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Stochastic control model for R&D race in a mixed duopoly with spillovers and knowledge stocks

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Abstract We consider the stochastic control model with finite time horizon for a mixed duopoly Research and Development (R&D) race between the profit-maximizing private firm and welfare-maximizing public firm. In our two-firm stochastic control R&D race model with input and output spillovers, the stochastic control variable is taken to be the private firm's rate of R&D expenditure and the hazard rate of success of innovation has dependence on the R&D effort and knowledge stock. Given the fixed R&D effort of the public firm, the optimal control is determined so as to maximize the private firm's value function subject to market uncertainty arising from the stochastic profit flow of the new innovative product. We conduct various sensitivity tests with varying model parameters to analyze the effects of input spillover, output spillover and knowledge stock on the optimal control policy and the value function of the profit-maximizing private firm. The R&D effort of the private firm is found to increase when the profit flow rate increases. Moreover, the optimal R&D effort level may decrease with increasing private firm's knowledge stock and output spillover. The effects of input spillover on the optimal control policy and value function are seen to be relatively small. We examine the robustness of various observed phenomena of the two-firm R&D race with varying values of the fixed R&D effort of the public firm. With regard to public policy issue, we examine the level of the fixed public firm's R&D effort so that social welfare is maximized.

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1 Introduction

There has been strong empirical evidence that reveals the importance of the role of the public firms (in a broader sense, including government funded research units) in enhancing social welfare in certain industrial sectors, such as the health care sector and agriculture sector. One well-known example is the Research and Development (R&D) race around the genome sequencing between Celera Genomics and public research units like the Human Genome Project (HGP). As a social policy to enhance social welfare, public funds have been pooled to support the international research project HGP due to the fear of potential monopolized use of the genome information by Celera Genomics. The monopoly by a private firm may pose a serious obstacle to the future progress of biotechnologies that may benefit the welfare of the whole society. The government may use the public firm or government funded research unit as an instrument for internal regulation of a particular industry and the implementation of public policy in mitigating the common pool problem—effort duplication as reflected in over-investment in R&D races under nearly zero spillovers and almost "winner-take-all" scenario. In the past decades, many researchers investigate the role of public firms in a mixed duopoly from both theoretical and public policy point of view. An early survey of the game theoretic models of mixed oligopoly can be found in Giovanni and Delbono (1990). Delbono and Denicolò (1993) argue that the presence of public firms has its special role in alleviating the problem of duplication of R&D effort where the overall social welfare is improved through cost reduction. In a mixed duopoly, they show that both the private firm and public firm invest less compared to that in a private duopoly. Interestingly, though the expected time of innovation is delayed, social welfare is improved. Ishibashi and Matsumura (2006) confirm the public sector's role in mitigating inefficiency of R&D spending and show that additional government control is needed for achieving social optimum. Later, Zikos (2007) suggests that the public firm should invest as a Stackelberg follower in order to fully exercise its desirable role. Marinucci (2014) constructs models on cooperative R&D networks among firms and public research institutions. He finds that the incentive for the players to form a partnership is decreasing with the spillover effect. In other words, subsidizing the formation of an R&D partnership among firms and public research institutions can be ineffective when the R&D spillover is sufficiently high.

The literature on R&D races has paid relatively little attention to analyze the impact of spillovers in mixed oligopolies where firms with different objectives coexist. In particular, one would ask what is the role of public policy in achieving social optimality of innovative activities undertaken by the public social welfare-maximizing firm and the private profit-seeking firm. Though spillovers have been extensively studied in R&D races among private firms (d'Aspremont and Jacquemin 1988; Kamien et al. 1992; Amir 2000; Martin 2002; Hausenschild 2003), there are only a few papers that focus on the impact of spillovers on the mixed market. Poyago-Theotoky (1998) takes one step further by considering easy imitation and analyzes public firm's role in tackling the problem of underinvestment due to the free rider phenomenon. However, she finds that the inclusion of public firms in the R&D race may or may not improve social welfare, depending on the size of the innovation. Such ambiguous effect attracts even more attention in examining the spillover effect in a mixed market framework. Naseem and Oehmke (2006) consider the category of spillover phenomena in pure private duopoly research and investigate the effects of input and output spillovers as well as their impact on social welfare. Gil Moltó et al. (2011) study the spillover effect from a policy maker's point of view. They focus on the use of subsidies and its relation with the degree of spillovers. In addition, the knowledge accumulation effect in which past accumulated knowledge may contribute to better chance of successful innovation has received less attention in studies of R&D races. Doraszelski (2003) disregards the memorylessness property of the exponential distribution in most earlier R&D models by analyzing the effect of knowledge accumulation in firms' R&D behavior. Leung and Kwok (2014) construct the R&D stochastic control model with the hazard rate of arrival of innovation that is dependent on knowledge stock. Steinmetz (2010) analyzes the effect of learning by doing and organizational forgetting in R&D on firms' incentives in innovations. He shows that the leader's innovation effort declines with the lead, and incentives are highest when competition is most keen. König et al. (2011) analyze a model in which innovation arises from recombinant knowledge of firms engaged in a network of R&D collaborations. The growth of firm knowledge stock is an increasing function of the individual knowledge stock and the knowledge of its R&D partners (see Eqs. 2.1, 2.2). They show how the R&D network influences knowledge growth, innovation and profits of the firms.

To address the above issues, we propose the two-firm stochastic control model that investigates the impact of input and output spillovers on R&D races under a mixed duopoly of public research institute and private firm. In our proposed stochastic control model, the stochastic revenue flow rates to the two firms generated from the innovative product are assumed to be scalar multiples of a stochastic fundamental state variable. We choose various appropriability factors in the revenue flow rates to model the level of output spillovers. The technological uncertainty of R&D innovation is modeled as a mixed Poisson process, where the hazard rates of arrival of innovation are path dependent (say, with dependence on the knowledge accumulation of research activities). This more realistic modeling of path dependence on past research effort eradicates the weakness in the memorylessness property of the simple exponential distribution assumption on random arrival of innovation in most earlier R&D races models. Under positive input spillovers of R&D efforts, the hazard rates of innovative success can be enhanced by the research efforts of the competing firm. The rate of R&D expenditure in the private firm is treated as a stochastic control variable in our stochastic control model. Given the fixed R&D effort of the public firm, the private firm chooses the corresponding optimal strategy of research effort so as to maximize its firm value function.

It would be instructive to explore the optimal set of public policies in R&D race in a mixed duopoly with public research institute and private firm. The implementation of public policies may be partially modeled by the levels of input and output spillovers from the public research institute to the private firm. We explore how the private firm's R&D spending level evolves under varying levels of spillovers and knowledge stock and different market conditions as proxied by the current level and volatility of the profit flow rate. There may exist the scenario where a lowering of private firm's R&D effort does occur due to a loss of incentive of the private firm as a result of strong output spillover and low profit flow rate. As a result, this leads to postponement of the expected arrival time of innovation.

In this paper, the mixed duopoly R&D race model with spillovers (input and output) and knowledge stock extends the basic modeling framework of Leung and Kwok's (2014) one-firm stochastic control model to the two-firm formulation. Usually, government spending on R&D effort of the public research institute is planned in advance, so it may be less flexible to adapt to changing market conditions and competition. We take the model assumption that the public firm's R&D investment level is fixed at initiation of the R&D race. The two-firm R&D race model has the private firm's R&D effort as the single stochastic control variable. Our model combines the effects of input spillover and output spillover in the asymmetric competition of the R&D race.

The paper is organized as follows. In the next section, we present the finite time R&D stochastic control model by specifying details on the modeling of knowledge accumulation, input and output spillovers. We then apply the Bellman optimality condition to derive the governing Hamilton–Jacobi–Bellman equation for the real option stochastic control model. In Sect. 3, we discuss the numerical scheme for finding the solution of the value function and optimal control of R&D expenditure of the private firm based on the finite difference algorithm together with the policy iteration procedure (Forsyth and Labahn 2007). Special precautions are taken to prescribe the numerical auxiliary conditions. We examine the mathematical conditions required for the convergence of the numerical calculations based on some earlier theoretical results in the literature (Barles 1997; Leung and Kwok 2014). In Sect. 4, we present sensitivity tests on input and output spillovers and knowledge accumulation through numerical studies on the optimal R&D expenditure of the private firm and public firm with respect to different sets of model parameters that characterize various market conditions and technological uncertainties. By identifying the criterion of social welfare maximization of the public firm, we examine the appropriate level of the fixed public firm's R&D effort so that social welfare is maximized. Summary and conclusive remarks are presented in the last section.

2 Stochastic control model for R&D race

We develop a two-firm R&D stochastic control model with finite time horizon for a mixed duopoly R&D race with the inclusion of input and output spillovers, together with knowledge accumulation effect.

2.1 Model settings

For the two firms in our mixed duopoly model, public firm (Firm 0) and private firm (Firm 1), both are engaged in R&D of developing an innovative product. While most real option models assume for the sake of analytic tractability, it would be more realistic to assume a finite life $T > 0$ of the innovative product. It is common that the government funding for supporting R&D of the public firm is planned in advance, so we assume the level of R&D effort of the public firm to be fixed. We let $u_0(\cdot)$ denote the fixed public firm's R&D effort and assume that it is known to the private firm. For the private firm, it targets to maximize its own firm value in view of the costs expended on its own R&D activities. Let $u_1(t)$ be the stochastic control variable for the rate of R&D expenditure of the private firm, where $u_1(t)$ is assumed to be chosen from a compact set $Q_1(t) \in [0, B]$ for $0 \leq t \leq T$, where B is the upper bound of R&D effort expended by the private firm. It is reasonable to set an upper bound on u_1 since a firm can only have a finite resource.

2.1.1 Knowledge stock

According to Doraszelski (2003), the past R&D efforts are accumulated as the firm's knowledge stock. In addition, the rival firm's R&D effort may also contribute to the firm's knowledge stock due to input spillover effect. Combining these two effects, we assume that Firm j's knowledge stock $z_j(t)$ evolves according to the following differential equation

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = u_j(t) + \beta_j u_{j'}(t), \quad j = 0, 1, \ 0 \leq t \leq T, \tag{2.1}$$

where β_j is a nonnegative constant, $\beta_j \in [0, 1]$. The first term on the right-hand side of Eq. (2.1) represents the contribution from the current research effort of Firm j $(j = 0, 1)$ itself, while the second term represents the input spillover from rival Firm j' $(j' \neq j$ and $j' = 0, 1)$. It is common that the private firm would try to minimize the spread of the firm's knowledge to the public, so we assume no input spillover from the private firm to the public firm with $\beta_0 = 0$ in our later model calculations. For the private firm, we set $\beta_1 \in [0.5, 1]$, corresponding to a relatively high input spillover from the public firm to the private firm. As a remark, complexity in the numerical calculations of the R&D race model would not be affected by the choices of the parameter values for β_0 and β_1.

2.1.2 Hazard rate of arrival of discovery

The uncertainty in the success of innovation is modeled by a Poisson process with the hazard rate of arrival of the innovative product as the Poisson parameter. The R&D effort of the private firm is the control variable, which is chosen optimally so as to maximize the firm value of the private firm. Both the current R&D effort and knowledge stock contribute to the likelihood of innovative success. We assume that the hazard rate $h_j(t)$ $(j = 0, 1)$ increases with Firm j's current R&D effort $u_j(t)$ and knowledge stock $z_j(t)$ as modeled by the following equation (Leung and Kwok 2014)

$$h_j(t) = a_j u_j(t) + b_j z_j(t), \tag{2.2}$$

where R&D effort factor $a_j > 0$ and the knowledge stock factor $b_j \geq 0$ are constant parameters and $0 \leq t \leq T$. The special case $b_j = 0$ corresponds to memoryless R&D process.

2.1.3 Profit flow rate and expected payoff functions

Our model assumes that the winner of the R&D race would commercialize the innovative product immediately and receive the profit flow from the sale of the new product. The profit flow rate is modeled as the product of a nonnegative constant appropriability factor and the stochastic profit flow rate x_t. Similar to most real option models, x_t is assumed to follow the geometric Brownian motion

$$\mathrm{d}x_t = \mu x_t \,\mathrm{d}t + \sigma x_t \,\mathrm{d}Z_t. \tag{2.3}$$

Here, μ is the constant drift rate, σ is the constant volatility parameter and Z_t is the standard Brownian motion. Let r denote the riskless interest rate. The usual no-bubble condition is assumed, where $\mu < r$.

For convenience, suppose we write the total patent of the new innovative product as $\Pi > 0$. Due to output spillover, Firm j as the winner of the race can only appreciate part of the total patent. The appropriability factor of the winner is defined as $\theta\Pi$, where θ denotes the appreciation rate of the winner on the patent. Moreover, we assume that the remaining part of the patent $(1-\theta)\Pi$ is distributed to Firm j' as the loser, assuming that the loser manages to imitate a similar product. One would expect $\theta \in [0.5, 1]$, so the appropriability factors observe $\theta\Pi \geq (1-\theta)\Pi$. For the "winner-take-all" scenario, we have $\theta = 1$, which then leads to $(1-\theta)\Pi = 0$.

At the current time t, suppose the innovative product has been active in the market, where $t < T$. Conditional on $x_t = x$, the expected payoff $W_j(x, t)$ of the winner Firm j and the expected payoff $L_{j'}(x, t)$ of the loser Firm j' are given by

$$W_j(x, t) = \frac{\theta x}{r - \mu}\left[1 - e^{-(r-\mu)(T-t)}\right],$$
$$L_{j'}(x, t) = \frac{(1-\theta)x}{r - \mu}\left[1 - e^{-(r-\mu)(T-t)}\right].$$

2.1.4 Cost function

Let cost $c_j(u_j)$ denote the rate of cost incurred on Firm j in R&D effort to be a nonnegative, continuous and strictly increasing function with respect to the R&D effort u_j. It is common to assume the cost function to be a power function. By setting the marginal cost to be the R&D effort, the cost function takes the following quadratic form

$$c_j(u_j) = c_j(0) + \frac{u_j^2}{2}, \tag{2.4}$$

where $c_j(0)$ is a fixed minimum cost for Firm j to maintain its R&D facilities even at zero R&D effort.

2.2 Hamilton–Jacobi–Bellman formulation

Given the fixed public firm's R&D expenditure u_0, our objective is to determine the optimal R&D control u_1 and the firm value function V_1 of the private firm. Conditional on $x_t = x$, $z_0(t) = z_0$ and $z_1(t) = z_1$, we write $V_1(x, t; z_0, z_1)$ as the value function of the private firm derived from the R&D project. The Bellman optimality condition gives the following relation that governs the value function:

$$V_1(x, t; z_0, z_1) = \lim_{\mathrm{d}t \to 0} \sup_{u_1 \in Q_1} \Big\{ G\,(x, t; z_0, z_1) - c_1(u_1) \Big\}.$$

Neglecting the scenario where both firms succeed in R&D within $(t,\ t + \mathrm{d}t)$, we have

$$\begin{aligned} G\,(x, t; z_0, z_1)\ \mathrm{d}t = {} & (h_0\ \mathrm{d}t)\,(1 - h_1\ \mathrm{d}t)\, L_1(x, t) + (1 - h_0\ \mathrm{d}t)\,(h_1\ \mathrm{d}t)\, W_1(x, t) \\ & + (1 - h_0\ \mathrm{d}t)\,(1 - h_1\ \mathrm{d}t)\, e^{-r\mathrm{d}t} E\big[V_1\,(x_{t+\mathrm{d}t}, t + \mathrm{d}t; z_0\,(t + \mathrm{d}t)\,, \\ & \times z_1\,(t + \mathrm{d}t)\,|x_t = x, z_0(t) = z_0, z_1(t) = z_1)\,\big]. \end{aligned}$$

The individual terms in G are derived based on the following considerations.

(i) With probability $(h_0\ \mathrm{d}t)(1 - h_1\ \mathrm{d}t)$, the public firm wins the R&D race within $(t,\ t + \mathrm{d}t)$. The resulting expected value for the private firm is $L_1(x, t)$.

(ii) With probability $(1 - h_0\ \mathrm{d}t)(h_1\ \mathrm{d}t)$, the private firm wins the R&D race within $(t,\ t + \mathrm{d}t)$ and its corresponding expected value is $W_1(x, t)$.

(iii) With probability $(1 - h_0\ \mathrm{d}t)(1 - h_1\ \mathrm{d}t)$, R&D efforts of both firms continue beyond $t + \mathrm{d}t$ since none of the two firms succeed in R&D within $(t,\ t + \mathrm{d}t)$. The discounted expected value of the R&D project for the private firm is given by

$$e^{-r\mathrm{d}t} E\big[V_1(x_{t+\mathrm{d}t}, t + \mathrm{d}t; z_0(t + \mathrm{d}t), z_1(t + \mathrm{d}t)|x_t = x, z_0(t) = z_0, z_1(t) = z_1)\big].$$

We apply the Ito Lemma to obtain

$$\begin{aligned} & E\big[V_1\,(x_{t+\mathrm{d}t}, t + \mathrm{d}t; z_0(t + \mathrm{d}t), z_1(t + \mathrm{d}t)|x_t = x, z_0(t) = z_0, z_1(t) = z_1)\,\big] \\ & = V_1 + \frac{\partial V_1}{\partial t}\ \mathrm{d}t + \mu x \frac{\partial V_1}{\partial x}\ \mathrm{d}t + \frac{\sigma^2 x^2}{2} \frac{\partial^2 V_1}{\partial x^2}\ \mathrm{d}t + (u_0 + \beta_0 u_1) \frac{\partial V_1}{\partial z_0}\ \mathrm{d}t \\ & \quad + (u_1 + \beta_1 u_0) \frac{\partial V_1}{\partial z_1}\ \mathrm{d}t + O(\mathrm{d}t^{\frac{3}{2}}). \end{aligned}$$

Since we have assumed no input spillover from the private firm, we have $\beta_0 = 0$. Moreover, u_0 is fixed as a constant, so we can replace z_0 by $u_0 t$. As a result, the dependence of V_1 on z_0 can be dropped. Substituting the above equation into the Bellman optimality condition and neglecting higher order terms when we take the limit of d$t \to 0$, we obtain the HJB formulation of the optimal R&D control model as follows

$$\sup_{u_1 \in Q_1} \left\{ \frac{\partial V_1}{\partial t} + \mu x \frac{\partial V_1}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 V_1}{\partial x^2} + (u_1 + \beta_1 u_0) \frac{\partial V_1}{\partial z_1} - (r + h_0 + h_1) V_1 \right.$$
$$\left. + \; h_0 L_1(x, t) + h_1 W_1(x, t) - c_1(u_1) \right\} = 0. \tag{2.5}$$

2.3 Auxiliary conditions

To complete the HJB formulation, it is necessary to impose the appropriate auxiliary conditions. Firstly, since the R&D process is sure to terminate at T, we have

$$V_1(x, t; z_1) \to 0 \;\text{ as }\; t \to T. \tag{2.6a}$$

Also, when x or z_1 becomes asymptotically small, the expected value of the project is almost zero. For the far-field boundary condition at $z_1 \to \infty$, we have

$$V_1(x, t; z_1) \to W_1(x, t) \;\text{ as }\; z_1 \to \infty. \tag{2.6b}$$

One may visualize that V becomes a linear function in x when $x \to \infty$ (Leung and Kwok 2014), so we have

$$V_1(x, t; z_1) \to C_1(t; z_1)x + C_2(t; z_1) \;\text{ as }\; x \to \infty. \tag{2.6c}$$

The solution procedure for finding the closed form expressions for $C_1(t; z_1)$ and $C_2(t; z_1)$ is outlined in Appendix.

We would like to comment on the specific features of the mixed duopoly model that are distinctive from the single-firm model studied in Leung and Kwok (2014). The impact of the R&D effort u_0 and the hazard rate of arrival of the innovation h_0 on the private firm's value function V_1 and its optimal R&D effort u_1 enters into the HJB formulation for $V_1(x, t; z_1)$ via the coefficient in $\frac{\partial V_1}{\partial z_1}$ and V_1 and the source term $h_0 L_1(x, t)$. Also, spillover effects between the two firms are modeled by the expected loser payoff function $L_1(x, t)$ and the winner counterpart $W_1(x, t)$.

3 Finite difference algorithms with policy iteration procedure

By following a similar numerical procedure for solving the stochastic control R&D model (Leung and Kwok 2014), we solve the HJB model formulation using the finite difference approach together with policy iteration procedure. The usual backward induction procedure in discounted expectation calculations is seen to be equivalent to forward time marching with respect to time to expiry τ in a finite difference scheme, where $\tau = T - t$. Here, we use τ as the temporal variable in V_1, L_1 and W_1 and rewrite Eq. (2.5) as follows:

$$F(V_1) = \sup_{u_1 \in Q_1} \left\{ -\frac{\partial V_1}{\partial \tau} + \mu x \frac{\partial V_1}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 V_1}{\partial x^2} + (u_1 + \beta_1 u_0) \frac{\partial V_1}{\partial z_1} \right.$$
$$\left. - (r + h_0 + h_1) V_1 + h_0 L_1(x, \tau) + h_1 W_1(x, \tau) - c_1(u_1) \right\} = 0. \quad (3.1)$$

The discretized computational domain is restricted to a finite domain: $[0, x_{\max}] \times [0, z_{\max}] \times [0, T]$, where $x_{\max}$ and $z_{\max}$ are chosen to be some sufficiently large values. The (j, k, n)th node in the discretized domain corresponds to $x_j = j\Delta x,\ z_k = k\Delta z$ and $\tau_n = n\Delta\tau$, where $j = 0, 1, 2, \ldots, j_{\max}$, $k = 0, 1, 2, \ldots, k_{\max}$ and $n = 0, 1, 2, \ldots, n_{\max}$. We let $V_{j,k}^n$ denote the numerical approximation to $V_1(x_j, z_k, \tau_n)$ and let $u_{j,k}^n$ denote the respective control strategy for u_1 at the nodal point (x_j, z_k, τ_n). We follow the discretization techniques developed in Forsyth and Labahn (2007), where fully implicit discretization is adopted and appropriate forward/backward differencing is applied to various spatial differential operators so that the condition of positive coefficients in the resulting finite difference scheme is enforced. The discretized version of the HJB equation becomes

$$F^*\left(V_{j,k}^{n+1}\right) = \sup_{u_{j,k}^{n+1} \in Q_1} \left\{ -\frac{V_{j,k}^{n+1} - V_{j,k}^n}{\Delta\tau} + \mu x_j \frac{V_{j+1,k}^{n+1} - V_{j,k}^{n+1}}{\Delta x} \right.$$
$$+ \frac{\sigma^2 x_j^2}{2} \frac{V_{j+1,k}^{n+1} - 2V_{j,k}^{n+1} + V_{j-1,k}^{n+1}}{(\Delta x)^2}$$
$$+ (u_1 + \beta_1 u_0) \frac{V_{j,k+1}^{n+1} - V_{j,k}^{n+1}}{\Delta z} - (r + h_0 + h_1) V_{j,k}^{n+1}$$
$$\left. + h_0 L_1(x_j, \tau_{n+1}) + h_1 W_1(x_j, \tau_{n+1}) - c_1(u_{j,k}^{n+1}) \right\} = 0. \quad (3.2)$$

The numerical auxiliary conditions along the computational boundaries and at expiry (corresponding to $\tau = 0$) are (i) $V_{j,k_{\max}}^n = W_1(x_j, \tau_{n+1})$, (ii) $V_{j_{\max},k}^n = C_1(\tau_n; z_k)x_{\max} + C_2(\tau_n; z_k)$, (iii) $V_{j,k}^0 = 0$, (iv) $V_{0,k}^n = 0$.

Let the stepwidth and time step parameters be chosen such that $\Delta x = \gamma_1\delta$, $\Delta\tau = \gamma_2\delta$, $\Delta z = \gamma_3\delta$, where γ_1, γ_2, γ_3 are positive constants independent of the small parameter δ. One can show that

$$F^*\left(V_{j,k}^{n+1}\right) - F(V_1) = O(\Delta x) + O(\Delta\tau) + O(\Delta z) = O(\delta).$$

Using the above discretization procedures, it can be shown that the resulting numerical scheme observes the properties of consistency (pointwise), monotonicity and l_∞-stability. Provided that the strong comparison property holds, the numerical solution of Eq. (3.2) converges to the viscosity solution of the continuous HJB formulation in Eq. (2.5). The proof of these theoretical results can be mimicked from a similar proof presented in Leung and Kwok (2014).

We proceed to solve for $V_{j,k}^{n+1}$ through marching backward in k and forward in n. For a fixed value of k and n, we solve recursively for the optimal control variables, where each iteration requires the numerical solution of a system of $j_{\max}+1$ algebraic equations of the form

$$-a_{j,k}^{n+1}\Delta\tau V_{j+1,k}^{n+1}+\left[1+\left(a_{j,k}^{n+1}+b_{j,k}^{n+1}+c_{j,k}^{n+1}\right)\Delta\tau\right]V_{j,k}^{n+1}-b_{j,k}^{n+1}\Delta\tau V_{j-1,k}^{n+1}=h_{j,k}^{n+1}.$$

The coefficients are given by

$$\begin{aligned}
a_{j,k}^{n+1} &= \left[\frac{\sigma^2 x_j^2}{2(\Delta x)^2}+\frac{\mu x_j}{\Delta x}\right],\ b_{j,k}^{n+1}=\frac{\sigma^2 x_j^2}{2(\Delta x)^2},\ d_{j,k}^{n+1}=\frac{u_{j,k}^{n+1}+\beta_1 u_0}{\Delta x},\\
c_{j,k}^{n+1} &= \frac{u_{j,k}^{n+1}+\beta_1 u_0}{\Delta x}+r+\left(a_1 u_{j,k}^{n+1}+b_1 z_k\right)\\
&\quad +\left[a_0 u_0+b_0 u_0\left(T-\tau_{n+1}\right)\right],\\
e_{j,k}^{n+1} &= \left(a_1 u_{j,k}^{n+1}+b_1 z_k\right)W_1(x_j,\tau_{n+1})\\
&\quad +\left[a_0 u_0+b_0 u_0(T-\tau_{n+1})\right]L_1(x_j,\tau_{n+1})-c_1\left(u_{j,k}^{n+1}\right),\\
h_{j,k}^{n+1} &= \left(d_{j,k}^{n+1}V_{j,k+1}^{n+1}+e_{j,k}^{n+1}\right)\Delta\tau+V_{j,k}^{n}.
\end{aligned}$$

Here, the coefficients $a_{j,k}^{n+1}$, $b_{j,k}^{n+1}$, $c_{j,k}^{n+1}$, $d_{j,k}^{n+1}$ are all nonnegative. The coefficients and $V_{j,k}^{n+1}$ are evaluated at the optimal control variables $u_{j,k}^{n+1*}$, which are determined by

$$\begin{aligned}
u_{j,k}^{n+1*} = \underset{u_{j,k}^{n+1}\in Q_1}{\operatorname{argmax}} &\left\{a_{j,k}^{n+1}\Delta\tau V_{j+1,k}^{n+1}-\left[1+\left(a_{j,k}^{n+1}+b_{j,k}^{n+1}+c_{j,k}^{n+1}\right)\Delta\tau\right]\right.\\
&\left. V_{j,k}^{n+1}+b_{j,k}^{n+1}\Delta\tau V_{j-1,k}^{n+1}+h_{j,k}^{n+1}\right\}.
\end{aligned}$$

We can express the above numerical scheme in the following matrix form

$$\sup_{u_{j,k}^{n+1}\in Q_1}\left\{-\mathbf{B}_k\mathbf{V}_k^{n+1}+\mathbf{h}_k^{n+1}\right\}=0,\quad k=0,1,\ldots,k_{\max}-1, \tag{3.3}$$

where

$$\begin{aligned}
\mathbf{V}_k^{n+1} &= \left(V_{0,k}^{n+1}\ V_{1,k}^{n+1}\ldots V_{j_{\max},k}^{n+1}\right)^T,\\
\mathbf{h}_k^{n+1} &= \left(h_{0,k}^{n+1}\ h_{1,k}^{n+1}\ldots h_{j_{\max}-1,k}^{n+1}\ C_1(\tau_{n+1};z_k)x_{\max}+C_2(\tau_{n+1};z_k)\right)^T,\\
[\mathbf{B}_k]_{l,m} &= \begin{cases}
1 & (l,m)=(j_{\max},j_{\max})\\
-b_{j,k}^{n+1}\Delta\tau & m=l-1, l=1,\ldots,j_{\max}-1,\\
-a_{j,k}^{n+1}\Delta\tau & m=l+1, l=0,\ldots,j_{\max}-1,\\
1+\left(a_{j,k}^{n+1}+b_{j,k}^{n+1}+c_{j,k}^{n+1}\right)\Delta\tau & m=l, l=0,\ldots,j_{\max}-1,\\
0 & \text{otherwise.}
\end{cases}
\end{aligned}$$

Let $(\mathbf{V}_k^{n+1})^i$ denote the ith iterate of the vector $\mathbf{V}_k^{n+1}$. The sequence of steps in the policy iteration are outlined as follows:

1. Set the initial guess of $\mathbf{V}_k^{n+1}$ to be $(\mathbf{V}_k^{n+1})^0 = \mathbf{V}_k^n$.
2. Assume the value of $(\mathbf{V}_k^{n+1})^i$ to be known, the ith iterate of the optimal control variable $(u_{j,k}^{n+1})^i$ is determined by

$$\left(u_{j,k}^{n+1}\right)^i = \underset{u_{j,k}^{n+1}\in Q_1}{\operatorname{argmax}}\left\{\left(-\mathbf{B}_k\left(\mathbf{V}_k^{n+1}\right)^i + \mathbf{h}_k^{n+1}\right)_j\right\},$$

   where $\left(-\mathbf{B}_k(\mathbf{V}_k^{n+1})^i + \mathbf{h}_k^{n+1}\right)_j$ denotes the jth component of the corresponding vector, $j = 1, 2, \ldots, j_{\max} - 1$.
3. Solve the linear system of equations

$$-(\mathbf{B}_k)^i\left(\mathbf{V}_k^{n+1}\right)^{i+1} + \left(\mathbf{h}_k^{n+1}\right)^i = 0,$$

   where $(\mathbf{B}_k)^i = \mathbf{B}_k|_{\left(u_{j,k}^{n+1}\right)^i}$ and $\left(\mathbf{h}_k^{n+1}\right)^i = \mathbf{h}_k^{n+1}|_{\left(u_{j,k}^{n+1}\right)^i}$.

The policy iteration is terminated when

$$\max_j \frac{\left(\mathbf{V}_k^{n+1}\right)^{i+1} - \left(\mathbf{V}_k^{n+1}\right)^i}{\left(\mathbf{V}_k^{n+1}\right)^{i+1}} < \text{tolerance value}.$$

The tridiagonal matrix $\mathbf{B}_k$ can be easily checked to be a M-matrix. In the policy iteration scheme presented above, the M-matrix property provides a sufficient condition for convergence of the policy iteration procedure so that the iterates $(\mathbf{V}_k^{n+1})^i$ converge to the unique solution of Eq. (3.3) for any initial guess $(\mathbf{V}_k^{n+1})^0$. A similar proof of the sufficient condition for convergence can be found in Leung and Kwok (2014).

4 Characteristics and sensitivity analysis of the optimal R&D effort

In this section, we would like to explore the characteristics of the optimal R&D effort of the private firm and public firm. The sensitivity analysis of the value functions with respect to optimal R&D efforts would inform the private firm manager and the policy maker about how to choose their optimal R&D strategies according to different levels of spillover effect and knowledge stock. Moreover, we analyze the impact on the optimal R&D effort with respect to varying market conditions, such as volatility and drift rate of the stochastic profit dynamics. In our numerical experiments, the following basic set of parameter values were used: $r = 0.05$, $\mu = 0.01$, $\sigma = 0.3$, $a_0 = a_1 = 1, b_0 = b_1 = 0.05, \beta_0 = 0, \beta_1 = 0.5, T = 10, \theta = 0.8, c_j(u_j) = 10 + \frac{u_j^2}{2}$,

$j = 0,\ 1, u_1 \in Q_1 = [0, 5]$ and $u_0 = 5$. When we examine the impact on the optimal R&D effort with respect to a particular parameter, the specific parameter may assume varying values other than the value specified in the above basic set of parameter values.

4.1 Characteristics of the optimal R&D efforts

Similar to the one-firm stochastic control R&D race model studied in Leung and Kwok (2014), we expect that the optimal R&D effort of the private firm is increasing with the current value of profit rate x while decreasing with the knowledge stock. Our numerical results reveal a similar pattern for the optimal R&D effort of the private firm in our two-firm mixed duopoly R&D race model. In Fig. 1a, we plot the optimal

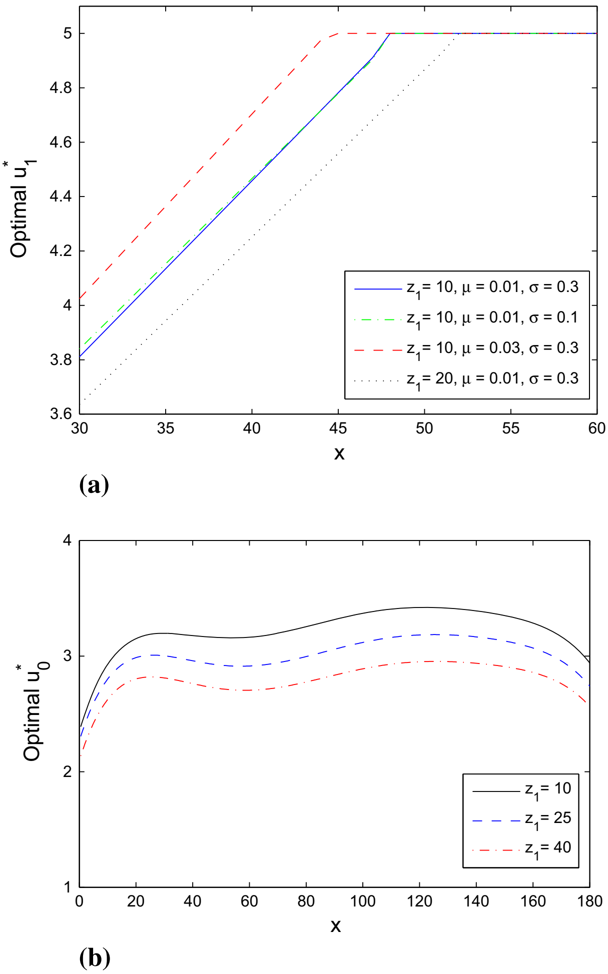

Fig. 1 **a** Plots of the optimal R&D effort of the private firm u_1^* (subject to maximum cap value of 5) against x with varying values of z_1, μ and σ; **b** Plots of the optimal R&D effort of the public firm u_0^* against x with varying values of z_1. **a** u_1^* against x, **b** u_0^* against x

control u_1^* of the private firm against the profit rate x with varying values of the knowledge stock z_1, mean of the profit flow rate μ and volatility of the profit flow rate σ. Given that the payoff to the private firm $V_1(x, t)$ upon arrival of the innovation is a linear function in x, the optimal control u_1^* increases linearly in x until it reaches the maximum cap level $\sup Q_1 = 5$. When the knowledge stock of the private firm increases, we observe a decrease in u_1^* since the private firm can benefit from the past knowledge accumulation. This phenomenon is revealed in Fig. 1a by the observation that the corresponding dotted curve of u_1^* for $z_1 = 20$ lies below the solid curve of u_1^* for $z_1 = 10$.

Moreover, the above observations are robust to various mean values and volatilities of the profit flow rate. To investigate how u_1^* may be affected by market uncertainty, we show the comparison of the optimal R&D effort of the private firm for different values of μ and σ in Fig. 1a. One would expect that the private firm should increase its R&D effort with a higher value of μ since the payoff to the private firm $V_1(x, t)$ is an increasing function of μ. As revealed in Fig. 1a, u_1^* increases as μ increases from 0.01 to 0.03 and u_1^* reaches the maximum level of 5 at a lower value of x when μ has a higher value. On the other hand, we observe that u_1^* is not quite sensitive to volatility σ since the payoff is a linear function in x with no optionality. We do observe a small increment in u_1^* when x is relatively small when σ reduces in value from 0.3 to 0.1. This indicates that the private firm chooses to expend less in R&D effort under a higher level of market volatility.

The plots shown in Fig. 1a are obtained with a fixed value of public firm's R&D effort of $u_0 = 5$. With varying levels of the public firm's fixed R&D effort u_0, the private firm's optimal R&D effort u_1^* would be adjusted accordingly. We do expect that a similar behavior of adopting the optimal control u_1^* would prevail with different values of u_0. We would like to recall that the aim of the public firm is to maximize social welfare. In the literature, social welfare is commonly modeled by the sum of consumer and producer surpluses minus costs. In our model, it is not straightforward to quantify surpluses as capacity additions. Pawlina and Kort (2006) argue that consumer surpluses can be modeled by firms' revenue flows. Here, we take social welfare S to be the sum of the firm values of both the public and private firms, $S = V_0 + V_1$. A similar approach has also been adopted by Delbono and Denicolò (1993), Poyago-Theotoky (1998) and Naseem and Oehmke (2006). In our model setup, once we have obtained the private firm's optimal control policy, we can calculate the value of social welfare S. With varying level of the public firm's R&D effort, we can determine the public firm's optimal R&D effort u_0^* that maximizes the social welfare value.

In Fig. 1b, we show the plots of the public firm's optimal R&D effort u_0^* against x with varying values of z_1. Unlike the private firm's optimal R&D effort u_1^* which is an increasing function of x, we observe that u_0^* increases at first with x, then drops slightly after it reaches its first peak. With a higher value of x, u_0^* increases again and stays at a certain high level until x becomes very large. Beyond some large value of x, the optimal R&D effort u_0^* decreases again. The dependence of u_0^* on x can be explained as follows. When x is relatively low, it is optimal for the public firm to increase its R&D effort in order to enhance social welfare through an earlier arrival of the new product. When the payoff rate x increases to around 25, the return is already attractive enough for the private firm to expend its maximum R&D effort. The public firm slightly reduces its R&D effort in order to avoid keen competition. This may explain the small drop in u_0^* after the first peak. When the payoff rate x continues to increase to a more attractive level, the public firm may choose to increase its R&D effort accordingly as it becomes more profitable to expend more R&D efforts for an earlier arrival of the new product that promises a high payoff. However, when x increases to a very high level, the public firm lowers its R&D effort again in order to avoid excessive sum of R&D efforts of the two firms. This explains the drop when x increases further to a very high value. Furthermore, we observe that u_0^* is a decreasing function of z_1 since less R&D effort is required with a higher knowledge stock z_1 in order to achieve the same hazard rate of success of innovation.

4.2 Sensitivity tests on various market conditions

We would like to examine the sensitivity of the optimal R&D effort of both firms with respect to different levels of input spillover, output spillover and knowledge stock. We performed various sensitivity tests by only changing one of the parameters in the standard set of parameter values in each set of calculations. The values of the input spillover rate β_1 and output spillover rate θ are chosen from 0.5 to 1, and the value of the knowledge stock factor b_j is chosen from 0 to 0.06, $j = 0$ or 1.

4.2.1 Input spillover effect

For the private firm, higher input spillover rate β_1 can accelerate its speed of knowledge accumulation and increases the private firm's chance of winning the R&D race. It is obvious that the private firm value increases with increasing input spillover rate. However, the impact of the input spillover rate on the optimal R&D effort of the private firm is not so obvious. In Fig. 2, we show the plots of the private firm's optimal R&D

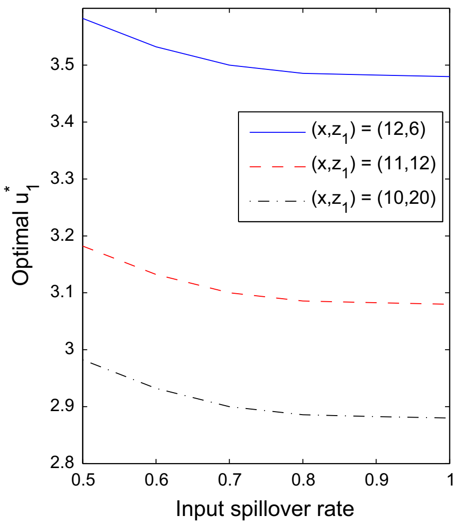

Fig. 2 Plots of the optimal R&D effort of the private firm u_1^* against input spillover rate β_1 at varying values of (x, z_1)

effort u_1^* against the input spillover rate β_1 at three different states of $(x,\ z_1)$ with $u_0 = 5$. We find that when the input spillover rate of the private firm β_1 increases in value from 0.5 to 1, an insignificant decline trend of the optimal R&D effort of the private firm is observed. Intuitively, the private firm can take advantage of the public firm's research effort under a higher level of input spillover. Therefore, the private firm would choose to reduce its R&D effort accordingly since it foresees an increase in the knowledge stock in the future due to higher input spillover. The same pattern of u_0^* against β_1 prevails with different states of (x, z_1).

4.2.2 Output spillover effect

Since output spillover allows the losing firm to imitate the production of the new product released by the winning firm and appreciate part of the payoffs, the effects of the output spillover rate on the firm values and optimal R&D efforts of the two firms are expected to be much more significant. In Fig. 3a, we show the plots of the optimal R&D effort of the private firm u_1^* to varying values of the appreciation rate θ at three different states of $(x,\ z_1)$ with $u_0 = 5$. We find that u_1^* increases almost linearly with the appreciation rate θ, which suggests that the private firm would expend more R&D effort to compete when it becomes harder to imitate the public firm's innovative product. When the appreciation rate is very high, especially when the firm can have perfect appreciation ($\theta = 1$), the competition is intensified as there is zero output spillover. We observe that the private firm enhances its R&D effort to a very high level. If the current payoff rate x is high enough, say $x = 20$, maximal level of R&D effort will be chosen for u_1^*. In contrast, when the appreciation rate is as low as $\theta = 0.5$,

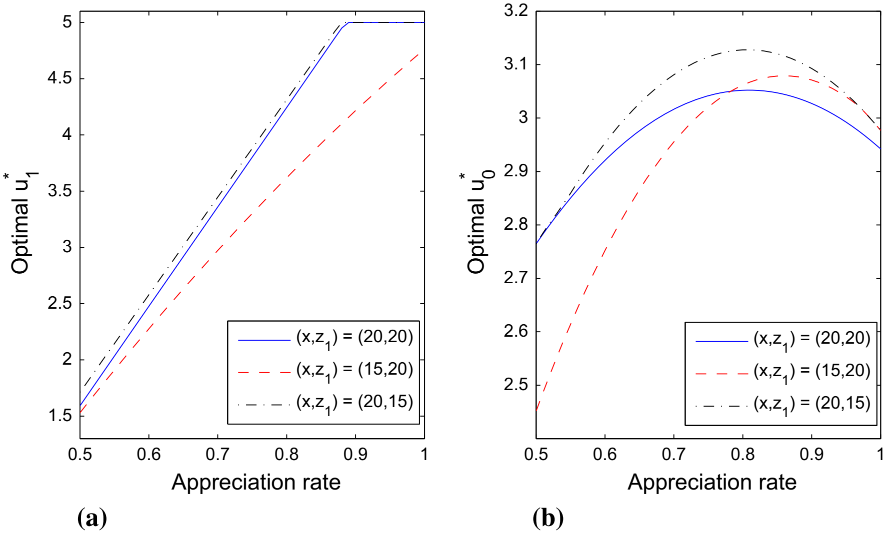

Fig. 3 **a** Plots of the optimal R&D effort of the private firm u_1^* against appreciation rate θ at varying values of (x, z_1); **b** Plots of the optimal R&D effort of the public firm u_0^* against appreciation rate θ at varying values of (x, z_1). **a** u_1^* against θ, **b** u_0^* against θ

the output spillover rate is at the highest level where the profits are equally shared by the two firms. The R&D effort of the private firm is found largely reduced to a low value of 1.5. Therefore, the private firm should simply play as the free rider.

In Fig. 3b, the optimal R&D effort of the public firm u_0^* is seen to be a non-monotonic function of the appreciation rate. We find that u_0^* increases with the appreciation rate at first and then decreases. The magnitude of variation is relatively small compared to the variation in u_1^* with varying values of θ. For different states of $(x,\ z_1)$, the maximum value of u_0^* is found at some intermediate level of the appreciation rate. The public firm foresees the response of the private firm to the output spillover, either adopting maximal R&D effort under perfect appreciation or taking a free ride attitude under a high level of output spillover, so the public firm reduces its R&D effort in view of excessive combined R&D efforts. When the output spillover rate is at some appropriate level, the public firm maximizes its R&D effort to achieve an earlier arrival of the new product. As a result, one would expect that the public policy maker can achieve higher social welfare by setting an appropriate level of output spillover in the mixed duopoly. That is, the public firm can play its role in alleviating inefficient R&D expenditure.

4.2.3 *Knowledge stock effect*

Since the knowledge stock also contributes to the hazard rate of the arrival of innovation, both firms can take advantage of the past knowledge accumulation to accelerate the arrival of the new product. Here, we expect that a high value of knowledge stock factor can help reducing both firms' R&D efforts. The plots in Fig. 4 are seen to agree

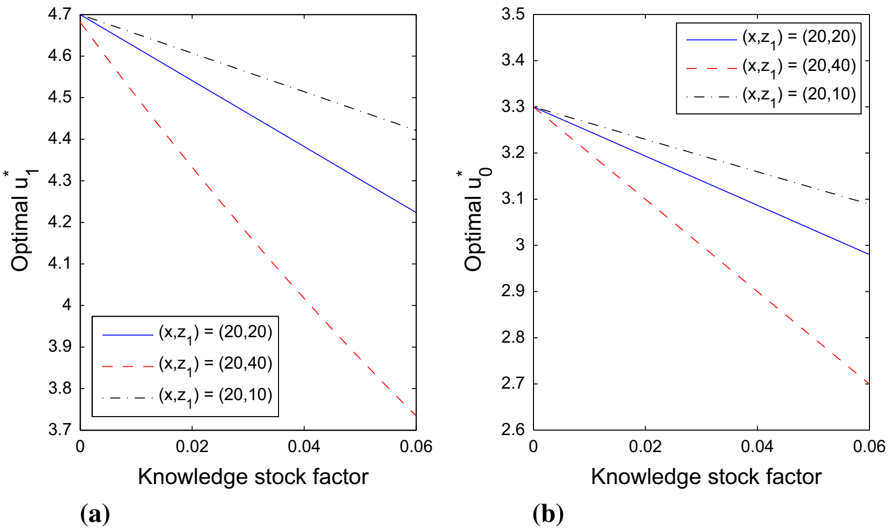

Fig. 4 **a** Plots of the optimal R&D effort of the private firm u_1^* against knowledge stock factor b_1 at varying values of (x, z_1); **b** Plots of the optimal R&D effort of the public firm u_0^* against knowledge stock factor b_0 at varying values of (x, z_1). **a** u_1^* against b_1, **b** u_0^* against b_0

with the above economic intuition. We find that the optimal R&D efforts of both firms decrease as the corresponding knowledge stock factor increases. When the knowledge accumulation factor b_0 or b_1 equals to 0, which means the knowledge accumulation effect is absent, both firms would set the highest value for the optimal R&D efforts. With a higher value of z_1, the effect of the knowledge stock becomes more evident, so u_1^* and u_0^* decline at a faster rate.

5 Conclusion

We have developed a stochastic control model under the setting of mixed duopoly for the R&D race with input and output spillovers. In addition, the effect of knowledge accumulation is considered so that the past knowledge stock also contributes to the hazard rate of arrival of the innovative product. We present the HJB formulation of the R&D race model and apply the finite difference scheme to solve for the private firm value and its optimal control of R&D expenditure simultaneously using the technique of policy iteration. The optimal R&D effort of the private firm increases almost linearly with the profit flow rate and decreases with the knowledge stock. However, the optimal R&D effort of the public firm does not reveal a monotonic pattern against the profit flow rate. This is because the public firm's R&D effort would be optimally chosen in response to the trade-off between acceleration of the arrival of the new product and excessive total R&D efforts expended in the society.

We also performed various sensitivity tests on the optimal R&D effort of the private firm and public firm with respect to various levels of spillover rate and knowledge stock factor. We find that the input spillover rate has a relatively smaller effect on the optimal R&D efforts of both firms. However, the knowledge stock and output spillover have more significant impact on the R&D effort. We find that both firms reduce their R&D expenditure to take advantage of the past accumulated knowledge when the knowledge stock factors increase. Also, when the appreciation rate is extremely high or low, overinvestment or free rider phenomenon is observed. The public firm has its role to mitigate inefficiency of R&D expenditure by setting an appropriate level of output spillover.

The interaction of the input and output spillover effects, knowledge stock accumulation and social role of the public firm is highly nonlinear. An increase in the level of the public firm's R&D effort may or may not improve social welfare. The resulting effects depend on various factors, such as the expected payoff from innovation (level of revenue flow), free rider phenomenon (spillover effects) and chance of innovative success (level of knowledge stock). The sensitive studies of the mixed duopoly R&D race model under various market conditions presented in this paper may shed some useful hints and insights for both the private firm manager and public fund decision makers on the optimal choices of R&D expenditures and appropriation of spillovers so as to maximize their respective value functions.

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6 Appendix: Derivation of $C_1(\tau; z_1)$ and $C_2(\tau; z_1)$, $\tau = T - t$

It is convenient to use $\tau = T - t$ as the temperal variable in C_1 and C_2. To determine $C_1(\tau; z_1)$ and $C_2(\tau; z_1)$, we substitute Eq. (2.6c) into the HJB formulation and obtain the following governing partial differential equations for $C_1(\tau; z_1)$ and $C_2(\tau; z_1)$:

$$\frac{\partial C_1}{\partial \tau} - (u_1 + \beta_1 u_0)\frac{\partial C_1}{\partial z_1} = -(r - \mu + h_0 + h_1)\,C_1 + \frac{h_0(1-\theta)\Pi}{r-\mu}\left[1 - e^{-(r-\mu)\tau}\right] + \frac{h_1\theta\Pi}{r-\mu}\left[1 - e^{-(r-\mu)\tau}\right],$$

$$\frac{\partial C_2}{\partial \tau} - (u_1 + \beta_1 u_0)\frac{\partial C_2}{\partial z_1} = -(r + h_0 + h_1)C_2 - c_1(u_1).$$

The above equations share the following general form:

$$\frac{\partial C}{\partial \tau} - a\frac{\partial C}{\partial z_1} = h(\tau; z_1)C + k(\tau; z_1), \tag{6.1}$$

where $h(\tau; z_1)$ and $k(\tau; z_1)$ are functions of τ and z_1. Here, a is constant (u_0 is constant and u_1 is also fixed by adopting sup Q_1). The general solution of Eq. (6.1) is given by

$$C(\tau; u) = H(\tau; u)\left[\phi(u) + \int_0^\tau \frac{k(s; u - as)}{H(s; u)}\,\mathrm{d}s\right],$$

where

$$H(\tau; u) = \exp\left(\int h(s; u - as)\,\mathrm{d}s\right), \quad u = z_1 + a\tau,$$

and $\phi(u)$ is an arbitrary function to be determined from appropriate auxiliary conditions. Since $C_1(0; z_1) = 0$ and $C_2(0; z_1) = 0$, we obtain $\phi_1(u) = \phi_2(u) = 0$. Therefore, the solution to Eq. (6.1) is found to be

$$C(\tau; u) = H(\tau; u)\int_0^\tau \frac{k(s; u - as)}{H(s; u)}\,\mathrm{d}s.$$

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