionality of the simulation procedure of the asset price process is reduced. For some choices of the underlying variance processes, the simulation procedure of the discrete realized variance can be effectively performed using an appropriate exact simulation method.

This paper is organized as follows. In the next section, we discuss the appropriate choices of various saddlepoint approximations that can be adopted for pricing options on discrete realized variance and volatility swaps, assuming that the corresponding cumulant generating function (CGF) of the discrete realized variance $I(0,T;N)$ is available. Special precaution is taken to ensure that the algebraic root of the saddlepoint equation lies inside the domain of definition of the CGF. In Section 3, we show the detailed mathematical procedure of pricing options on discrete realized variance using the saddlepoint approximation method under Lévy models and stochastic volatility models with simultaneous jumps. In Section 4, we develop the conditional saddlepoint approximation method, so named since this is accomplished by simulating a random path of the stochastic variance process and followed by applying the appropriate saddlepoint approximation to derive the semi-analytic approximate pricing formulas. Numerical tests on the performance and reliability of the saddlepoint approximation formulas for pricing options on discrete realized variance and volatility swaps are reported in Section 5. Conclusive remarks are presented in the last section.

2 Saddlepoint approximation methods

The saddlepoint approximation method was first introduced by Daniels (1954) as an analytic method for approximating the density of the sample mean of a set of independent and identically distributed random variables whose CGF is known. The derivation and implementation of the saddlepoint approximation rely on a set of statistical techniques and mathematical tools, like exponential tilting, Edgeworth expansion, steepest descent, etc. The literature on the saddlepoint approximation method in statistics is quite voluminous (Lieberman, 1994; Strawderman, 2000; Studer, 2001). Good comprehensive treatise of the saddlepoint approximation methods can be found in Jensen (1995) and Butler (2004). Lugannani and Rice (1980) propose an effective saddlepoint approximation formula for the calculations of the tail probabilities. Wood *et al.* (1993) argue that the normal base distribution may fail to perform well in some cases. They propose an extension of the Lugannani-Rice formula that is applicable to an arbitrary base distribution. More recently, Aït-Sahalia and Yu (2006) and Glasserman and Kim (2009) use the saddlepoint approximation methods to derive analytic approximations to the transition densities and cumulative distribution functions of Markov processes and affine jump-diffusion processes.

The literature on the use of saddlepoint approximation methods in option pricing is relatively thin. Rogers and Zane (1998) apply the saddlepoint approximation to price European options under Lévy dynamics using the Lugannani-Rice formula. Xiong *et al.* (2005) extend Rogers-Zane's approach to price options under stochastic volatility and stochastic interest rates. Carr and Madan (2009) manage to represent the call option price as a single tail probability, which is interpreted as the probability of staying above the log strike price for the log asset price minus an independent exponential random variable. Based on a similar technique proposed by Wood *et al.* (1993), they derive the corresponding Lugannani-Rice saddlepoint approximation with a non-Gaussian base distribution. Besides pricing European vanilla options, several papers also adopt the saddlepoint approximation methods to price collateralized debt obligations (CDOs) (Antonov *et al.*, 2005; Yang *et al.*, 2006; Huang *et al.*, 2011) and analyze portfolio credit loss distributions (Martin, 2006; Huang *et al.*, 2007).

In the derivation procedure of the saddlepoint approximation of the transition density function and tail probabilities of a random variable, we commonly assume the existence of an analytic form of the CGF so that analytic expressions for the derivatives of the CGF of various orders can be obtained. Also, it is commonly assumed that the CGF $\kappa(z)$ is finite in some open strip $\{z : \alpha_- < \text{Re}(z) < \alpha_+\}$ in the complex plane that contains the imaginary axis, where $\alpha_- < 0$ and $\alpha_+ > 0$; and both α_- and α_+ can be infinite. In this paper, we also develop an alternative viable approach so that the saddlepoint approximation can be derived even when $\kappa(z)$ is defined only in the left half complex plane not including the imaginary axis. When the analytic expression of the CGF is not available, Aït-Sahalia and Yu (2006) demonstrate that useful analytic approximations can be obtained by replacing the characteristic function by an analytic expansion formula in small time. Alternatively, one may follow the numerical procedure developed by Glasserman and Kim (2009) to obtain the numerical approximation of the characteristic function.

First, we state several Lugannani-Rice type saddlepoint approximation formulas under the usual assumption that the domain of definition of $\kappa(z)$ contains the imaginary axis in the complex plane. We then illustrate an alternative version of the steepest descent approach when the algebraic root of the saddlepoint equation lies outside the domain of definition of $\kappa(z)$.

2.1 Exponentially tilted distribution

Let $\kappa(\theta)$ and $\kappa_0(\theta)$ denote the CGF of the random discrete realized variance I and $I - K$, respectively, where K is a fixed constant (related to the fixed strike in the call payoff). The two CGFs are related by

$$\kappa_0(\theta) = \kappa(\theta) - K\theta.$$

We write $X = I - K$ and $F_0(x)$ as the distribution function of X. Let $F_\theta(x)$ denote the distribution function of the exponentially θ-tilted distribution of X, where

$$dF_\theta(x) = e^{\theta x - \kappa_0(\theta)} dF_0(x). \tag{2.1}$$

The CGF of the θ-tilted distribution is related to the original one by

$$\kappa_\theta(t) = \kappa_0(t + \theta) - \kappa_0(\theta).$$

We would like to derive the saddlepoint approximation to the tail expectation $E[X^+] = E[X\mathbf{1}_{\{X>0\}}]$. Note that

$$
\begin{aligned}
E[X\mathbf{1}_{\{X>0\}}] &= E\left[\frac{\partial}{\partial\theta}e^{\theta X}\mathbf{1}_{\{X>0\}}\Big|_{\theta=0}\right] \\
&= \frac{\partial}{\partial\theta}\left[e^{\kappa_0(\theta)}\int_0^\infty e^{\theta x-\kappa_0(\theta)}dF_0(x)\right]\Big|_{\theta=0} \\
&= \kappa_0'(0)[1-F_0(0)] - \frac{\partial F_\theta(0)}{\partial\theta}\Big|_{\theta=0}. \qquad (2.2)
\end{aligned}
$$

Let $\hat{F}_\theta(x)$ denote the Lugannani-Rice approximation to $F_\theta(x)$, which is defined to be

$$
F_\theta(x) \approx \hat{F}_\theta(x) = N(w_\theta) + n(w_\theta)\left(\frac{1}{w_\theta} - \frac{1}{u_\theta}\right), \qquad (2.3a)
$$

where $N(\cdot)$ and $n(\cdot)$ denote the standard normal distribution function and density function, respectively, and

$$
\begin{aligned}
w_\theta &= \operatorname{sgn}(\hat{t}_x - \theta)\{2[(\hat{t}_x-\theta)x - \kappa_0(\hat{t}_x) + \kappa_0(\theta)]\}^{1/2}, \\
u_\theta &= (\hat{t}_x - \theta)\sqrt{\kappa_0^{(2)}(\hat{t}_x)}.
\end{aligned}
$$

We use $\kappa_0^{(n)}(t)$ to denote the n^{th} order derivative of $\kappa_0(t)$. Here, $\hat{t}_x$ denotes the unique solution (with dependence on x) to the saddlepoint equation:

$$
\kappa_0'(t) = x.
$$

By differentiating Eq. (2.3a) with respect to θ, we obtain

$$
\frac{\partial\hat{F}_\theta(x)}{\partial\theta} = n(w_\theta)\left\{[x-\kappa_0'(\theta)]\left(\frac{1}{w_\theta^3} - \frac{1}{u_\theta}\right) - \frac{1}{(\hat{t}_x-\theta)^2\sqrt{\kappa_0^{(2)}(\hat{t}_x)}}\right\}. \qquad (2.3b)
$$

To obtain the analytic expressions for $\hat{F}_0(0)$ and $\frac{\partial\hat{F}_\theta(0)}{\partial\theta}\Big|_{\theta=0}$ that approximate the two terms $F_0(0)$ and $\frac{\partial F_\theta(0)}{\partial\theta}\Big|_{\theta=0}$ in Eq. (2.2), we set $\theta = 0$ and $x = 0$ in Eqs. (2.3a, 2.3b). Putting these results together, we obtain the saddlepoint approximation to the tail expectation $E[X^+]$. In a similar manner, the saddlepoint approximation to the tail expectation at the tail of the other side is given by

$$
E[(-X)^+] \approx -\kappa_0'(0)\hat{F}_0(0) - \frac{\partial\hat{F}_\theta(0)}{\partial\theta}\Big|_{\theta=0}. \qquad (2.4)
$$

As a remark, in the context of option pricing, $E[\,\cdot\,]$ denotes the expectation taken under a risk neutral pricing measure Q.

2.2 Approximation to the Fourier inversion integrals

Other types of saddlepoint approximation methods may be derived from the Fourier inversion representation of the tail expectation. The steepest descent method can then be subsequently used to find an approximate value of the principal contribution to the Fourier integral. Again, various approximation approaches may be adopted. Taking the call option on discrete realized variance as an example, the corresponding tail expectation can be expressed as

$$E[X\mathbf{1}_{\{X>0\}}] = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \frac{e^{\kappa_0(t)}}{t^2}\,dt, \quad \tau \in (0,\alpha_+), \text{ for some } \alpha_+ > 0. \tag{2.5}$$

The most popular approach is to expand $\kappa_0(t)$ at its saddlepoint $\hat{t}$ that uniquely solves the saddlepoint equation: $\kappa_0'(t) = 0$. For example, Antonov *et al.* (2005) derive the following saddlepoint approximation formula for the tail expectation:

$$E[X^+] \approx \frac{1}{2\pi i}\int_{\hat{t}-i\infty}^{\hat{t}+i\infty} \frac{e^{\kappa_0(\hat{t})+\frac{1}{2}\kappa_0^{(2)}(\hat{t})(t-\hat{t})^2}}{t^2}\left[1+\frac{1}{6}\kappa_0^{(3)}(\hat{t})(t-\hat{t})^3+\cdots\right]dt. \tag{2.6a}$$

In a similar manner, Martin (2006) approximates the local quadratic behavior of the exponent term around the saddlepoint $\hat{t}$ and derives the corresponding Lugannani-Rice type formula as follows:

$$E[X^+] = \begin{cases} \mu_X P(X>0) - \frac{\mu_X}{\hat{t}} f_X(0) & \hat{t} \neq 0 \\ \kappa_0^{(2)}(\hat{t}) f_X(0) & \hat{t} = 0 \end{cases}, \tag{2.6b}$$

where μ_X is the mean of X and $f_X(t)$ is the density function of X.

The above methods of expanding $\kappa_0(t)$ around $\hat{t}$ have been widely adopted in the literature. However, it may occur that the solution $\hat{t}$ to the saddlepoint equation: $\kappa_0'(t) = 0$ lies outside the range in which $\kappa_0(t)$ is defined. In this case, the above mathematical procedure fails. For instance, suppose $\kappa(t)$ is defined only for non-positive values of t, the aforementioned problem arises when we attempt to solve the saddlepoint equation: $\kappa_0'(t) = \kappa'(t) - K = 0$ for $K > \kappa'(0)$. Fortunately, the difficulty can be resolved by using another version of the steepest descent method presented below.

Following Yang *et al.* (2006), we write the integrand as $e^{\kappa_0(t)-2\ln t}$. The corresponding saddlepoint equation is modified to the form: $\kappa_0'(t) - 2/t = 0$. Provided that $\kappa(t)$ is well defined, the saddlepoint equation now has two roots, one positive and the other negative. Recall that the contour in Eq. (2.5) is taken to be along a vertical line parallel to the imaginary axis, one has to choose the positive root if we prefer to follow the usual saddlepoint approximation procedure. Suppose one is forced to choose the negative root that lies inside the domain of definition of $\kappa(t)$, we may transfer the vertical contour to the negative half plane and apply the usual procedure of performing expansion of the exponent around the saddlepoint to obtain the approximation value. As part of the procedure, it is necessary to take care of the contribution from the residue of the integrand at the origin. As an illustration, we consider the saddlepoint approximation to the following tail expectations:

$$\Xi_1 = E[X\mathbf{1}_{\{X>0\}}] = \frac{1}{2\pi i}\int_{\tau_1-i\infty}^{\tau_1+i\infty} \frac{e^{\kappa_0(t)}}{t^2}dt, \quad \tau_1 \in (0,\alpha_+) \text{ where } \alpha_+ > 0; \tag{2.7a}$$

$$\Xi_2 = -E[X\mathbf{1}_{\{X<0\}}] = \frac{1}{2\pi i}\int_{\tau_2-i\infty}^{\tau_2+i\infty} \frac{e^{\kappa_0(t)}}{t^2}dt, \quad \tau_2 \in (\alpha_-,0) \text{ where } \alpha_- < 0. \tag{2.7b}$$

Following the approach in Yang *et al.* (2006), the saddlepoint approximation to Ξ_j is given by

$$\Xi_j \approx \hat{\Xi}_j = \frac{e^{\kappa_0(\hat{t}_j)}/\hat{t}_j^2}{\sqrt{2\pi\left[\frac{2}{\hat{t}_j^2} + \kappa_0^{(2)}(\hat{t}_j)\right]}}, \; j = 1,2, \tag{2.8a}$$

where $\hat{t}_1 > 0$ ($\hat{t}_2 < 0$) is the positive (negative) root within (α_-, α_+) of the saddlepoint equation:

$$\kappa_0'(t) - 2/t = 0.$$

The second order saddlepoint approximation to Ξ_j is given by

$$\tilde{\hat{\Xi}}_j = \hat{\Xi}_j(1 + R_j), \quad j = 1,2, \tag{2.8b}$$

where the adjustment term R_j is given by

$$R_j = \frac{1}{8}\frac{\kappa_0^{(4)}(\hat{t}_j) + 12\hat{t}_j^{-4}}{[\kappa_0^{(2)}(\hat{t}_j) + 2\hat{t}_j^{-2}]^2} - \frac{5}{24}\frac{[\kappa_0^{(3)}(\hat{t}_j) - 4\hat{t}_j^{-3}]^2}{[\kappa_0^{(2)}(\hat{t}_j) + 2\hat{t}_j^{-2}]^3}, \; j = 1,2.$$

Note that $\Xi_1 - \Xi_2 = \mu_X$, a result that is consistent with the put-call parity in option pricing theory. Suppose both roots $\hat{t}_1$ and $\hat{t}_2$ exist, we can use either the saddlepoint approximation $\hat{\Xi}_1$ ($\tilde{\hat{\Xi}}_1$) or $\mu_X + \hat{\Xi}_2$ ($\mu_X + \tilde{\hat{\Xi}}_2$) to approximate the value of the call option. To achieve better performance, the rule of thumb is to use the former if $\mu_X < 0$ [equivalently, $K > \kappa'(0)$] or the latter if $\mu_X > 0$. Apparently, some extra efforts may be required to determine which saddlepoint to be adopted. As a remark, suppose we only have the CGF defined on the negative part of the real axis, a vertical contour in the left half complex plane should be adopted for deriving the saddlepoint approximation.

In a similar manner, Yang *et al.*'s approach can also be adopted in the valuation of volatility swaps, thanks to the identity:

$$E[\sqrt{I}] = \frac{1}{4\sqrt{\pi}i}\int_{\tau-i\infty}^{\tau+i\infty} \frac{e^{\kappa(t)}}{t^{3/2}}\mathrm{d}t, \quad 0 < \tau < \alpha. \tag{2.9}$$

Provided that a positive root $\hat{t} > 0$ exists which solves the saddlepoint equation:

$$\kappa'(t) - \frac{3}{2t} = 0,$$

the corresponding saddlepoint approximation is given by

$$E[\sqrt{I}] \approx \frac{\sqrt{2}}{4} \frac{e^{\kappa(\hat{t})}/\hat{t}^{3/2}}{\sqrt{\kappa^{(2)}(\hat{t}) + \frac{3}{2}\hat{t}^2}}. \tag{2.10a}$$

One can also derive the second order approximation formula:

$$E[\sqrt{I}] \approx \frac{\sqrt{2}}{4} \frac{e^{\kappa(\hat{t})}/\hat{t}^{3/2}}{\sqrt{\kappa^{(2)}(\hat{t}) + \frac{3}{2}\hat{t}^2}}(1 + R), \tag{2.10b}$$

where the adjustment term R is given by

$$R = \frac{1}{8} \frac{\kappa^{(4)}(\hat{t}) + 9\hat{t}^{-4}}{[\kappa^{(2)}(\hat{t}) + 3\hat{t}^{-2}/2]^2} - \frac{5}{24} \frac{[\kappa^{(3)}(\hat{t}) - 3\hat{t}^{-3}]^2}{[\kappa^{(2)}(\hat{t}) + 3\hat{t}^{-2}/2]^3}.$$

Hereafter, we categorize the saddlepoint approximation methods discussed above into two types based on the form of the corresponding saddlepoint equation. In this paper, the class of methods associated with the saddlepoint equation, $\kappa_0'(t) = 0$, are called the *classical saddlepoint approximation* (CSPA) methods, while the class of methods whose saddlepoint equation has an additional right-hand term, like $\frac{2}{t}$ or $\frac{3}{2t}$, are called the *alternative saddlepoint approximation* (ASPA) methods.

2.3 Further extension of the approximation formulas

There are other alternative approaches of deriving the saddlepoint approximation to the tail expectations and $E[\sqrt{I}]$ that are beyond the methods discussed above. For example, the original Lugannani-Rice formula uses the normal distribution as the base distribution. However, the normal distribution is not always a good choice. In particular, accuracy of approximation would deteriorate significantly when we consider pure jump processes with no diffusion in the underlying asset price. One may consider the extension of the Lugannani-Rice formula using an appropriate base distribution that resembles closer to the underlying asset price process. Besides, computational efficiency of the saddlepoint approximation method is largely dependent on the root-finding procedure in solving the saddlepoint equation. When the numerical solution to the saddlepoint equation is time consuming, computational efficiency of the saddlepoint approximation method would be degraded. Lieberman (1994) and Glasserman and Kim

(2009) provide various analytical approximate expressions for the solution to the saddlepoint equation of the form: $\kappa'(t) - K = 0$. The availability of an improved analytical approximation formula for the solution to the saddlepoint equation would add versatility to the saddlepoint approximation method.

3 Small time asymptotic approximation of moment generating functions

The effective implementation of the saddlepoint approximation methods relies crucially on the availability of the analytic form of the moment generating function. For most asset price models, it is in general difficult to derive closed form expression for the moment generating function or the Laplace transform of the discrete realized variance. In this section, we first present a brief review of the small time asymptotic approximation (STAA) method due to Keller-Ressel and Muhle-Karbe (2010). We then explain how it can be applied to derive accurate approximate moment generating functions (defined in the left half of the complex plane) for discrete realized variance under the exponential Lévy models and stochastic volatility models with jumps.

Suppose the log asset price is governed by the following semimartingale process:

$$\mathrm{d}X_t = b_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t + \int k_t(x)(N(\mathrm{d}t,\mathrm{d}x) - F(\mathrm{d}x)\mathrm{d}t), \quad X_0 = 0, \tag{3.1}$$

where W_t is a standard Brownian motion, $N(\mathrm{d}t, \mathrm{d}x)$ is a Poisson random measure with absolutely continuous compensator $F(\mathrm{d}x)\mathrm{d}t$. Also, the parameter functions b, σ and k are predictable integrands that satisfy the following constraint:

$$\int_0^T E\left[b_t^2 + \sigma_t^2 + \int k_t(x)F(\mathrm{d}x)\right]\mathrm{d}t < \infty. \tag{3.2}$$

When t is small, X_t can be approximated by the square-integrable Lévy process $\bar{X}_t$

$$\mathrm{d}\bar{X}_t = b_0\,\mathrm{d}t + \sigma_0\,\mathrm{d}W_t + \int k_0(x)(N(\mathrm{d}t,\mathrm{d}x) - F(\mathrm{d}x)\mathrm{d}t), \quad \bar{X}_0 = 0, \tag{3.3}$$

obtained from X_t by "freezing" the integrands b, σ and k of X_t at the respective values at time zero. Under some additional technical assumptions, we have the following theorem that describes the asymptotic distributional properties of the quadratic variation process and discrete realized variance of X_t (Keller-Ressel and Muhle-Karbe, 2011).

Theorem. *Let X_t be a semimartingale of the form (3.1).*

(a) *Suppose the set of payoff functions (indexed by T) $g_T : \mathbb{R}_+ \to \mathbb{R}$, $T \geq 0$, are continuous, uniformly bounded and satisfy $\|g_T - g_0\|_\infty \to 0$ as $T \to 0^+$. Moreover, suppose that g_0 is Lipschitz continuous. We have the following small time asymptotic approximation result:*

$$\lim_{T\to 0^+} E\left[g_T\left(\frac{1}{T}[X,X]_T\right)\right] = g_0(\sigma_0^2). \tag{3.4a}$$

(b) *Suppose that the set of payoff functions (indexed by T) $g_{n,T} : \mathbb{R}_+ \to \mathbb{R}$, $T \geq 0$, $n \in \mathbb{N}$, are continuous, uniformly bounded and satisfy $\|g_{n,T} - g_{n,0}\|_\infty \to 0$ as $T \to 0^+$ for each n. Moreover, suppose that $g_{n,0}$ is Lipschitz continuous. We observe the following small time asymptotic limiting property:*

$$\lim_{T\to 0^+} E\left[g_{n,T}\left(I(0,T;n)\right)\right] = E[g_{n,0}(Z_n)], \tag{3.4b}$$

where Z_n has the gamma distribution with shape parameter $n/2$ and scale parameter $2\sigma_0^2/n$.

Note that the above theorem is applicable to most of the prevailing asset price models, including the Lévy models and stochastic volatility models. In what follows, we mainly discuss the application of this small time asymptotic approximation method under the exponential Lévy models and stochastic volatility models with jumps.

3.1 Lévy models

Since the Lévy process is known to have independent and stationary increments, so the increments $X_{t_k} - X_{t_{k-1}}$, $k = 1, 2, \cdots, N$, as defined in Eq. (1.1) are independent. In addition, they become identically distributed when the time step is taken to be uniform. The characteristic function of the underlying Lévy process X_t admits the Lévy-Khinchine representation: $E[e^{uX_t}] = e^{t\psi(u)}$, where the characteristic exponent $\psi(u)$ is given by

$$\psi(u) = bu + \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} [e^{ux} - 1 - uh(x)]F(\mathrm{d}x), \quad u \in \mathbb{C}, \tag{3.5}$$

with $h(\cdot)$ being an appropriately chosen truncated function. Moreover, we assume the convergence strip of $\psi(u)$ to be in some open strip $\{u : \alpha_- < \mathrm{Re}(u) < \alpha_+\}$ in the complex plane, where $\alpha_- < 0$ and $\alpha_+ > 0$. The Lévy process is fully characterized by the triplet $(b, \sigma^2, F)_h$ with dependence on h. We take the time steps to be uniform, where $t_i - t_{i-1} = \Delta$, $i = 1, \cdots, N$.

Given the independent increment property of X_t, the calculation of the moment generating function (MGF) of $I(0,T;N)$ amounts to the calculation of the Laplace transform of the squared process X_t^2. Observe that

$$E[e^{-uI}] = (E[e^{-uA/NX_\Delta^2}])^N,$$

where X_Δ^2 denotes the increment of X_t^2 over time period Δ. For brevity, we write $Y_t = X_t^2$ and $Y = X_\Delta^2$. As a remark, there exists the following integral representation of the Laplace transform of Y_t for all $u \in \mathbb{R}_+$:

$$M_{Y_t}(-u) = E[e^{-uX_t^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\psi(ix\sqrt{2u})-x^2/2} dx.$$

Though the saddlepoint approximation method only requires knowledge of the MGF on the real axis, the above integral representation is not quite useful since differentiation with respect to u of the CGF [defined by $\ln M_{Y_t}(u)$] is quite cumbersome. It would be almost infeasible to derive any tractable analytical expression for higher order derivatives of the CGF.

To obtain an approximation that is more analytically tractable than the above integral representation, we make use of the small time asymptotic approximation. Let $g_T,\ g_{n,T} : x \mapsto e^{ux}$ for any fixed $u \in \mathbb{C}_-$ and consider approximating the Laplace transform of Y by that of $[X, X]_\Delta$. Apparently, Y can be regarded as the realized variance over $[0, \Delta]$ with $n = 1$. For each $u \in \mathbb{C}_-$, we then derive the asymptotic limit of the Laplace transform of $[X, X]_\Delta$ and Y as $e^{u\Delta\sigma^2}$ and $\frac{1}{(1-2u\Delta\sigma^2)^{1/2}}$, respectively. To proceed, we define the "discretization ratio" as follows:

$$\Lambda_n(g) := \lim_{T\to 0^+} \frac{E\left[\exp\left(uI(0, T; n)\right)\right]}{E\left[\exp\left(u\frac{1}{T}[X, X]_T\right)\right]} = E[\exp(uZ_n)]e^{-u\sigma^2}. \tag{3.6}$$

We assume that this ratio is preserved for different values of T so that it can be used as an adjustment to deduce the approximate Laplace transform of Y as follows:

$$\hat{M}_Y(u) = M_{[X,X]_\Delta}(u) \frac{e^{-u\Delta\sigma^2}}{(1 - 2u\Delta\sigma^2)^{1/2}}. \tag{3.7}$$

In particular, when X_t is a pure jump process with $\sigma = 0$, Eq. (3.7) reduces to $\hat{M}_Y(u) = M_{[X,X]_\Delta}(u)$. In other words, we take the MGF of quadratic variation directly as an approximation without any adjustment under the pure jump model. Moreover, the STAA becomes exact if X_t is a Gaussian process. It is well known that the quadratic variation process is still a Lévy process whose characteristic triplet can be derived from that of the original process. Indeed, we have

$$M_{[X,X]_\Delta}(u) = E[e^{u[X,X]_\Delta}] = \exp\left(\Delta\left[\sigma^2 u + \int (e^{ux^2} - 1)F(\mathrm{d}x)\right]\right), \tag{3.8}$$

where $\mathrm{Re}(u) \leq 0$ [Kallsen *et al.* (2009) and Carr *et al.* (2005)]. Apparently, the MGF of the quadratic variation is still not in a closed form representation when the jump integral cannot be explicitly integrated. In practice, we find that it is more preferable to choose this form since differentiation with respect to u is rather straightforward when performed under this representation.

The combination of Eqs. (3.7) and (3.8) naturally leads to an approximate expression for the CGF of Y:

$$\hat{\kappa}(u) = \Delta g(u) - \frac{1}{2}\ln(1 - 2\Delta\sigma^2 u). \tag{3.9a}$$

The corresponding higher order derivatives are given by

$$\hat{\kappa}^{(n)}(u) = \Delta g^{(n)}(u) + \frac{(n-1)!}{2}\left(\frac{2\Delta\sigma^2}{1 - 2\Delta\sigma^2 u}\right)^n, \; n = 1, 2, \cdots, \tag{3.9b}$$

where

$$g(u) = \int (e^{ux^2} - 1)F(dx), \quad g^{(n)}(u) = \int e^{ux^2} x^{2n} F(dx), \; n = 1, 2, \cdots.$$

3.2 Stochastic volatility models with jumps

It is more appropriate to adopt stochastic variance as one of the risk factors in the underlying asset price process when we consider pricing of variance products and volatility derivatives. Unfortunately, it is almost intractable to derive an analytic form of the MGF of the discrete realized variance $I(0, T; N)$. One primitive approach is to use the quadratic variation $I(0, T; \infty)$ as a proxy of $I(0, T; N)$ since the MGF of $I(0, T; \infty)$ under an affine stochastic volatility model can be derived analytically by solving a Ricatti system of ordinary differential equations. However, the limitations of this quadratic variation approximation have been discussed earlier. For more detailed numerical experiments on this issue, see Bühler (2006), Keller-Ressel and Muhle-Karbe (2010), or Zheng and Kwok (2013).

In what follows, we derive the small time asymptotic approximation of the MGF of $I(0, T; N)$ under the general framework of stochastic volatility models with jumps. While different s-tochastic volatility models have different specifications of the stochastic volatility process V_t, the log asset price process can be described by the following semimartingale dynamics under a risk neutral measure Q:

$$\mathrm{d}\ln S_t = b_t\,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W_t^S + \int x(N(\mathrm{d}t, \mathrm{d}x) - F(\mathrm{d}x)\,\mathrm{d}t), \tag{3.10}$$

where W_t^S is a standard Brownian motion and

$$b_t = r - d - \frac{V_t}{2} + \int (e^x - x - 1)F(\mathrm{d}x),$$

such that the discounted dividend-adjusted asset price process $S_t e^{-(r-d)t}$ is a Q-martingale. Here, r is the constant risk-free rate and d is the constant continuous dividend yield. By assuming that V_t follows the square root process with Poisson jumps, we can retrieve the following familiar form of Heston's stochastic volatility model with simultaneous jumps in asset returns and variance (SVSJ model):

$$\begin{cases} \dfrac{\mathrm{d}S_t}{S_t} &= (r-d-\lambda m)\,\mathrm{d}t + \sqrt{V_t}(\rho\,\mathrm{d}W_t^V + \sqrt{1-\rho^2}\,\mathrm{d}B_t) + (e^{J^S}-1)\,\mathrm{d}N_t, \\ \mathrm{d}V_t &= \kappa(\theta - V_t)\,\mathrm{d}t + \varepsilon\sqrt{V_t}\,\mathrm{d}W_t^V + J^V\mathrm{d}N_t, \end{cases} \tag{3.11}$$

where W_t^V and B_t are a pair of independent standard Brownian motions, N_t is a Poisson process with constant intensity λ that is independent of the two Brownian motions, ρ is the correlation coefficient between S_t and V_t, ε is the volatility of V_t, κ is the drift parameter and θ is the long-term mean of the variance process V_t. We let J^S and J^V denote the random jump sizes of the log asset price and its variance, respectively. These random jump sizes are assumed to be independent of W_t^V, B_t and N_t. While one has the freedom to specify the customized jump distributions, we would like to proceed with the canonical jump size distributions: $J^V \sim \exp(1/\eta)$ and $J^S|J^V \sim \mathcal{N}(\nu + \rho_J J^V, \delta^2)$. These distributions correspond to the exponential distribution with mean η and the Gaussian distribution with mean $\nu + \rho_J J^V$ and variance δ^2 conditional on J^V, respectively. Also, we take $m = E[e^{J^S} - 1]$ so that the process $S_t e^{-(r-d)t}$ is a Q-martingale. In Eq. (3.11), we have used the distributional equivalence: $\mathrm{d}W_t^S = \rho\,\mathrm{d}W_t^V + \sqrt{1-\rho^2}\,\mathrm{d}B_t$ and the compound Poisson jump term is written as

$$\int xN(\mathrm{d}t, \mathrm{d}x) = J^S \mathrm{d}N_t.$$

We now apply the small time asymptotic approximation to the MGFs of the quadratic variation process $I(0,T;\infty) = \frac{1}{T}[\ln S_T, \ln S_T]$ and the discrete realized variance $I(0,T;N)$. Recall that the asymptotic limit of $I(0,T;N)$ as $T \to 0^+$ is a gamma distribution with shape parameter $N/2$ and scale parameter $2V_0/N$. The corresponding MGF is given by $\left(1 - \frac{2V_0 u}{N}\right)^{-N/2}$. Therefore, for any $u \le 0$, we obtain

$$\lim_{T\to 0^+} M_{I(0,T;\infty)}(u) = e^{uV_0}, \tag{3.12a}$$

$$\lim_{T\to 0^+} M_{I(0,T;N)}(u) = \left(1 - \frac{2V_0 u}{N}\right)^{-N/2}. \tag{3.12b}$$

Assuming that the difference $M_{I(0,T;N)}(u) - M_{I(0,T;\infty)}(u)$ is invariant with respect to T, we use the above difference as a control and propose the following approximate MGF formula:

$$\hat{M}_{I(0,T;N)}(u) = M_{I(0,T;\infty)}(u) + \left(1 - \frac{2V_0 u}{N}\right)^{-N/2} - e^{uV_0}, \quad u \in \mathbb{C}_-. \tag{3.13}$$

Note that the above approximation formula holds under the general stochastic volatility framework as specified by Eq. (3.10). Here, we do not use the same ratio adjustment as we did in Eq. (3.7), though it is less cumbersome with regard to the computation of the CGF and its derivatives. The reason for adopting the approximation in Eq. (3.13) instead of the alternative approximation in Eq. (3.7) is explained as follows. From the perspective of computational stability, when u takes a very negative value, $e^{-u\Delta\sigma^2}$ grows exponentially and this leads to erosion of the approximation in Eq. (3.7). Under the Lévy model, there is a canceling factor in $M_{[X,X]_\Delta}(u)$ and the approximation remains to be stable. Unfortunately, we do not have such a property under the stochastic volatility models. As a result, approximation formulas like Eq. (3.7) may likely blow up for very negative values of u.

After some tedious calculations, the approximate CGF and its higher order derivatives are given by

$$\begin{aligned}
\hat{\kappa}_{I(0,T;N)}(u) &= \ln \hat{M}_{I(0,T;N)}(u), \\
\hat{\kappa}'_{I(0,T;N)}(u) &= \frac{M'_{I(0,T;\infty)}(u) + f_1(u)}{M_{I(0,T;N)}(u)}, \\
\hat{\kappa}^{(2)}_{I(0,T;N)}(u) &= \frac{M^{(2)}_{I(0,T;\infty)}(u) + f_2(u)}{\hat{M}_{I(0,T;N)}(u)} - \frac{[M'_{I(0,T;\infty)}(u) + f_1(u)]^2}{[\hat{M}_{I(0,T;N)}(u)]^2},
\end{aligned}$$

where

$$f_n(u) = V_0^k \frac{\frac{N}{2}\left(\frac{N}{2}+1\right)\cdots\left(\frac{N}{2}+n\right)}{\left(\frac{N}{2}\right)^n}\left(1 - \frac{2V_0 u}{N}\right)^{-N/2-n}, \ n = 1, 2, \cdots.$$

Higher order derivatives are also available, except that they involve more tedious expressions.

Given the analytic expression of the MGF of $I(0,T;N)$, one faces the question of choosing which saddlepoint approximation method to be used. As mentioned before, since the CGF of $I(0,T;N)$ is only defined on the left half plane, the CSPA methods have no saddlepoint on $(-\infty, 0)$ when $K > \kappa'(0)$, and hence this approach may not work for all range of strikes. On the other hand, the ASPA methods can work effectively under this unusual scenario since a negative saddlepoint is guaranteed. In other words, we may conclude that the saddlepoint approximation formulas shown in Eqs. (2.8a, 2.8b) are better choices that work for all range of strikes.

4 Enhanced simulation methods under stochastic volatility models with jumps

Despite the merits of the small time asymptotic approximation method for the MGF of $I(0,T;N)$ under the stochastic volatility models as shown in Section 3.2, the successful implementation of the saddlepoint approximation method also relies crucially on the existence of the analytic form of the MGF of $I(0,T;\infty)$. For a more general stochastic volatility model with non-affine structure, there is no guarantee for a tractable form of the MGF of $I(0,T;\infty)$. As a result, we may encounter difficulty in applying the above results presented in Section 3.2. As an alternative approach, we would like to introduce the conditional saddlepoint approximation approach that makes use of the conditional MGF of $I(0,T;N)$ given a realization path of the stochastic volatility process. The success of this approach relies on the property of conditional independency of the log asset price returns given a realization of the stochastic volatility path.

Suppose the joint dynamics of S_t and its instantaneous variance V_t under a risk neutral pricing measure Q is specified by the following generalized stochastic differential equations:

$$\begin{cases} \dfrac{\mathrm{d}S_t}{S_t} &= (r-d-\lambda m)\,\mathrm{d}t + \sqrt{V_t}(\rho\,\mathrm{d}W_t^V + \sqrt{1-\rho^2}\,\mathrm{d}B_t) + (e^{J^S}-1)\,\mathrm{d}N_t, \\ \mathrm{d}V_t &= \alpha(V_t)\,\mathrm{d}t + \beta(V_t)\,\mathrm{d}W_t^V + J^V\mathrm{d}N_t, \end{cases} \tag{4.1}$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ are twice differentiable functions on $\mathbb{R}_+$, and $\beta(\cdot)$ is a nonzero function. The SVSJ model as specified in Eq. (3.11) is a special case under the above generalized model. Drimus and Farkas (2012) show that conditional on a realization path of V_t the log returns under the simple Heston stochastic volatility model are independent to each other and follow the normal distribution. In fact, the same property still holds under the asset price dynamics specified by Eq. (4.1). It can be shown that (see Appendix A)

$$\begin{aligned} \ln\frac{S_{t_k}}{S_{t_{k-1}}} = &\int_{t_{k-1}}^{t_k}\left(r-d-m\lambda-\frac{V_t}{2}\right)\mathrm{d}t + \sqrt{1-\rho^2}\int_{t_{k-1}}^{t_k}\sqrt{V_t}\,\mathrm{d}B_t + \sum_{m=1}^{N_{\Delta t_k}} J_m^S \\ &+\rho\left\{f(V_{t_k}) - f(V_{t_{k-1}}) - \int_{t_{k-1}}^{t_k}\left[f(V_t)\alpha(V_t) + \frac{1}{2}f''(V_t)\beta^2(V_t)\right]\mathrm{d}t\right\} \\ &-\rho\int_{t_{k-1}}^{t_k}[f(V_t+J^V) - f(V_t)]\,\mathrm{d}N_t, \end{aligned} \tag{4.2}$$

where $\Delta t_k = t_k - t_{k-1}$, the random varaibles J_m^S are independent copies of J^s, and

$$f(x) = \int_0^x \frac{\sqrt{z}}{\beta(z)}\mathrm{d}z.$$

Assuming the usual canonical jump size distribution as specified in Section 3.2, and conditional on $\mathcal{F}_T^V$ (σ-algebra generated by V_t, $0 \le t \le T$), we have

$$\ln \frac{S_{t_k}}{S_{t_{k-1}}}\Big|_{\mathcal{F}_T^V} \sim \mathcal{N}(\mu_k, \sigma_k^2). \tag{4.3}$$

The respective conditional mean and variance are given by

$$\begin{aligned}
\mu_k &= \int_{t_{k-1}}^{t_k} \left(r - d - m\lambda - \frac{V_t}{2} \right) \mathrm{d}t - \rho \int_{t_{k-1}}^{t_k} [f(V_t + J^V) - f(V_t)] \, \mathrm{d}N_t \\
&\quad + \rho \left\{ f(V_{t_k}) - f(V_{t_{k-1}}) - \int_{t_{k-1}}^{t_k} \left[f'(V_t)\alpha(V_t) + \frac{1}{2} f''(V_t)\beta^2(V_t) \right] \mathrm{d}t \right\} \\
&\quad + \sum_{m=1}^{N_{\Delta t_k}} (\nu + \rho_J J_m^V), \\
\sigma_k^2 &= (1 - \rho^2) \int_{t_{k-1}}^{t_k} V_t \, \mathrm{d}t + \delta^2 N_{\Delta t_k},
\end{aligned}$$

where J_m^V are independent copies of J^V.

Since the log returns are conditionally normal and independent, the discrete realized variance is given by the sum of a sequence of independent squared normal random variables. By virtue of the normality property, its CGF can be calculated analytically. In fact, the MGF of $I(0, T; N)$ is given by

$$M(u) = \prod_{k=1}^{N} \frac{\exp(\frac{u}{T}\mu_k^2(1 - 2\sigma_k^2\frac{u}{T})^{-1})}{(1 - 2\sigma_k^2\frac{u}{T})^{1/2}}, \quad u < \min_{1 \le k \le N} \left\{ \frac{T}{2\sigma_k^2} \right\}. \tag{4.4}$$

It is relatively straightforward to find the corresponding CGF and its higher order derivatives, where

$$\kappa(u) = \sum_{k=1}^{N} \frac{\mu_k^2 u/T}{1 - 2\sigma_k^2 u/T} - \frac{1}{2} \ln(1 - 2\sigma_k^2 u/T); \tag{4.5a}$$

$$\kappa^{(n)}(u) = \sum_{k=1}^{N} \left(\frac{n\mu_k^2/T}{1 - 2\sigma_k^2 u/T} + \sigma_k^2/T \right) \frac{(n-1)!(2\sigma_k^2/T)^{n-1}}{(1 - 2\sigma_k^2 u/T)^n}, \; n = 1, 2, \cdots. \tag{4.5b}$$

Following a similar idea originated in Drimus and Farkas (2012), the key simulation procedures in the conditional saddlepoint approximation method are summarized as follows:

1. Simulate a path of the stochastic variance process $\{V_t : 0 \le t \le T\}$.

2. Compute the quantities μ_k and σ_k^2.

3. Apply the saddlepoint approximation method to derive the price of the volatility derivative.

4. Repeat steps 1 to 3 for a sufficiently large number of simulation runs and take the sampled average price.

Unlike the direct Monte Carlo simulation method under the stochastic volatility model, the conditional saddlepoint approximation method achieves dimension reduction of the simulation. This analytic-simulation approach improves computational efficiency of the simulation method significantly. Moreover, when the stochastic variance is specified as Heston's square root process or the 3/2 stochastic volatility model, one can make use of the computational advantages of the exact simulation methods [Broadie and Kaya (2006) and Baldeaux (2012)].

Heston's model with jumps

In Heston's stochastic volatility model with jumps, the variance process is given by

$$\mathrm{d}V_t = \kappa(\theta - V_t)\mathrm{d}t + \varepsilon\sqrt{V_t}\mathrm{d}W_t^V + J^V\mathrm{d}N_t, \tag{4.6}$$

which implies that $\alpha(V_t) = \kappa(\theta - V_t)$, $\beta(V_t) = \varepsilon\sqrt{V_t}$ and $f(V_t) = V_t/\varepsilon$. Consequently, the simplified conditional mean and variance are given by

$$\begin{aligned}
\mu_k &= (r - d - m\lambda)(t_k - t_{k-1}) + \frac{\rho}{\varepsilon}\Big[V_{t_k} - V_{t_{k-1}} - \kappa\theta(t_k - t_{k-1})\Big] \\
&\quad + \Big(\frac{\rho\kappa}{\varepsilon} - \frac{1}{2}\Big)\int_{t_{k-1}}^{t_k} V_t\,\mathrm{d}t + \nu N_{\Delta t_k} + \Big(\rho_J - \frac{\rho}{\varepsilon}\Big)\sum_{m=1}^{N_{\Delta t_k}} J_m^v, \\
\sigma_k^2 &= (1 - \rho^2)\int_{t_{k-1}}^{t_k} V_t\,\mathrm{d}t + \delta^2 N_{\Delta t_k}.
\end{aligned}$$

The simulation procedures for the above μ_k and σ_k^2 are summarized as follows. In the first step, we simulate a Poisson process with arrival rate λ and record all the jump times till T. Suppose we are now at time t_k and have already simulated μ_j and σ_j^2 for $1 \leq j \leq k$. To proceed with μ_{k+1} and σ_{k+1}^2, we perform the following steps:

1. Set $\tau_0 = t_k$ and $I = 0$.

2. Determine the next jump time τ. If $\tau > t_{k+1}$, then set $\tau = t_{k+1}$.

3. Disregard the jump part, and subsequently simulate the variance value V_τ and the integrated variance $\int_{\tau_0}^{\tau} V_t\,\mathrm{d}t$ given V_{τ_0} using the exact simulation method as described in Broadie and Kaya (2006). Update the integrated variance by setting $I = I + \int_{\tau_0}^{\tau} V_t\,\mathrm{d}t$.

4. When $\tau = t_{k+1}$, compute μ_{k+1} and σ_{k+1}^2 using the simulated values of $V_{t_{k+1}}$ and I. Otherwise, generate J^V by sampling from an exponential distribution with mean η. Update the variance value by setting $V_\tau = V_\tau + J^V$. Set $\tau_0 = \tau$ and $V_{\tau_0} = V_\tau$. Go to step 2.

3/2 stochastic volatility model

In the 3/2 stochastic volatility model, the variance process is governed by (Drimus, 2012)

$$\mathrm{d}V_t = \kappa V_t(\theta - V_t)\mathrm{d}t + \varepsilon V_t^{3/2}\mathrm{d}W_t^V, \tag{4.7}$$

which corresponds to $\alpha(V_t) = \kappa V_t(\theta - V_t)$, $\beta(V_t) = \varepsilon V_t^{3/2}$, and $f(V_t) = \frac{\ln V_t}{\varepsilon}$. In general, one can make a simultaneous jump model extension as we did for Heston's model with jumps. The conditional saddlepoint approximation framework can be generalized to accommodate jumps in the asset price process in a similar manner. However, the inclusion of the jump component may prevent one from constructing an exact simulation scheme. To illustrate the effective use of exact simulation, we consider the jump-free 3/2 stochastic volatility model. Under this scenario, the expressions of μ_k and σ_k^2 can be simplified as follows:

$$\begin{aligned}
\mu_k &= \int_{t_{k-1}}^{t_k} \left(r - d - m\lambda - \frac{V_t}{2} \right) \mathrm{d}t \\
&\quad + \frac{\rho}{\varepsilon} \left\{ \ln V_{t_k} - \ln V_{t_{k-1}} - \int_{t_{k-1}}^{t_k} \left[\kappa(\theta - V_t) - \frac{\varepsilon^2}{2} V_t \right] \mathrm{d}t \right\}, \\
\sigma_k^2 &= (1-\rho^2) \int_{t_{k-1}}^{t_k} V_t \, \mathrm{d}t.
\end{aligned}$$

Here, μ_k and σ_k^2 have similar structural properties as those under Heston's model. That is, μ_k and σ_k^2 depend on the variance process via the values of $V_{t_{k-1}}$, V_{t_k} and $\int_{t_{k-1}}^{t_k} V_t \, \mathrm{d}t$. To perform exact simulation of the above quantities, we follow the approach due to Baldeaux (2012) and consider the reciprocal process $\tilde{V}_t = 1/V_t$. By Itô's Lemma, it is easy to derive the dynamics of $\tilde{V}_t$:

$$\mathrm{d}\tilde{V}_t = \kappa\theta \left(\frac{\kappa + \varepsilon^2}{\kappa\theta} - \tilde{V}_t \right) - \varepsilon \sqrt{\tilde{V}_t} \, \mathrm{d}W_t^V, \tag{4.8}$$

which happens to take the same form of the square root process in Heston's model. As a result, we can perform exact simulation of the values of $\tilde{V}_{t_k}$, $k = 1, 2 \cdots, N$, from the non-central chi-square distribution. To generate a sample of $\int_{t_{k-1}}^{t_k} \tilde{V}_t^{-1} \mathrm{d}t$ given $\tilde{V}_{t_{k-1}}$ and $\tilde{V}_{t_k}$, we can use the conditional Laplace transform of $\int_{t_{k-1}}^{t_k} \tilde{V}_t^{-1} \, \mathrm{d}t$ derived in Baldeaux (2012). Indeed, the whole procedure resembles a close analogy to the exact simulation scheme for the square root process in Broadie and Kaya (2006).

5 Sample calculations and comparison of numerical accuracy

In this section, we present various numerical tests that were performed for the assessment of accuracy of our saddlepoint approximation formulas. We consider pricing put options on discrete realized variance and volatility swaps under Kou's double exponential model and the stochastic volatility model with simultaneous jumps (SVSJ). In Section 5.1, we show the performance of the saddlepoint approximation formulas tested under Kou's model with different contractual specifications on the sampling frequency, strike rate, maturity and different values of the model parameters σ and λ. Moreover, we also present the results of the small time asymptotic approximation (STAA) by Keller-Ressel and Muhle-Karbe (2010) for comparison of accuracy. In Section 5.2, we consider pricing of put options on discrete realized variance and volatility swaps under the SVSJ model. We present the numerical tests performed using both the saddlepoint approximation method and the analytic-simulation conditional saddlepoint approximation approach. The numerical results obtained from the conditional central limit theory approximation (CLTA) by Drimus and Farkas (2012) are used for comparison of accuracy.

5.1 Kou's double exponential model

Kou's double exponential model is adopted for the underlying asset price process in our numerical tests that were performed to assess accuracy of the ASPA methods. We consider a wide range of maturities and strikes of the put options on daily sampled realized variance. In our sample calculations, we take the riskfree interest rate to be $r = 3\%$ and the initial stock price to be $S_0 = 1$. We use the Monte Carlo simulation results as the benchmark for comparison of accuracy. The number of simulation paths was taken to be 10^6. As an effective method to reduce the standard derivation in the Monte Carlo simulation results, we adopt a control variate technique where the realized variance is taken to be the control variate.

The risk neutral dynamics of S_t under Kou's double exponential model is specified by

$$\frac{\mathrm{d}S_t}{S_t} = (r - m\lambda)\mathrm{d}t + \sigma\,\mathrm{d}W_t + (e^Y - 1)\,\mathrm{d}N_t, \tag{5.1}$$

where N_t is a Poisson process with intensity λ that is independent of W_t, and Y denotes the independent random jump size that has an asymmetric double exponential distribution as specified by

$$Y = \begin{cases} \xi_+ & \text{with probability } p \\ -\xi_- & \text{with probability } 1-p \end{cases},$$

where $\xi_\pm$ are exponential random variables with means $1/\eta_\pm$, respectively. By the martingale property of the underlying asset price process, one can easily infer that

$$m = E[e^Y - 1] = \frac{p}{\eta_+ - 1} - \frac{1-p}{\eta_- + 1}.$$

The MGF of the log return $\ln \frac{S_{t+\Delta}}{S_t}$ is known to be

$$M_\Delta(u) = \exp\left(\Delta\left[(r - m\lambda - \sigma^2/2)u + \frac{\sigma^2 u^2}{2} + \lambda u\left(\frac{p}{\eta_+ - u} - \frac{1-p}{\eta_- + u}\right)\right]\right), \tag{5.2}$$

for $-\eta_- < u < \eta_+$. The Lévy measure is given by $\lambda f_Y(x)\,\mathrm{d}x$, where

$$f_Y(x) = p\eta_+ e^{-\eta_+ x}\mathbf{1}_{\{x\geq 0\}} + (1-p)\eta_- e^{\eta_- x}\mathbf{1}_{\{x<0\}}.$$

The model parameter values of Kou's model listed in Table 1 are taken from Sepp (2004).

σ	λ	η_+	η_-	p
0.3	3.97	16.67	10	0.15

Table 1: Model parameter values of Kou's double exponential model.

Numerical results

In Table 2, we present the prices of one-year deep out-of-the-money, at-the-money and deep in-the-money put options on discrete realized variance with different sampling frequencies. The rows labelled "SPA1" and "SPA2" list the option prices calculated with the use of the first order ASPA [formula (2.8a)] and the second order ASPA [formula (2.8b)], respectively, both with the choice of $j = 2$. The row labelled "STAA" presents the numerical results obtained from the small time asymptotic approximation. The last two rows labelled "MCS" and "SE" show the Monte Carlo simulation results and the corresponding standard errors (shown as bracket quantities), respectively. Both the STAA and MCS results are used for comparison of accuracy.

It is obvious that the ASPA methods (both the first order and second order) perform well under Kou's double exponential model. We do observe a deterioration of accuracy when the sampling frequency becomes lower and the option becomes more out-of-the-money for all the approximation methods that have been tested. Surprisingly, the first order ASPA performs better than the second order ASPA in some cases. As a positive remark to the saddlepoint approximation approach, the second order ASPA method consistently outperforms the STAA method across all sampling frequencies and strike rates.

frequency (N)	weekly (52)			daily (252)		
strike	0.1295	0.1618	0.1942	0.1294	0.1618	0.1941
SPA1	1.2373	3.0351	5.3350	1.1637	3.0299	5.3649
SPA2	1.2461	3.0255	5.2777	1.1546	2.9740	5.2461
STAA	1.1496	2.9636	5.2441	1.1308	2.9611	5.2409
MCS	1.3332	3.1425	5.4032	1.1729	2.9998	5.2748
(SE)	(0.0015)	(0.0022)	(0.0027)	(0.0013)	(0.0020)	(0.0025)

Table 2: Comparison of numerical results obtained using different approximation methods for the prices of one-year put options on discrete realized variance with various strike rates and sampling frequencies under Kou's double exponential model. The strike rates are chosen to be $0.8\mu, \mu, 1.2\mu$, where μ is the at-the-money strike. All put option prices are multiplied by a notional value of 100. The corresponding standard errors (SE) in the Monte Carlo simulation are shown in brackets.

Next, we examine the performance of these approximation methods for pricing put options on discrete realized variance with different maturities, especially for short-maturity put options. Also, we examine the impact of some of the key model parameters on accuracy of these approximations. In Figure 1, we show the plots of the percentage errors of various approximation methods against moneyness for one-month and one-year put options on the daily sampled realized variance. It is observed that these approximation methods are more accurate when they are used for pricing options of shorter maturity. The second order ASPA method is seen to provide more stable performance in accuracy compared to that of the first order counterpart. All these approximation methods generally give better results for in-the-money puts.

Finally, we examine the impact of the two key model parameters in Kou's model, σ and λ, on accuracy of the approximation methods. In Figure 2(a), we reduce the volatility parameter σ in Kou's model from 0.3 to 0.1 so as to reduce the diffusion effect of the model. The performance of the first order ASPA method is not quite satisfactory compared to that of the second order counterpart. On the other hand, when we reduce the effect of the jump component by decreasing the intensity λ from 3.97 to 1, we observe that accuracy of the STAA method deteriorates quite significantly for out-of-the-money puts while the first and second order ASPA methods perform much better [see Figure 2(b)]. Overall speaking, the performance of the second order saddlepoint approximation formula is less sensitive to different choices of the model parameter values.

5.2 Stochastic volatility model with simultaneous jumps

In Table 3, we present the model parameters that are chosen in our sample calculations for the stochastic volatility model with simultaneous jumps (SVSJ). These parameters are calibrated to S&P500 option prices on November 2, 1993 (Duffie *et al.*, 2000). In addition, we take $r = 3.19\%$ and $S_0 = 1$.

κ	3.46	ν	-0.086
θ	$(0.0894)^2$	η	0.05
ε	0.14	λ	0.47
ρ	-0.82	ρ_J	0 or -0.38
$\sqrt{V_0}$	0.087	δ	0.0001

Table 3: Model parameter values of the SVSJ model. We take $\rho_J = 0$ for the saddlepoint approximation method using the small time approximation of the MGFs and $\rho_J = -0.38$ for the conditional saddlepoint aproximation method.

Saddlepoint approximation

As shown in Section 3.2, the approximate CGF of $I(0, T; N)$ and its higher order derivatives can be expressed in analytic forms if the MGF of the quadratic variation process $I(0, T; \infty)$ is known in closed form. This is possible under the SVSJ model only if we impose $\rho_J = 0$, which means the jump size of the asset return J^S is assumed to follow an independent normal distribution with mean ν and variance δ^2. This lack of dependency between the jump size distributions in general has only minor effect on the SVSJ model in capturing the real asset price dynamics [see Sepp (2008)]. The derivation of the analytic formula of the MGF of $I(0, T; \infty)$ is presented in Appendix B.

In Table 4, we present the prices of put options on daily sampled realized variance with varying maturities and strike prices. The numerical results indicate that the saddlepoint approximation methods can produce fairly accurate results for the given range of strike prices and maturities. Specifically, the approximation results for the short-maturity (5 days) and out-of-the-money (OTM) put options are fairly good. It is interesting that the second order saddlepoint approximation does not necessarily outperform the first order saddlepoint approximation. The numerical results in Table 4 show that "SPA2" would in general outperform "SPA1" for short-maturity or out-of-the-money put options. When maturity is lengthened and moneyness becomes more in-the-money, "SPA1" performs better than "SPA2".

maturity (days)	5	10	15	20	40	60
strike (OTM)	0.9037	0.9222	0.9399	0.9568	1.0174	1.0683
SPA1	0.2885	0.2556	0.2530	0.2577	0.2840	0.3071
SPA2	0.2851	0.2545	0.2500	0.2500	0.2741	0.2986
MC	0.2794	0.2463	0.2404	0.2441	0.2732	0.2992
SE	0.0008	0.0007	0.0006	0.0006	0.0006	0.0007
strike (ATM)	1.1296	1.1527	1.1748	1.1960	1.2717	1.3354
SPA1	0.4579	0.4334	0.4367	0.4459	0.4865	0.5188
SPA2	0.4500	0.4255	0.4262	0.4336	0.4729	0.5079
MC	0.4490	0.4286	0.4309	0.4406	0.4828	0.5154
SE	0.0010	0.0009	0.0008	0.0008	0.0008	0.0008
strike (ITM)	1.3555	1.3833	1.4098	1.4352	1.5261	1.6024
SPA1	0.6483	0.6352	0.6455	0.6597	0.7129	0.7517
SPA2	0.6367	0.6240	0.6322	0.6450	0.6978	0.7385
MC	0.6402	0.6330	0.6429	0.6574	0.7094	0.7465
SE	0.0012	0.0010	0.0009	0.0008	0.0008	0.0009

Table 4: The prices of put options on daily sampled realized variance with varying strike prices and maturities under the SVSJ model with $\rho_J = 0$. The strike prices are chosen to be $0.8\mu, \mu, 1.2\mu$, where μ is the at-the-money strike. All strike prices and option prices are multiplied by a notional value of 100.

Next, we would like to investigate more closely how the saddlepoint approximation formulas perform for varying moneyness and the sensitivity with respect to the model parameters. In Figures 3(a,b), we plot the put option prices against moneyness from 0.8 (out-of-the-money) to 1.2 (in-the-money) for two different sets of model parameters. In Figure 3(a), we use the original SVSJ model parameters given in Table 3, while different values of λ and ε are used to generate Figure 3(b). The plots reveal that the performance of the saddlepoint approximation results do change with changes in the model parameters. In most cases, the percentage errors are nevertheless within a bounded range ($\pm 3\%$) for SPA2. One may be concerned about the performance of "SPA1" for deep out-of-the-money options. The relatively high percentage error is driven mainly by the small value of the option price of the deep out-of-the-money put. When measured in absolute errors, both "SPA1" and "SPA2" produce rather stable approximation results.

Conditional saddlepoint approximation

We use the Euler-Maruyama scheme to discretize the stochastic differential equation for the variance process and benchmark our conditional saddlepoint approximation results to those of the conditional quadrature method (details of which are presented in the Appendix C). The number of simulation paths is taken to be 100,000. Moreover, the realized variance process has been used as the control variate.

frequency (N)	weekly (52)			daily (252)		
strike	0.0145	0.0182	0.0218	0.0145	0.0181	0.0218
ASPA1	0.4032	0.6366	0.8899	0.3998	0.6309	0.8815
(SE)	(0.0009)	(0.0011)	(0.0014)	(0.0009)	(0.0011)	(0.0013)
ASPA2	0.4033	0.6368	0.8901	0.3999	0.6309	0.8816
(SE)	(0.0009)	(0.0011)	(0.0014)	(0.0009)	(0.0011)	(0.0013)
CSPA	0.4033	0.6368	0.8901	0.3999	0.6309	0.8816
(SE)	(0.0009)	(0.0011)	(0.0014)	(0.0009)	(0.0011)	(0.0013)
CLTA	0.4034	0.6370	0.8904	0.3999	0.6309	0.8816
(SE)	(0.0009)	(0.0011)	(0.0014)	(0.0009)	(0.0011)	(0.0013)
QUAD	0.4033	0.6368	0.8901	0.3999	0.6309	0.8816
(SE)	(0.0009)	(0.0011)	(0.0014)	(0.0009)	(0.0011)	(0.0013)

Table 5: Comparison of numerical results obtained using different approximation methods for the prices of one-year put options on realized variance with different sampling frequencies and strike rates under the SVSJ model. The strike rates are chosen to be $0.8\mu, \mu, 1.2\mu$, where μ is the at-the-money strike. All option prices and strike prices are multiplied by a notional value of 100. The corresponding standard errors (SE) are shown in brackets.

In Table 5, the labels "ASPA1" and "ASPA2" refer to the first and second order conditional ASPA methods, respectively. We take the saddlepoint according to the rule of thumb as discussed in Section 2.2. For the other rows, "CSPA" stands for the conditional CSPA method, "CLTA" refers to the conditional central limit theory approximation, and "QUAD" refers to the conditional quadrature method (which serves as a benchmark). The bracket quantities in rows labeled "SE" are the respective standard errors in the simulation. The at-the-money strike rate is set to be the expectation of the realized variance, which is calculated using the analytical formula in Zheng and Kwok (2013).

It is observed that all these approximation methods deliver good performance. It seems that the conditional CLTA method also performs quite well. As discussed earlier, the conditional CLTA method only works for relatively large number of observations (options with relatively long maturities). We demonstrate below how this method may underperform for put options with short maturities.

In Figure 4, we show the plots of the percentage errors of these approximation methods against moneyness for one-week put options on daily sampled realized variance with different model parameters. The set of parameters in Table 3 are used to generate the plots in Figure 4(a) while the parameter values of both λ and ε are increased in the plots in Figure 4(b). For one-week put options of short maturity, our saddlepoint approximation formulas outperform the conditional CLTA method in two aspects. Firstly, accuracy of the saddlepoint approximation results remains stable when there are substantial changes in value of the key model parameters. Secondly, the saddlepoint approximation results are very accurate, both for the conditional CSPA method and the second order conditional ASPA method (benchmark against the results from the conditional QUAD method).

Finally, we consider the prices of put options and volatility swaps with different maturities. In Figure 5(a), the plots of the put prices from these approximation methods almost overlap with each other. One can observe easily that the conditional CLTA may not be a good approximation for the short-maturity put options as the saddlepoint approximations. In Figure 5(b), the fair strike prices of volatility swaps with different maturities are presented. It clearly shows how the second order conditional ASPA method [given by Eq. (2.10b)] improves over its first order counterpart [given by Eq. (2.10a)]. As a remark, the conditional CLTA method may have difficulty in pricing volatility swaps since the non-negative realized variance is approximated by a normal random variable as part of the procedure.

6 Conclusion

We have demonstrated the versatility of the saddlepoint approximation techniques for deriving analytic approximation formulas for pricing derivatives whose payoffs depend on discrete realized variance of the price process of an underlying risky asset. For pricing discrete variance options under Lévy models and stochastic volatility models, we derive an approximation to the cumulant generating function of the squared returns based on some theoretical result on the small time asymptotic distributions of the quadratic variation and discrete realized variance. Furthermore, we propose an alternative saddlepoint approximation method since the classical Lugannani-Rice approach may fail when used in pricing variance options. This is because the root of the saddlepoint equation may lie outside the domain of definition of the cumulant generating function. For pricing variance options and volatility swaps under stochastic volatility models with simultaneous jumps, we also develop an analytic-simulation procedure. In our conditional saddlepoint approximation method, we first perform simulation on the variance process and then compute the saddlepoint approximation to the prices of the variance options and volatility swaps conditional on the simulated variance values. We have performed extensive numerical tests to assess the performance of our proposed saddlepoint approximation through comparison with the Monte Carlo simulation results and other approximation methods. The errors in computing prices of various volatility derivatives using our saddlepoint approximation methods are in general shown to be within numerical tolerance level of a few percents.

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Appendix A Derivation of Eq. (4.2)

For the asset price process S_t specificed in Eq. (4.1), the log return over (t_{k-1}, t_k) can be expressed as

$$\ln \frac{S_{t_k}}{S_{t_{k-1}}} = \int_{t_{k-1}}^{t_k} \left(r - d - \lambda m - \frac{V_t}{2}\right) \mathrm{d}t + \rho \int_{t_{k-1}}^{t_k} \sqrt{V_t}\, \mathrm{d}W_t^V + \sqrt{1-\rho^2} \int_{t_{k-1}}^{t_k} \sqrt{V_t}\, \mathrm{d}B_t + \sum_{k=1}^{N_{\Delta t_k}} J_m^S. \tag{A.1}$$

Applying Itô's Lemma to $f(V_t)$, we have

$$\begin{aligned} \mathrm{d}f(V_t) &= \left[f'(V_t)\alpha(V_t) + \frac{1}{2} f''(V_t)\beta^2(V_t)\right] \mathrm{d}t + f'(V_t)\beta(V_t)\, \mathrm{d}W_t^V + [f(V_t + J^V) - f(V_t)]\, \mathrm{d}N_t \\ &= \left[f'(V_t)\alpha(V_t) + \frac{1}{2} f''(V_t)\beta^2(V_t)\right] \mathrm{d}t + \sqrt{V_t}\, \mathrm{d}W_t^V + [f(V_t + J^V) - f(V_t)]\, \mathrm{d}N_t. \end{aligned}$$

Integrating the above equation from t_{k-1} to t_k and rearranging the terms, we have

$$\begin{aligned} \int_{t_{k-1}}^{t_k} \sqrt{V_t}\, \mathrm{d}W_t^V &= f(V_{t_k}) - f(V_{t_{k-1}}) - \int_{t_{k-1}}^{t_k} \left[f'(V_t)\alpha(V_t) + \frac{1}{2} f''(V_t)\beta^2(V_t)\right] \mathrm{d}t \\ &\quad - \int_{t_{k-1}}^{t_k} [f(V_t + J^V) - f(V_t)]\, \mathrm{d}N_t. \end{aligned}$$

Substituting the above expression into Eq. (A.1), we obtain Eq. (4.2).

Appendix B MGF of $I(0, T; \infty)$ under SVSJ model

When the asset price dynamics is governed by the SVSJ model [see Eq. (3.11)] with the random jump size distribution as specified by

$$J^S \sim \mathcal{N}(\nu, \delta^2), \quad J^V \sim \exp(1/\eta),$$

the MGF of $I(0, T; \infty) = \int_0^T V_t\, \mathrm{d}t + (J^S)^2 N_T$ can be derived analytically. Write $I_t = I(0, t; \infty)$ and $\tau = T - t$, and denote $U(V_t, I_t, \tau) = E_t^Q[e^{uI_T}]$ with u being fixed. By the Feymann-Kac Theorem, $U(V_t, I_t, \tau)$ satisfies the following partial integro-differential equation:

$$\frac{\partial U}{\partial \tau} = \kappa(\theta - V)\frac{\partial U}{\partial V} + \frac{\varepsilon^2 V}{2}\frac{\partial^2 U}{\partial V^2} + V\frac{\partial U}{\partial I} + \lambda E\big[U(V + J^V, I + (J^S)^2, \tau) - U(V, I, \tau)\big]. \tag{B.1}$$

The solution to Eq. (B.1) admits the exponential affine form:

$$U(V_t, I_t, \tau) = \exp\left(B(\tau)V_t + uI_t + \Gamma(\tau) + \Lambda(\tau)\right), \tag{B.2}$$

where the coefficient functions $B(\tau)$, $\Gamma(\tau)$ and $\Lambda(\tau)$ can be determined by solving the following Riccati differential equation system:

$$\begin{cases} B' = -\kappa B + \frac{\varepsilon^2}{2}B^2 + u, \\ \Gamma' = \kappa\theta B, \\ \Lambda' = \lambda\big(E[\exp(BJ^V + (J^S)^2 u) - 1]\big), \end{cases} \tag{B.3}$$

with the initial conditions: $B(0) = \Gamma(0) = \Lambda(0) = 0$. The solution to the above Riccati system gives [Sepp (2008), Zheng and Kwok (2013)]

$$\begin{cases} B(\tau) &= \dfrac{2u(1-e^{-\zeta\tau})}{\xi_+ e^{-\zeta\tau} + \xi_-}, \\ \Gamma(\tau) &= -\dfrac{\kappa\theta}{\varepsilon^2}\left(\xi_+\tau + 2\ln\dfrac{\xi_+ e^{-\zeta\tau} + \xi_-}{2\zeta}\right), \\ \Lambda(\tau) &= -\lambda\tau + \lambda\dfrac{\exp(u\nu^2/(1-2u\delta^2))}{\sqrt{1-2u\delta^2}}\dfrac{1}{\xi_- - 2\eta u} \\ & \quad \left[\xi_-\tau + \dfrac{4\eta u}{\xi_+ + 2\eta u}\ln\dfrac{(\xi_+ + 2\eta u)e^{-\zeta\tau} + \xi_- - 2\eta u}{2\zeta}\right], \end{cases} \tag{B.4}$$

where $\zeta = \sqrt{\kappa^2 - 2\varepsilon^2 u}$ and $\xi_\pm = \zeta \mp \kappa$. The solution is valid for any $u < \frac{1}{2\delta^2}$.

Appendix C Conditional quadrature method

Given the conditional MGF of $I(0,T;N)$ in Eq. (4.4) and the integral representation of the tail expectation as shown in Eq. (2.7b), we can obtain the integral representation of the price of the put option on discrete realized variance as follows:

$$E[K - I(0,T;N)]^+ = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{\kappa(\tau+iu)-(\tau+iu)K}}{(\tau+iu)^2}\right)\mathrm{d}u, \tag{C.1}$$

for any $\tau < 0$. The above complex integral can be directly evaluated by numerical integration methods such as the Gauss-Kronrod quadrature method. In order that this method works, it is necessary to perform analytic continuation of the moment generating function in the complex domain, possibly not a straightforward procedure in some cases. Also, we have a similar formula for volatility swaps:

$$E[\sqrt{I}] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - e^{\kappa(-u)}}{u^{3/2}} \, \mathrm{d}u, \tag{C.2}$$

which can also be directly evaluated by numerical quadrature method. With the availability of the integral representation of the price function of the volatility derivative, by following a similar analytic-simulation procedure as in the conditional saddlepoint approximation method, the computational procedure in the conditional quadrature method can be summarized as follows:

1. Simulate a path of the stochastic variance process;
2. Compute the quantities μ_k and σ_k^2;
3. Apply the numerical quadrature method to evaluate the price of the volatility derivative represented as a complex integral;
4. Repeat steps 1 to 3 for sufficiently large number of simulation runs and take the sampled average price.

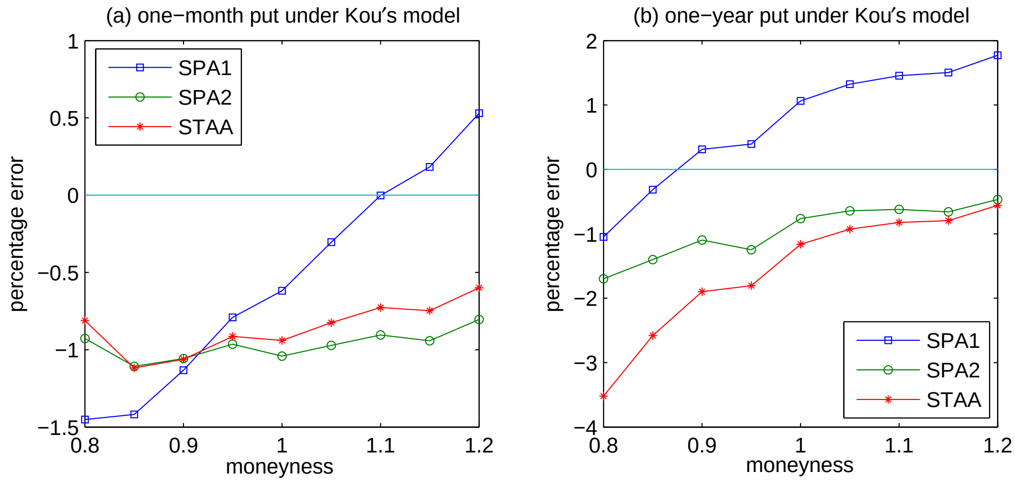

Figure 1: Plots of the percentage errors of different approximation methods against moneyness for pricing one-month ($N = 20$) and one-year ($N = 252$) put options on the daily sampled realized variance under Kou's double exponential model. The performance of the STAA method is comparable to that of SPA2. All these approximation methods provide better accuracy for in-the-money put options.

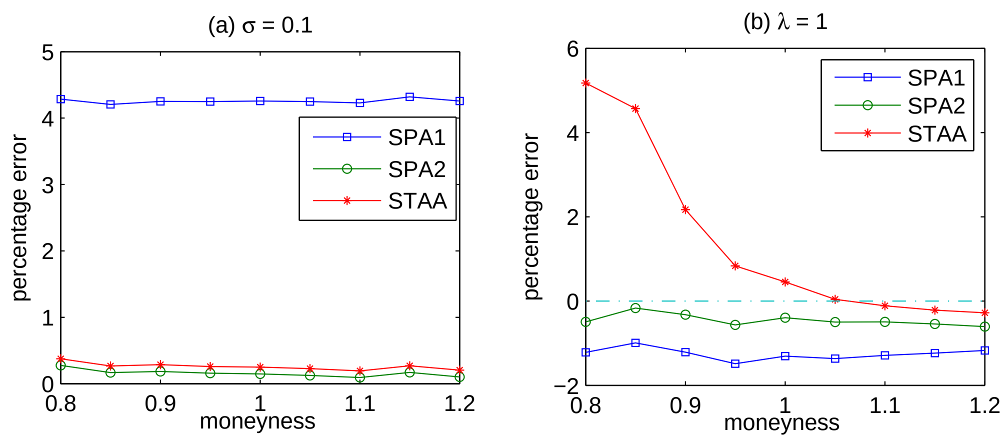

Figure 2: Plots of the percentage errors of different approximation methods against moneyness for one-month ($N = 20$) put options on the daily sampled realized variance with different values of volatility σ and jump intensity λ in Kou's model. Accuracy of the STAA method deteriorates quite significantly for out-of-the-money put options.

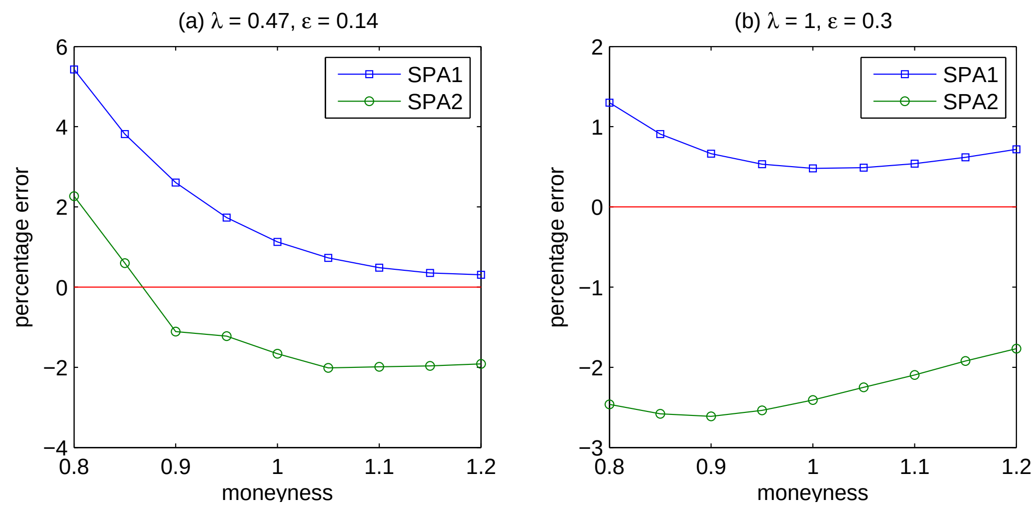

Figure 3: Plots of percentage errors of the two saddlepoint approximation methods against moneyness for one-month ($N = 20$) put options on daily sampled realized variance with different model parameters in the SVSJ model. The performance of the saddlepoint approximation methods (both SPA1 and SPA2) are sensibly dependent on the model parameter values of the SVSJ model.

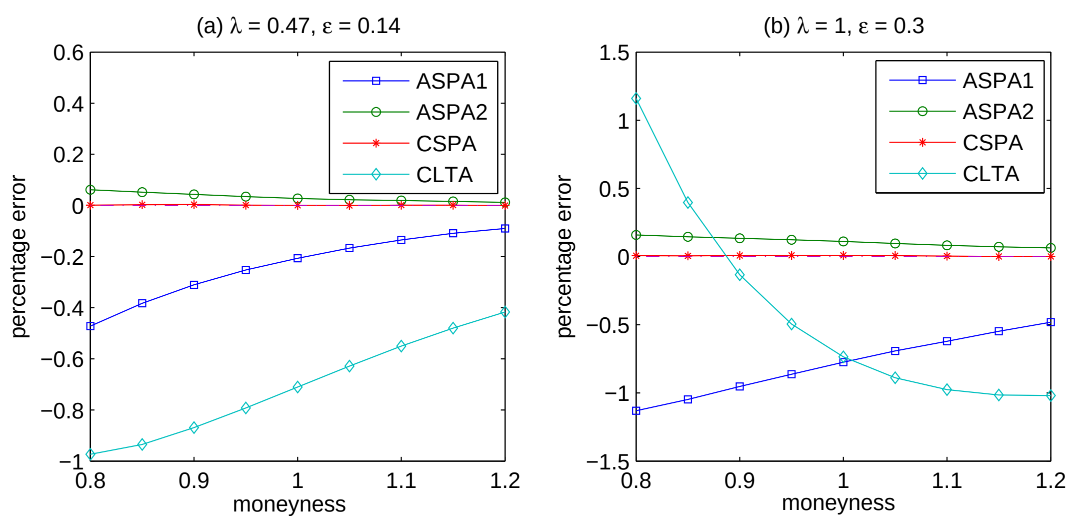

Figure 4: Plots of percentage errors of different approxmation methods against moneyness for one-week ($N = 5$) put options on the daily sampled realized variance with different model parameters in the SVSJ model. The conditional CSPA method produces put option prices that are in good agreement with those of the second order conditional ASPA method. The conditional CLTA method is seen to be the least accurate compared to the other analytic approximation methods.

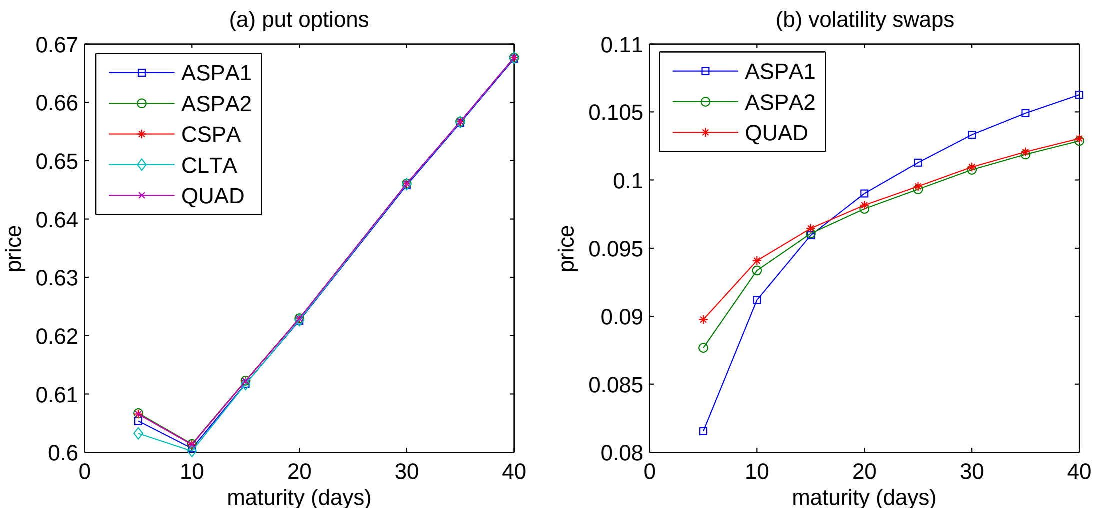

Figure 5: Plots of the prices of put options and volatility swaps based on daily sampled realized variance against maturity. The put option prices are multiplied by notional value of 100. All analytic approximation methods provide almost identical numerical results for put option values, except that the conditional CLTA method is slightly less accurate for options with maturity less than 10 days. The conditional CLTA method cannot be directly applied to pricing volatility swaps. Both the second order conditional ASPA and QUAD methods provide values of volatility swaps that are in good agreement with each other.