Pricing options on discrete realized variance with partially exact and bounded approximations

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Abstract

We derive efficient and accurate analytic approximation formulas for pricing options on discrete realized variance (DRV) under affine stochastic volatility models with jumps using the partially exact and bounded (PEB) approximations. The PEB method is an enhanced extension of the conditioning variable approach commonly used in deriving analytic approximation formulas for pricing discrete Asian style options. By adopting either the conditional normal or gamma distribution approximation based on some asymptotic behavior of the DRV of the underlying asset price process, we manage to obtain PEB approximation formulas that achieve high level of numerical accuracy in option values even for short-maturity options on DRV.

Keywords: options on discrete variance, stochastic volatility, conditioning variable method, partially exact and bounded approximations.

1 Introduction

We consider pricing of options on discrete realized variance (DRV) of the price process of an underlying risky asset. Let S_t denote the price process of the risky asset. Given N monitoring dates $0 = t_0 < t_1 < \cdots < t_N = T$ over the time period $[0, T]$, the DRV of S_t with respect to

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the above tenor is defined to be

$$I_T^{(N)} = \sum_{k=1}^{N} \left(\ln \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 = \sum_{k=1}^{N} (X_{t_k} - X_{t_{k-1}})^2, \tag{1.1}$$

where $X_t = \ln S_t$ is the log asset price process. It is common to define the annualized DRV by multiplying $I_T^{(N)}$ by A/N, where the annualization factor A is taken to be 252 for daily monitoring. In this paper, under the dynamics of stochastic volatility and jumps for S_t, we derive analytic approximation formulas using the enhanced conditioning variable approach for pricing financial options on $I_T^{(N)}$. The terminal payoff functions of these options take the form: $\max(I_T^{(N)} - K, 0)$ and $\max(K - I_T^{(N)}, 0)$, where K is the strike price.

Analytic approximation of the prices of options on DRV poses mathematical challenges due to the exotic path dependence of $I_T^{(N)}$, as exemplified by the sum of the quadratic terms involving $(X_{t_k} - X_{t_{k-1}})^2$. There have been several recent papers on pricing options on $I_T^{(N)}$. For the asymptotic approach proposed by Keller-Ressel and Muhle-Karbe (2013), the prices of options on continuous realized variance (CRV) are adjusted by the asymptotic formula for the short-time limit of the discretization gap between the CRV and DRV. They also develop the Laplace transform pricing method for options on DRV under exponential Lévy models without use of any approximation. Sepp (2012) considers an analytic approximation for the characteristic function of the DRV when the price dynamics is governed by the Heston model via the combination of the distribution of the CRV under the Heston model with that of the DRV under the Black-Scholes model. Good numerical accuracy is achieved for pricing near-the-money options on DRV over varying maturities. Zheng and Kwok (2014b) develop the saddlepoint approximation formulas for pricing options on DRV and volatility derivatives under both Lévy models and stochastic volatility models with jumps. As part of their procedure, they use the small time asymptotic expansion of the Laplace transform of the DRV (Keller-Ressel and Muhle-Karbe, 2013) as a control in the approximation of the cumulants of the DRV. The numerical accuracy of their saddlepoint approximation formulas is shown to be within a few percents of relative errors. Drimus and Farkas (2013) derive the discretization adjustment term added to the price of an option on CRV that serves to adjust the discretization effect in the discrete sampling of the realized variance. They show that conditional on the realization of the instantaneous variance process, the residual randomness arising from discrete sampling can be approximated by a normal random variable. Their treatment of the discretization effect arising from discrete sampling is applicable for general stochastic volatility processes. Motivated by the similarities between options on CRV and Asian options, Drimus (2012) adopts the conditioning variable approach (Rogers and Shi, 1995) to derive the lower bound for the prices of options on CRV under log-OU models.

In this paper, we derive efficient and accurate analytic approximation formulas for pricing options on DRV under affine stochastic volatility models with jumps using an enhanced version

of the conditioning variable approach. We consider the PEB approximation method (Lord, 2006), which adds adjustment terms by finding an analytic approximation to the residual component in the conditioning variable method. A similar approach has also been used by Zeng and Kwok (2014) to derive pricing bounds and approximation for discrete arithmetic Asian options under time-changed Lévy processes. The application of the PEB method in pricing options on DRV relies on the adoption of either the normal distribution approximation (Drimus and Farkas, 2013) or gamma distribution approximation (Keller-Ressel and Mulhe-Karbe, 2013) that is based on some asymptotic behavior of the DRV of the underlying asset price process.

The rest of the paper is organized as follows. In the next section, we derive the lower bound of the price of a call option on $I_T^{(N)}$ using the conditioning variable approach. The lower bound is expressible in terms of an integral that involves the joint characteristic functions of $(X_{t_k} - X_{t_{k-1}})$ and I_T, where I_T is the terminal value of the CRV of the underlying asset price process. Numerical tests reveal that the lower bound based on the conditioning variable approach does not provide sufficiently accurate approximation to the price of an option on DRV. In Section 3, we derive the PEB approximation for the residual terms in the conditioning variable approximation. Instead of following the usual procedure of constructing an approximation to the distribution of $I^{(N)}|I$ (dependency on T is suppressed for notational convenience) by matching the conditional moments (Lord, 2006; Zeng and Kwok, 2014), we propose to approximate $I^{(N)}$ by $\hat{I}^{(N)}$ such that $\hat{I}^{(N)}|I$ follows some common type of distribution that is highly tractable. One choice is based on the generalized Central Limit Theorem, where $\hat{I}^{(N)}|I$ is normally distributed with conditional mean I and conditional variance $\frac{2}{N}I^2$. The other choice is based on the small-time asymptotic behavior of the DRV, where $\hat{I}^{(N)}|I$ follows a gamma distribution with shape parameter $N/2$ and scale parameter $2I/N$. We show how to apply the PEB approximation formulas for pricing options on DRV whose underlying asset price process follows the Heston stochastic volatility model with jumps in asset price. By virtue of the affine structure of the Heston model with jumps, the joint characteristic function of $X_{t_k} - X_{t_{k-1}}$ and I_T is readily available in an affine form. This leads to nice computational efficiency in the evaluation of the Fourier integrals in the PEB approximation formulas. The effective implementation of the PEB approximation for pricing options on DRV under the Heston model with jumps are illustrated via various numerical tests presented in Section 4. The PEB approximation based on the gamma distribution is seen to provide high level of accuracy even for short maturity options. The last section contains summary and conclusive remarks.

2 Optimal lower bound based on conditioning on CRV

Our PEB approximation procedure consists of two steps. The first step is to derive the lower bound of the undiscounted price of the option on DRV based on the conditioning variable approach. In the second step, we derive the adjustment terms that approximate the residual component. In this section, we focus on the derivation of the lower bound, where the conditioning variable is chosen to be $I_T = [\ln S, \ln S]_T$, the time-T CRV of the underlying asset price process S_t. We show how to compute the lower bound when the characteristic function of I_T is available in an analytic form.

2.1 Lower bound of the call price based on conditioning variable

Let $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ be the natural filtration generated by the asset price process S_t and A be an event of the form $\{I_T > c\}$ with $c > 0$ such that $A \in \mathcal{F}_T$. For the undiscounted price of the call option on the DRV with strike price K, we have (Drimus, 2012)

$$
\begin{aligned}
\mathbb{E}[(I_T^{(N)} - K)^+] &= \mathbb{E}[(I_T^{(N)} - K)^+ 1_{A^c}] + \mathbb{E}[(I_T^{(N)} - K)^+ 1_A] \\
&= \mathbb{E}[(I_T^{(N)} - K)^+ 1_{A^c}] + \mathbb{E}[(K - I_T^{(N)})^+ 1_A] + \mathbb{E}[(I_T^{(N)} - K) 1_A] \quad (2.1) \\
&\geq \mathbb{E}[(I_T^{(N)} - K) 1_{\{I_T > c\}}].
\end{aligned}
$$

The last term gives a lower bound for the undiscounted call option price and the corresponding maximum value among all choices of nonnegative values of c provides the best lower bound. Note that I_T is chosen to be the conditioning variable since $I_T^{(N)}$ and I_T are highly correlated and we take the advantage that I_T is tractable. For convenience, we write

$$
g(c) = \mathbb{E}[(I_T^{(N)} - K) 1_{\{I_T > c\}}] = \int_c^\infty \mathbb{E}[I_T^{(N)} - K | I_T = y] f_I(y) \, \mathrm{d}y, \quad (2.2)
$$

where f_I is the density function of I_T. We observe that the conditional expectation $g(c)$ increases in value with increasing c when c is small and eventually drops to zero as c becomes sufficiently large due to the rapid decay of f_I. Therefore, we expect that $g(c)$ achieves its maximum value at some finite value c^* that is close to K. The critical value c^* satisfies the first order condition:

$$
g'(c) = -\mathbb{E}[I_T^{(N)} - K | I_T = c] f_I(c) = 0. \quad (2.3)
$$

2.2 Evaluation of the lower bound with known characteristic function via Fourier inversion method

We consider the implementation of the first step in the PEB approximation when the analytic form of the joint characteristic function of the squared increment and CRV is known. Though

our argument works even when the joint characteristic function does not have a closed form, such analytic tractability indeed greatly facilitates the efficient computation of the lower bound. For a typical quadratic term $(X_{t_k} - X_{t_{k-1}})^2$ in $I_T^{(N)}$, $k = 1, 2, \ldots, N$, evaluation of the conditional expectation can be performed via the following transformation:

$$\mathbb{E}[(X_{t_k} - X_{t_{k-1}})^2 | I_T = c] = -\frac{\partial^2}{\partial \phi^2} \mathbb{E}[e^{i\phi(X_{t_k} - X_{t_{k-1}})} | I_T = c]\Big|_{\phi=0}.$$

We write $\Delta_k = X_{t_k} - X_{t_{k-1}}$ and let $f_{\Delta_k,I}$ and $\Phi_{\Delta_k,I}$ be the joint density function and joint characteristic function of Δ_k and I_T, respectively. By virtue of the Parseval identity and interchanging order of integration, we obtain

$$
\begin{aligned}
E[e^{i\phi\Delta_k} | I_T = c] &= \frac{1}{f_I(c)} \int_{-\infty}^{\infty} e^{i\phi x} f_{\Delta_k,I}(x, c) \, dx \\
&= \frac{1}{f_I(c)} \int_{-\infty}^{\infty} e^{i\phi x} \frac{1}{4\pi^2} \int_{i\beta_i-\infty}^{i\beta_i+\infty} \int_{i\alpha_i-\infty}^{i\alpha_i+\infty} \Phi_{\Delta_k,I}(\alpha, \beta) e^{-i\alpha x - i\beta c} \, d\alpha \, d\beta \, dx \\
&= \frac{1}{f_I(c)} \int_{i\beta_i-\infty}^{i\beta_i+\infty} \int_{i\alpha_i-\infty}^{i\alpha_i+\infty} \frac{1}{4\pi^2} \Phi_{\Delta_k,I}(\alpha, \beta) \int e^{i(\phi-\alpha)x} \, dx \, e^{-i\beta c} \, d\alpha \, d\beta \\
&= \frac{1}{f_I(c)} \int_{i\beta_i-\infty}^{i\beta_i+\infty} \frac{1}{2\pi} \Phi_{\Delta_k,I}(\phi, \beta) e^{-i\beta c} \, d\beta,
\end{aligned}
$$

where $\alpha = \alpha_r + i\alpha_i$ and $\beta = \beta_r + i\beta_i$ are complex Fourier transform variables. The respective imaginary part α_i and β_i of the pair of transform variables α and β are chosen to be some appropriate fixed constants to ensure convergence of the generalized Fourier transform integral. Summing all the individual expectations of $e^{i\phi\Delta_k}$ conditional on $I_T = c$, where $k = 1, 2, \cdots, N$, and substituting into eq. (2.3), the first order condition can be expressed as

$$g'(c) = \int_0^{\infty} \frac{1}{\pi} \sum_{k=1}^{N} \Re \left[\frac{\partial^2}{\partial \phi^2} \Phi_{\Delta_k,I}(\phi, \beta_r + i\beta_i) \Big|_{\phi=0} e^{-i(\beta_r + i\beta_i)c} \right] d\beta_r + K f_I(c) = 0,$$

where $\Re(\cdot)$ stands for the real part. It is known that most stochastic volatility models do not admit closed form analytic expression for f_I. However, the characteristic function Φ_I of I exist in closed form for most affine jump-diffusion models, including the Heston stochastic volatility model with jumps. In that case, it is convenient to express $f_I(c)$ as a Fourier inversion integral of Φ_I such that

$$f_I(c) = \frac{1}{\pi} \int_0^{\infty} \Re \left[\Phi_I(\beta_r + i\beta_i) e^{-i(\beta_r + i\beta_i)c} \right] d\beta_r.$$

As a result, the first order condition can be expressed in the following compact form:

$$g'(c) = \frac{1}{\pi} \int_0^{\infty} \Re \left[\Psi(\beta_r + i\beta_i) e^{-i(\beta_r + i\beta_i)c} \right] d\beta_r = 0, \tag{2.4}$$

where

$$\Psi(\beta) = \sum_{k=1}^{N} \frac{\partial^2}{\partial \phi^2} \Phi_{\Delta_k,I}(\phi, \beta)\Big|_{\phi=0} + K \Phi_I(\beta).$$

Since the integral representation of $g''(c)$ is readily available and it takes the form

$$g''(c) = \frac{1}{\pi} \int_0^\infty \Re\left[-i(\beta_r + i\beta_i)\Psi(\beta_r + i\beta_i)e^{-i(\beta_r + i\beta_i)c}\right] \, d\beta_r,$$

one may solve eq. (2.4) for the critical value c^* via Newton's iteration method with the initial guess $c = K$. One may check for $g''(c) < 0$ to ensure that c^* is a maximizer of $g(c)$.

Finally, we can calculate the optimal lower bound for the option price as follows:

$$
\begin{aligned}
g(c^*) &= \sum_{k=1}^N \mathbb{E}[\Delta_k^2 1_{\{I_T > c^*\}}] - K\mathbb{P}(I_T > c^*) \\
&= -\frac{1}{\pi} \int_0^\infty \Re\left[\sum_{k=1}^N \frac{\partial^2}{\partial\phi^2}\Phi_{\Delta_k, I}(\phi, \beta_r + i\beta_i)\Big|_{\phi=0} \frac{e^{-ic^*(\beta_r + i\beta_i)}}{i(\beta_r + i\beta_i)}\right] \, d\beta_r \\
&\quad - K\frac{1}{\pi} \int_0^\infty \Re\left[\Phi_I(\beta_r + i\beta_i)\frac{e^{-ic^*(\beta_r + i\beta_i)}}{i(\beta_r + i\beta_i)}\right] \, d\beta_r \\
&= -\frac{1}{\pi} \int_0^\infty \Re\left[\Psi(\beta_r + i\beta_i)\frac{e^{-ic^*(\beta_r + i\beta_i)}}{i(\beta_r + i\beta_i)}\right] \, d\beta_r.
\end{aligned}
\tag{2.5}
$$

Note that it is necessary to restrict β_i to be negative.

2.3 Alternative derivation

It is relatively easy to establish a lower bound for the undiscounted call option price via an approximation of $I_T^{(N)}$ by the conditional variable $\mathbb{E}[I_T^{(N)}|I_T]$. Indeed, by the Jensen inequality, we deduce that

$$\mathbb{E}[(I_T^{(N)} - K)^+] = \mathbb{E}[\mathbb{E}[(I_T^{(N)} - K)^+|I_T]] \geq \mathbb{E}[(\mathbb{E}[I_T^{(N)}|I_T] - K)^+].$$

Interestingly, the lower bound on the right hand side is simply the optimal lower bound $g(c^*)$ as given by eq. (2.5).

The above observation relies on the assumption that $g'(c)$ defined by eq. (2.3) changes sign only once at some internal point. Though a rigorous proof of this assumed property of $g'(c)$ may not be straightforward, one can visualize this analytic property intuitively. Since $I_T^{(N)}$ and I_T are strongly correlated, and when I_T is set at some value of c that is significantly smaller than K, the conditional density of $I_T^{(N)}$ is supposed to be concentrated around c. As a result, the negative of the conditional expectation $-E[I_T^{(N)} - K|I_T = c]$ is positive, that gives $g'(c) > 0$ if $f_I(c)$ is refrained from hitting zero. As c increases in value and gets closer to K, the negative of the conditional expectation decreases in value and eventually $g'(c)$ hits the zero value at some critical value c^*. When c increases beyond c^* further, the conditional expectation remains positive, so $g'(c)$ stays negative. Finally, $|g'(c)|$ decreases in magnitude and $g'(c)$ approaches

the zero value from below at some asymptotically large value of c since $f_I(c)$ converges to zero rapidly as c tends to infinity. An illustrative plot of $g'(c)$ is shown in Figure 1. Based on the above assumed analytic property of $g'(c)$, one can then obtain

$$
\begin{aligned}
\mathbb{E}[(\mathbb{E}[I_T^{(N)}|I_T] - K)^+] &= \int_0^\infty (\mathbb{E}[I_T^{(N)}|I_T = c] - K)^+ f_I(c) \; \mathrm{d}c \\
&= \int_0^\infty \left(\mathbb{E}[I_T^{(N)}|I_T = c] f_I(c) - K f_I(c) \right)^+ \mathrm{d}c \\
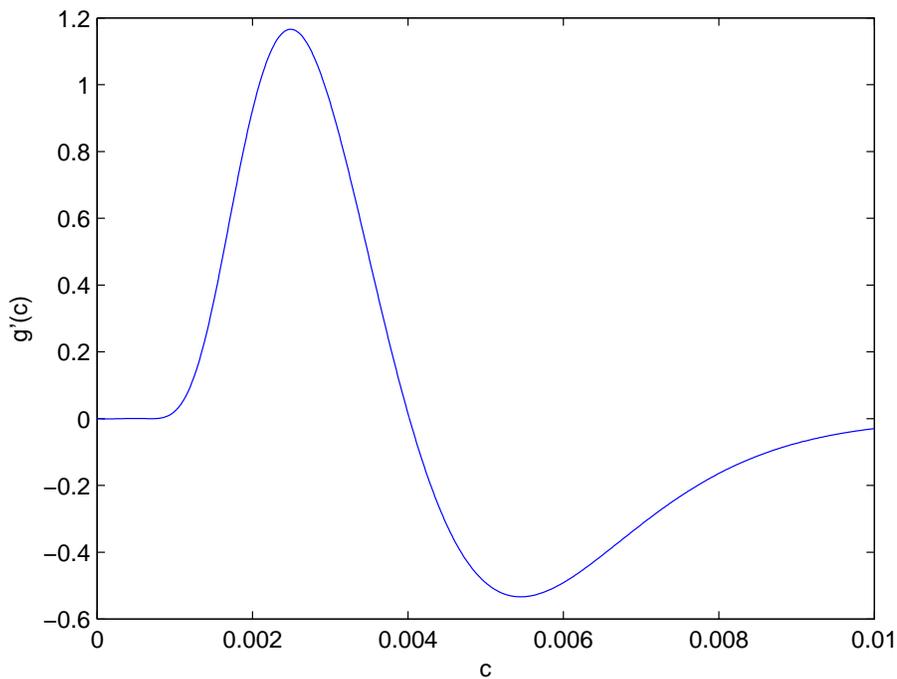&= \int_0^\infty [-g'(c)]^+ \; \mathrm{d}c = g(c^*).
\end{aligned}
$$

Figure 1: An illustrative plot of $g'(c)$. As deduced from the analytic property of $g'(c)$, there always exists a unique positive root of $g'(c)$.

3 Partially exact and bounded approximation

The lower bound derived from the conditioning variable approach works quite well for arithmetic Asian options based on conditioning on the geometric average counterpart (Zeng and Kwok, 2014), whereas the lower bound $g(c^*)$ defined by eq. (2.5) is seen to fail to provide sufficiently accurate approximation formulas for short-maturity options on DRV. One major difference is that while we observe dominance of arithmetic average over geometric average,

there is a lack of strict dominance of the DRV over the continuous counterpart or vice versa. Due to this lack of dominance, optionality on the CRV may not be carried over to optionality on the discrete counterpart. This explains the significant gap between the lower bound and the exact price of an option on DRV. Indeed, the discrepancy between the DRV and CRV becomes more profound when maturity or sampling period becomes shorter. Therefore, the lower bound approximation becomes more unreliable for short-maturity options on DRV. As a remark, the crude approximation of $I_T^{(N)}$ by I_T in the option valuation provides an even worse approximation than the lower bound $g(c^*)$ derived by conditioning.

Henceforth, we drop the subscript T in both $I_T^{(N)}$ and I_T for notational convenience in our later exposition when no ambiguity arises. To provide a better approximation, it is natural to consider an analytic approximation to the residual terms

$$\mathbb{E}[(I^{(N)} - K)^+ 1_{\{I \leq c^*\}}] + \mathbb{E}[(K - I^{(N)})^+ 1_{\{I > c^*\}}] \tag{3.1}$$

in the decomposition of the option price shown in eq. (2.1). In the literature on pricing arithmetic Asian options, this approach is termed the *partially exact and bounded* (PEB) approximation. The essence of the PEB approximation is to consider an approximation to the conditional distribution of $I^{(N)}|I$ so that evaluation of the two residual terms can be performed efficiently. The common technique in the PEB approximation for pricing arithmetic Asian options is to fit a lognormal or normal distribution to the difference of $I^{(N)}|I - I$ by matching the respective conditional moments (Lord, 2006; Zeng and Kwok, 2014). In the implementation of the second step of the PEB approximation for the call option on DRV, we propose two analytic approximation methods based on the normal distribution and gamma distribution approximations derived from some asymptotic behavior of the realized variance of the underlying asset price process.

3.1   Conditional normal distribution approximation

Based on the generalized Central Limit Theorem and asymptotic analysis of the DRV of an asset price process under stochastic volatility, Drimus and Farkas (2013) show that one may approximate $I^{(N)}|I$ by $\hat{I}^{(N)}|I$ for a sufficiently large value of N, where

$$\hat{I}^{(N)}|I \sim \mathcal{N}\left(I, \frac{2}{N}I^2\right). \tag{3.2}$$

Here, $\mathcal{N}(\mu, \sigma^2)$ denotes a normal distribution with mean μ and variance σ^2. Though their result is derived under the stochastic volatility framework, it can be seen that it remains to work well under stochastic volatility with jumps.

In our derivation of the PEB approximation for the call option on DRV, it is beneficial to introduce another approximation with the order of approximation error consistent with

that of the Drimus-Farkas approximation. Let $\Phi_{\hat{I}^{(N)},I}(\alpha, \beta)$ denote the joint characteristic function of $\hat{I}^{(N)}$ and I. Since $\hat{I}^{(N)}|I$ is given by eq. (3.2), by introducing the approximation: $e^{-\alpha^2 I^2/N} \approx 1 - \frac{\alpha^2 I^2}{N}$ under $O(N^{-2})$ approximation, we have

$$
\begin{aligned}
\Phi_{\hat{I}^{(N)},I}(\alpha, \beta) &= \mathbb{E}[e^{i\alpha\hat{I}^{(N)}+i\beta I}] = \mathbb{E}[\mathbb{E}[e^{i\alpha\hat{I}^{(N)}+i\beta I}|I]] = \mathbb{E}[e^{i\alpha I - \frac{\alpha^2 I^2}{N}} e^{i\beta I}] \\
&\approx \mathbb{E}\left[e^{i(\alpha+\beta)I}(1 - \frac{\alpha^2 I^2}{N})\right] = \Phi_I(\alpha + \beta) + \frac{\alpha^2}{N}\Phi_I^{(2)}(\alpha + \beta),
\end{aligned}
\tag{3.3}
$$

where Φ_I denotes the characteristic function of I and $\Phi_I^{(2)}$ refers to the second order derivative of Φ_I. The above approximation has the same order as that of the Drimus-Farkas approximation.

Next, we derive an analytic approximation of the two residual terms by writing them as Fourier integrals via the Parseval Theorem. For the first residual term, we propose

$$
\begin{aligned}
\mathbb{E}[(I^{(N)} - K)^+ 1_{\{I \leq c^*\}}] &\approx \mathbb{E}[(\hat{I}^{(N)} - K)^+ 1_{\{I \leq c^*\}}] \\
&= \frac{1}{4\pi^2} \int_{ib-\infty}^{ib+\infty} \int_{ia-\infty}^{ia+\infty} e^{-i\alpha K - i\beta c^*} \frac{\Phi_{\hat{I}^{(N)},I}(\alpha, \beta)}{i\beta\alpha^2} \, d\alpha \, d\beta,
\end{aligned}
$$

where $a < 0$ and $b > 0$ are chosen such that the integration contours are within the domain of convergence of the two-dimensional generalized Fourier transform. In the next step, we apply an analytic approximation of the joint characteristic function $\Phi_{\hat{I}^{(N)},I}(\alpha, \beta)$ given by eq. (3.3). For convenience, we write $z = \alpha + \beta$ so that

$$
\begin{aligned}
\mathbb{E}[(I^{(N)} - K)^+ 1_{\{I \leq c^*\}}] &\approx \frac{1}{2\pi} \int_{iu-\infty}^{iu+\infty} e^{-izc^*} \Phi_I(z) \frac{1}{2\pi} \int_{ia-\infty}^{ia+\infty} \frac{e^{-i\alpha(K-c^*)}}{i(z-\alpha)\alpha^2} \, d\alpha \, dz \\
&+ \frac{1}{2\pi} \int_{iu-\infty}^{iu+\infty} e^{-izc^*} \frac{\Phi_I^{(2)}(z)}{N} \frac{1}{2\pi} \int_{ia-\infty}^{ia+\infty} \frac{e^{-i\alpha(K-c^*)}}{i(z-\alpha)} \, d\alpha \, dz,
\end{aligned}
\tag{3.4}
$$

where $u = a + b > a$ specifies the horizontal contour of the complex integral with respect to z. In a similar manner, we may approximate the second residual term by

$$
\begin{aligned}
\mathbb{E}[(K - I^{(N)})^+ 1_{\{I > c^*\}}] &\approx \mathbb{E}[(K - \hat{I}^{(N)})^+ 1_{\{I > c^*\}}] \\
&= \frac{1}{4\pi^2} \int_{i\hat{b}-\infty}^{i\hat{b}+\infty} \int_{i\hat{a}-\infty}^{i\hat{a}+\infty} e^{-i\alpha K - i\beta c^*} \frac{\Phi_{\hat{I}^{(N)},I}(\alpha, \beta)}{-i\beta\alpha^2} \, d\alpha \, d\beta,
\end{aligned}
$$

where $\hat{a} > 0$ and $\hat{b} < 0$ are chosen to ensure that the integration contours are within the domain of convergence of the two-dimensional generalized Fourier transform. Again, by applying the approximation in eq. (3.3) and letting $z = \alpha + \beta$, we obtain

$$
\begin{aligned}
\mathbb{E}[(K - I^{(N)})^+ 1_{\{I > c^*\}}] &\approx -\frac{1}{2\pi} \int_{i\hat{u}-\infty}^{i\hat{u}+\infty} e^{-izc^*} \Phi_I(z) \frac{1}{2\pi} \int_{i\hat{a}-\infty}^{i\hat{a}+\infty} \frac{e^{-i\alpha(K-c^*)}}{i(z-\alpha)\alpha^2} \, d\alpha \, dz \\
&- \frac{1}{2\pi} \int_{i\hat{u}-\infty}^{i\hat{u}+\infty} e^{-izc^*} \frac{\Phi_I^{(2)}(z)}{N} \frac{1}{2\pi} \int_{i\hat{a}-\infty}^{i\hat{a}+\infty} \frac{e^{-i\alpha(K-c^*)}}{i(z-\alpha)} \, d\alpha \, dz,
\end{aligned}
\tag{3.5}
$$

where $\hat{u} = \hat{a} + \hat{b} < \hat{a}$ specifies the horizontal contour of the complex integral with respect to z.

Interestingly, the corresponding integrands in the Fourier integrals in eqs. (3.4) and (3.5) are identical. The two Fourier integrals differ only in the choices of the contours, where one is along a horizontal contour below the real axis oriented in the positive direction while the other is along a horizontal contour above the real axis oriented in the negative direction. This is not surprising since the two quantities in the two residual terms have the same analytic form but differ in sign. We include the vertical contours at the two extreme ends on the right and left side of the complex plane that join the two horizontal contours to form a closed contour C. The values of the contour integrals along the two vertical contours at the positive and negative far-end side of the complex plane are seen to assume zero value in the asymptotic limit.

We now combine the Fourier integrals in eqs. (3.4) and (3.5) that approximate the two residual terms. We choose a common contour for the integral with respect to z. That is, we choose the horizontal contour to be from $i\tilde{u} - \infty$ to $i\tilde{u} + \infty$, where $a < \tilde{u} < \hat{a}$. Also, we use the Cauchy Residue Theorem to evaluate the inner contour integral with respect to the closed contour C. Since we have chosen $a < \tilde{u} < \hat{a}$, where $a < 0$ and $\hat{a} > 0$, the poles are included inside the closed contour C. By combining the respective first terms in eqs. (3.4) and (3.5), we obtain

$$
\begin{aligned}
A &= \frac{1}{2\pi} \int_{i\tilde{u}-\infty}^{i\tilde{u}+\infty} e^{-izc^*} \Phi_I(z) \frac{1}{2\pi} \oint_C \frac{e^{-i\alpha(K-c^*)}}{i(z-\alpha)\alpha^2} \, d\alpha \, dz \\
&= \frac{1}{2\pi} \int_{i\tilde{u}-\infty}^{i\tilde{u}+\infty} e^{-izc^*} \Phi_I(z) \frac{1 + iz(c^* - K) - e^{iz(c^*-K)}}{z^2} \, dz.
\end{aligned}
\tag{3.6a}
$$

In a similar manner, by combining the respective second terms in eqs. (3.4) and (3.5), we obtain

$$
\begin{aligned}
B &= \frac{1}{2\pi} \int_{i\tilde{u}-\infty}^{i\tilde{u}+\infty} e^{-izc^*} \frac{\Phi_I^{(2)}(z)}{N} \frac{1}{2\pi} \oint_C \frac{e^{-i\alpha(K-c^*)}}{i(z-\alpha)} \, d\alpha \, dz \\
&= \frac{1}{2\pi} \int_{i\tilde{u}-\infty}^{i\tilde{u}+\infty} e^{-izc^*} \frac{\Phi_I^{(2)}(z)}{N} \left[-e^{-iz(K-c^*)} \right] dz, \\
&= \frac{K^2}{2\pi N} \int_{i\tilde{u}-\infty}^{i\tilde{u}+\infty} e^{-izK} \Phi_I(z) \, dz \quad \text{(applying integration by parts twice)} \\
&= \frac{K^2}{N} f_I(K).
\end{aligned}
\tag{3.6b}
$$

We manage to express the approximation of the two residual terms as the sum of an one-dimensional integral and an explicitly known term.

Last but not least, we would like to discuss the financial interpretation of the above two terms. It is easily visualized that the term A is simply equal to the following quantity:

$$
\mathbb{E}[(I - K)^+ 1_{\{I \leq c^*\}}] + \mathbb{E}[(K - I)^+ 1_{\{I > c^*\}}].
$$

In other words, keeping the single term A alone in the analytic approximation would be equivalent to approximating the two residual terms by simply replacing $I^{(N)}$ by I. Since the optimal solution $c^* \approx K$, we expect that both $\{K < I \leq c^*\}$ and $\{K \geq I > c^*\}$ are small probability events. Therefore, the correction contributed by A would be small and secondary. The second term B is seen to be identical to the discretization adjustment term presented in Drimus and Farkas (2013). This discretization adjustment arises when Drimus and Farkas try to account for the discrete sampling effect of DRV in the approximation of $\mathbb{E}[(I^{(N)} - K)^+]$ by $\mathbb{E}[(I - K)^+]$. It is interesting to observe that B has dependence on N but no dependence on c^* while A has the reverse properties of functional dependence. The term B provides the discretization gap between $I^{(N)}$ and I that is not captured by the optimal lower bound. In general, the contribution of B as an adjustment term added to the optimal lower bound is more significant compared to that of A.

3.2   Conditional gamma distribution approximation

The conditional normal distribution is based on the asymptotic behavior of $I^{(N)}$ as $N \to \infty$. When we consider pricing of short-maturity options on DRV, the asymptotic behavior of the DRV as $T \to 0$ is more relevant. In this regard, Keller-Ressel and Mulhe-Karbe (2013) propose the asymptotic gamma distribution of the DRV as $T \to 0$. More specifically, it can be shown that the annualized CRV tends to V_0 as $T \to 0$ while the DRV converges in distribution to a gamma distribution with shape parameter $N/2$ and scale parameter $2V_0/N$, where V_0 is the initial value of the instantaneous variance. Motivated by this elegant theoretical result, we propose to approximate $I^{(N)}$ by $\hat{I}^{(N)}$, which has a gamma distribution with shape parameter $N/2$ and scale parameter $2I/N$ conditional on I, where

$$\hat{I}^{(N)}|I \sim \text{gamma}(N/2, 2I/N). \tag{3.7}$$

The above gamma approximation has the same conditional mean and variance as the normal approximation in the previous subsection. Specifically, the gamma approximation is advantageous over the normal distribution in the following two aspects. Firstly, it becomes exact in asymptotic limit as $T \to 0$. Secondly, the gamma approximation retains nonnegativity of $I^{(N)}|I$.

As the first step in deriving the analytic approximation of the residual terms using the conditional gamma distribution approximation, we express the residual terms as nested conditional expectation:

$$\mathbb{E}[\mathbb{E}[(K - I^{(N)})^+|I]1_{\{I > c^*\}}] + \mathbb{E}[\mathbb{E}[(I^{(N)} - K)^+|I]1_{\{I \leq c^*\}}]. \tag{3.8}$$

Substituting the explicit form of the gamma density function and applying the put-call parity

relation, the inner expectation can be evaluated as follows:

$$
\begin{aligned}
\mathbb{E}[(K - I^{(N)})^+|I] &\approx \int_0^K (K - y)\frac{y^{N/2-1}e^{-\frac{Ny}{2I}}}{\Gamma(N/2)(2I/N)^{N/2}}\,\mathrm{d}y \\
&= \frac{1}{\Gamma(\frac{N}{2})}\left[(K - I)\gamma\left(\frac{N}{2}, \frac{KN}{2I}\right) + K\exp\left(\left(\frac{N}{2} - 1\right)\ln\frac{KN}{2I} - \frac{KN}{2I}\right)\right] \\
\mathbb{E}[(I^{(N)} - K)^+|I] &= E[I^{(N)}|I] - K + E[(K - I^{(N)})^+|I] \\
&\approx I - K + \frac{1}{\Gamma(\frac{N}{2})}\left[(K - I)\gamma\left(\frac{N}{2}, \frac{KN}{2I}\right) + K\exp\left(\left(\frac{N}{2} - 1\right)\ln\frac{KN}{2I} - \frac{KN}{2I}\right)\right],
\end{aligned}
$$

where $\Gamma(\cdot)$ is the gamma function and $\gamma(s, x) = \int_0^x z^{s-1}e^{-t}\,\mathrm{d}z$ is the lower incomplete gamma function. Putting the above results together, the correction term C_g that is added to the optimal lower bound based on the conditional gamma distribution approximation is given by

$$
C_g = \int_0^\infty G(y)f_I(y)\,\mathrm{d}y + \int_0^{c^*}(y - K)f_I(y)\,\mathrm{d}y, \tag{3.9}
$$

where

$$
G(y) = \frac{1}{\Gamma(\frac{N}{2})}\left[(K - y)\gamma\left(\frac{N}{2}, \frac{KN}{2y}\right) + K\exp\left(\left(\frac{N}{2} - 1\right)\ln\frac{KN}{2y} - \frac{KN}{2y}\right)\right].
$$

Unlike the earlier derivation of the conditional normal distribution approximation, we do not use the method of double Fourier transform in the above derivation of the conditional gamma distribution approximation. The major reason is that without making an approximation like eq. (3.3), the joint characteristic function of $I^{(N)}$ and I is intractable. As a result, the double Fourier transform method cannot be applied.

The above correction formula also conforms well with financial intuition. The first integral in C_g is seen to be $\mathbb{E}[(K - \hat{I}^{(N)})^+]$ under the conditional gamma distribution approximation. The second term can be interpreted as $\mathbb{E}[\hat{I}^{(N)} - K] - \mathbb{E}[(\hat{I}^{(N)} - K)1_{\{I>c^*\}}]$ under the same approximate distribution. The sum gives $\mathbb{E}[(\hat{I}^{(N)} - K)^+] - \mathbb{E}[(\hat{I}^{(N)} - K)1_{\{I>c^*\}}]$, which is exactly the residual given by eq. (3.1) with $I^{(N)}$ being replaced by $\hat{I}^{(N)}$ under the conditional gamma distribution. The small-time asymptotic approximation approach by Keller-Ressel and Mulhe-Karbe (2013) attempts to approximate the "discretization gap" between the price of an option on DRV and that of the continuous counterpart. Our PEB approximation considers approximating $I^{(N)}$ by $\hat{I}^{(N)}$ under the approximate gamma distribution in the residual terms. As a result, while the small time asymptotic approximation is only guaranteed to perform well for small T, our PEB approximation would provide high level of accuracy over a much wider value range of T.

Connection to the normal distribution approximation

We would like to connect the above conditional normal distribution approximation and conditional gamma distribution approximation through the well-known normal approximation to

the gamma distribution. By virtue of the Central Limit Theorem, it is well known that the gamma distribution with shape parameter k and scale parameter θ converges to the normal distribution with mean $k\theta$ and variance $k\theta^2$ when k is sufficiently large. We would like to show that when N is sufficiently large, the gamma distribution given by eq. (3.7) converges to the normal distribution given by eq. (3.2). We consider the Taylor expansion in powers of $1/N$ of the moment generating function of the gamma distribution:

$$
\begin{aligned}
M_g(z) &= \left(1 - \frac{2I}{N}z\right)^{-N/2} = \exp\left(-\frac{N}{2}\ln\left(1 - \frac{2I}{N}z\right)\right) \\
&= \exp\left(-\frac{N}{2}\left[-\frac{2I}{N}z - \frac{1}{2}\left(\frac{2I}{N}z\right)^2 - O(N^{-3})\right]\right) \\
&= \exp\left(Iz + \frac{I^2}{N}z^2 - O(N^{-2})\right).
\end{aligned}
$$

Suppose we ignore the higher order terms $O(N^{-2})$ in the above Taylor expansion, it becomes identical to the moment generating function of the normal distribution in eq. (3.2). This connection helps explain why the performances of the two approximations for long-maturity options on DRV are almost indistinguishable (see Table 2). Finally, we remark that since we have made the simplification shown in eq. (3.3), simply replacing the gamma density in eq. (3.9) with its normal approximation would not lead to the same formulas as shown in eqs. (3.6a) and (3.6b).

3.3 Heston stochastic volatility model with jumps

Though the PEB approximation procedure proposed above does not depend on any specific model, the success of the implementation of the procedure relies on the availability of the joint characteristic function of Δ_k and I in analytic form. Thanks to the affine structure of the Heston stochastic volatility model with jumps, we are able to express $\Phi_{\Delta_k,I}(\alpha,\beta)$ in an exponential affine form (details shown below).

In the Heston stochastic volatility model with jumps in asset price, the joint dynamics of the asset price S_t and the instantaneous variance V_t are specified by

$$
\begin{aligned}
\frac{dS_t}{S_t} &= (r - q)\,dt + \sqrt{V_t}\left(\rho\,dW_t^2 + \sqrt{1-\rho^2}\,dW_t^1\right) + \left(e^J - 1\right)dP_t, \\
dV_t &= \kappa(\theta - V_t)\,dt + \varepsilon\sqrt{V_t}\,dW_t^2,
\end{aligned}
\tag{3.10}
$$

where W_t^1 and W_t^2 are two independent Brownian motions, P_t is a Poisson process with intensity λ, the jump size J is assumed to have a normal distribution with mean ν and variance δ^2, ρ is the correlation coefficient, r and q are the constant riskfree rate and dividend yield, respectively. The Heston model exhibits the affine structure under which the characteristic function of the

triplet (X_t, I_t, V_t) has an exponential affine form (Kallsen *et al.*, 2011). The joint characteristic function of the triplet is given by

$$\mathbb{E}_t[e^{uX_T + wI_T + bV_T + c}] = \exp(uX_t + wI_t + B(\tau, \mathbf{q})V_t + D(\tau, \mathbf{q})), \tag{3.11}$$

where the parameter functions B and D are determined by solving a Riccati system of ordinary differential equations, the details of which can be found in Appendix A. Here, $\mathbf{q} = (u, w, b, c)^T$ denotes the initial values of the transform variables. It then follows that

$$
\begin{aligned}
\Phi_{\Delta_k, I}(\alpha, \beta) &= \mathbb{E}\left[\mathbb{E}_{t_k}[e^{i\beta I_T}]e^{i\alpha\Delta_k}\right] \\
&= \mathbb{E}\left[e^{i\alpha\Delta_k + i\beta I_{t_k} + B(T - t_k, \mathbf{q}_1)V_{t_k} + D(T - t_k, \mathbf{q}_1)}\right] \\
&= \mathbb{E}\left[e^{i\beta I_{t_{k-1}} + B(\Delta t_k, \mathbf{q}_2)V_{t_{k-1}} + D(\Delta t_k, \mathbf{q}_2)}\right] \\
&= e^{B(t_{k-1}, \mathbf{q}_3)V_0 + D(\Delta t_k, \mathbf{q}_3)}, \tag{3.12}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{q}_1 &= (0, 0, i\beta, 0)^T, \\
\mathbf{q}_2 &= (i\alpha, B(T - t_k, \mathbf{q}_1), i\beta, D(T - t_k, \mathbf{q}_1))^T, \\
\mathbf{q}_3 &= (0, B(\Delta t_k, \mathbf{q}_2), i\beta, D(\Delta t_k, \mathbf{q}_2))^T.
\end{aligned}
$$

4   Numerical calculations

In this section, we present the numerical calculations that were performed to examine accuracy of the proposed partially exact and bounded approximations. Though the PEB method can be applied for pricing options on DRV under a general model assumption, it is particularly effective for the affine stochastic volatility models with jumps since they admit closed form characteristic functions. For illustrative purposes, our numerical examples are confined to the Heston stochastic volatility model with compound Poisson jumps [see eq. (3.10)].

The model parameter values for the Heston model with jumps in our numerical calculations are adopted from Duffie *et al.* (2000) and they are shown in Table 1. Furthermore, we choose $r = 0.0319$, $q = 0$ and $S_0 = 1$.

κ	3.46	ν	-0.086
θ	$(0.0894)^2$	λ	0.47
ε	0.14	δ	0.0001
ρ	-0.82	$\sqrt{V_0}$	0.087

Table 1: The basic set of parameter values of the Heston model with jumps in asset price.

Monte Carlo simulation

Since there is no exact pricing formulas for options on DRV under the Heston model with jumps, we use the numerical results from Monte Carlo simulation for the benchmark comparison. The most straightforward approach to implement the simulation is the use of the first-order Euler scheme to simulate the joint dynamics of the underlying price process and the instantaneous variance process. However, it is well known that the Euler discretization scheme of the instantaneous variance process may possibly generate negative values and an improper handling of the negative values may lead to severely biased results. This effect becomes particularly noticeable since the price of a variance option is typically quite small in magnitude. To reduce the bias and obtain reliable benchmark results, we adopt the following modified Euler scheme proposed by Lord *et al.* (2010):

$$
\begin{aligned}
\ln S_{t+\Delta t} &= \ln S_t + \left(r - q - \frac{V_t^+}{2}\right)\Delta t + \sqrt{V_t^+ \Delta t}\Big(\rho Z_2 + \sqrt{1-\rho^2}\,Z_1\Big) + \sum_{i=1}^{N_{\Delta t}} J_i \\
V_{t+\Delta t} &= V_t + \kappa \Delta t(\theta - V_t^+) + \varepsilon\sqrt{V_t^+ \Delta t}\,Z_2,
\end{aligned}
\tag{4.1}
$$

where $V_t^+ = \max(V_t, 0)$, Z_2 and Z_1 are two independent standard normal random variables, and J_i are independent copies of the random jump size. For convergence analysis of the above simulation scheme, we refer the interested readers to Lord *et al.* (2010). To hasten the rate of convergence of the simulation, we use the DRV as a control variate. The details of this technique can be found in Broadie and Jain (2008).

Analysis of numerical accuracy

We present the numerical results for testing accuracy of the lower bound approximation and the partially exact and bounded approximation. We calculate the prices of the call options on daily sampled realized variance with varying sampling periods and strike prices. We made three choices of maturities, $N = 20$, $N = 126$ and $N = 252$. They represent one month (short), half a year (intermediate) and a year (long), respectively. For each maturity, we choose three representative strike prices that correspond to deep in-the-money (ITM), at-the-money (ATM) and deep out-of-the-money (OTM) call options. We also list the prices of the call options on the CRV, which can be regarded as a crude approximation to the prices of the discrete counterparts. The benchmark Monte Carlo simulation results are generated by simulating 8×10^5 paths with step size $\Delta t = \frac{1}{252} \times \frac{1}{16}$ according to the scheme (4.1) with the aid of the control variate technique.

Maturity	Strike	Cont (RE)	LB (RE)	PEBn (RE)	PEBg (RE)	MC (SE)
N = 20	7.049	2.938(10.4%)	2.956(9.84%)	3.423(4.41%)	3.309(0.93%)	3.278(0.002)
	8.812	2.685(7.01%)	2.703(6.39%)	2.879(0.27%)	2.908(0.74%)	2.887(0.002)
	10.574	2.595(3.23%)	2.595(3.23%)	2.624(2.16%)	2.679(0.11%)	2.682(0.002)
N = 126	45.087	18.817(1.25%)	18.773(1.48%)	19.041(0.08%)	19.033(0.12%)	19.055(0.008)
	56.358	14.721(1.29%)	14.698(1.45%)	14.903(0.07%)	14.898(0.10%)	14.914(0.010)
	67.630	11.696(0.88%)	11.671(1.10%)	11.788(0.10%)	11.791(0.09%)	11.801(0.010)
N = 252	90.836	34.210(0.62%)	34.160(0.76%)	34.382(0.12%)	34.379(0.13%)	34.423(0.013)
	113.545	23.131(0.89%)	23.088(1.07%)	23.328(0.04%)	23.341(0.01%)	23.338(0.017)
	136.254	14.652(2.28%)	14.642(2.35%)	15.077(0.55%)	15.059(0.43%)	14.994(0.019)

Table 2: All the option prices are interpreted as basis points. That is, the calculated results have been multiplied by 10^4. "Cont" refers to the prices of the call options on the CRV, "LB" means the lower bound approximation given by eq. (2.5), "PEBn" means the PEB approximation with normal distribution, "PEBg" means the PEB approximation with gamma distribution, and "MC" refers to the Monte Carlo simulation results using the Euler scheme eq. (4.1). The numbers in brackets after numerical option prices represent the relative errors (RE) with the Monte Carlo simulation results as the benchmark for comparison. The numbers in brackets after the Monte Carlo simulation values represent the standard error (SE) in the Monte Carlo simulation calculations.

The numerical results in Table 2 reveal that the performance of the lower bound approximation is quite similar to the crude approximation using the price of the call option on the CRV. For short-maturity options, though numerical accuracy is not quite satisfactory in general, the lower bound approximation slightly outperforms the "Cont" approximation. Both the PEB approximation methods with the normal or gamma distribution approximation have shown significant improvement over the lower bound approximation. However, the PEB method with the normal distribution approximation fails to deliver a consistent accurate approximation for the one-month call options. On the other hand, the PEB method with the gamma distribution approximation provides very accurate results for the short-maturity options. This is expected since the gamma distribution approximation is exact in the asymptotic limit when $T \to 0$. The gamma distribution approximation remains to perform equally well for relatively long maturities, which supports our theoretic result that the gamma distribution approximation converges to the normal one when N is sufficiently large. The numerical experiment once again confirms the significant discrepancy between the DRV and CRV when the time to maturity is small. The two PEB approximation methods, especially the gamma distribution approximation, prove to be an efficient and accurate analytic approximation method for pricing options on DRV under all ranges of maturities.

Figure 2 shows the percentage error in numerical pricing of options on DRV of the lower

bound (LB) and PEB approximation methods for varying moneyness and maturities. The volatility of variance is set to be a relatively large value of 0.9. For short-maturity options ($N = 20$), the normal and gamma distribution approximations are seen to exhibit comparable performance, while the LB approximation remains to be inferior. When the maturity of the option is lengthened to be half a year ($N = 126$), the percentage errors in all three approximations are within 1%. In general, we find that it is reliable to use the gamma distribution approximation for short-maturity options and the normal distribution approximation for long-maturity options.
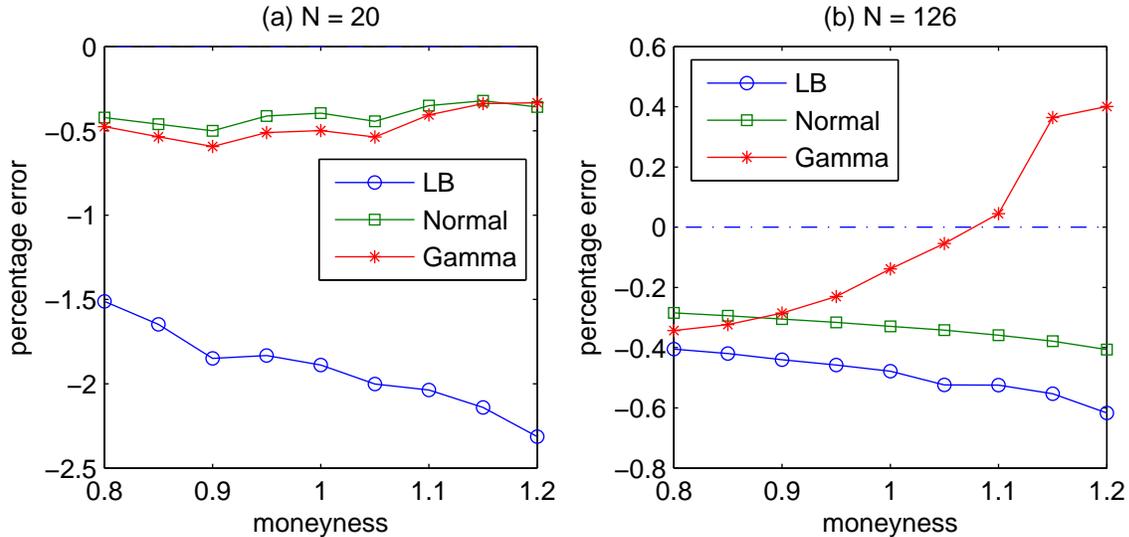

Figure 2: Plot of percentage error in numerical pricing of short-dated ($N = 20$) and long-dated ($N = 126$) call options on daily sampled realized variance of the three approximation methods against moneyness. The volatility of variance parameter ε is set to be 0.9.

5 Conclusion

The conditioning variable approach with PEB approximation is known to be an effective analytic approximation method for pricing path dependent options. We propose a significant extension of the PEB approximation for pricing options on DRV. Our numerical tests demonstrate that the PEB approximation formulas provide very good performance for pricing options on DRV under the Heston stochastic volatility model with jumps, without the shortcoming exhibited in other analytic approximation methods where accuracy may deteriorate substantially in pricing options with short maturities. The high level of numerical accuracy is attributed to the adoption of either the normal or gamma approximation of the distribution of DRV

conditional on the CRV. Thanks to the affine structure of the Heston model with jumps, the PEB approximation is seen to be particularly effective for pricing options on DRV under the Heston model with jumps. Since the gamma distribution approximation is exact in asymptotic limit as maturity tends to zero, the PEB method with the gamma distribution approximation is more reliable when pricing short-maturity options on DRV. For options with medium to long maturities, the gamma distribution is closely connected to the normal distribution. The approximation using either distribution is seen to be highly reliable and provide numerical accuracy within 1% error for most reasonable ranges of model parameter values.

Appendix A   Derivation of parameter functions B, D in eq. (3.11)

Let $U(X_t, I_t, V_t, \tau) = \mathbb{E}_t[e^{uX_T+wI_T+bV_T+c}]$, where $\tau = T - t$. It is seen that U satisfies the following Kolmogorov backward equation:

$$
\frac{\partial U}{\partial \tau} = \left(r - q - m\lambda - \frac{V}{2}\right)\frac{\partial U}{\partial X} + \kappa(\theta - V)\frac{\partial U}{\partial V} + \frac{V}{2}\frac{\partial^2 U}{\partial X^2} + \frac{\varepsilon^2 V}{2}\frac{\partial^2 U}{\partial V^2} + V\frac{\partial U}{\partial I}
$$
$$
+ \rho\varepsilon V\frac{\partial^2 U}{\partial X \partial V} + \lambda\mathbb{E}\big[U(X + J, V, I + J^2, \tau) - U(X, V, I, \tau)\big]. \tag{A.1}
$$

By substituting the solution form: $U(X_t, I_t, V_t, \tau) = e^{uX_t+wI_t+B(\tau,\mathbf{q})V_T+D(\tau,\mathbf{q})}$ into eq. (A.1), we obtain the following Riccati system of ordinary differential equations (ODEs):

$$
\frac{\partial B}{\partial \tau} = -\frac{1}{2}(u - u^2) - (\kappa - \rho\varepsilon u)B + \frac{\varepsilon^2}{2}B^2 + w
$$
$$
\frac{\partial D}{\partial \tau} = (r - q)u + \kappa\theta B + \lambda\big\{\mathbb{E}[\exp(uJ + wJ^2)] - 1] - mu\big\}. \tag{A.2}
$$

Using a similar technique as in Zheng and Kwok (2014a), we obtain the solution to the above system of ODEs as follows:

$$
B(\tau, \mathbf{q}) = \frac{b(\xi_- e^{-\zeta\tau} + \xi_+) - (u - u^2 - 2w)(1 - e^{-\zeta\tau})}{(\xi_+ + \varepsilon^2 b)e^{-\zeta\tau} + \xi_- - \varepsilon^2 b},
$$
$$
D(\tau, \mathbf{q}) = (r - q)u\tau + c - \frac{\kappa\theta}{\varepsilon^2}\left[\xi_+\tau + 2\ln\frac{(\xi_+ + \varepsilon^2 b)e^{-\zeta\tau} + \xi_- - \varepsilon^2 b}{2\zeta}\right]
$$
$$
- \lambda(mu + 1)\tau + \frac{\lambda\tau}{\sqrt{1 - 2\delta^2 w}}\exp\left(\frac{\delta^2 u^2 + 2\nu(u + \nu w)}{2(1 - 2\delta^2 w)}\right),
$$

where

$$
\zeta = \sqrt{(\kappa - \rho\varepsilon u)^2 + \varepsilon^2(u - u^2 - 2w)},
$$
$$
\xi_\pm = \zeta \mp (\kappa - \rho\varepsilon u).
$$

Here, the analytic formula of D is valid provided that $\mathfrak{R}(w) < \frac{1}{2\delta^2}$.

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