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Early Exercise Policies of American Floating Strike and Fixed Strike Lookback Options

Hong Yu

Department of Information Systems, School of Computing, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260

Yue Kuen Kwok

Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong

Lixin Wu

Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong

Abstract

Using appropriate similarity transform, we present the partial differential equation formulation of both floating strike and fixed strike American lookback option models. We examine the early exercise policies of the floating strike and fixed strike American lookback options, while the realized extrmum of the asset price can be monitored continuously or discretely. The characterizations of the optimal exercise prices of American lookback options are also discussed. For the numerical valuation of the American lookback options, several approaches for deriving efficient and accurate numerical results are addressed.

*Key words:*American lookback options, early exercise policies, floating strike and fixed strike, numerical algorithms.

1 Introduction

Lookback options are path dependent options whose payoff depends on the maximum or the minimum of the underlying asset price attained over a certain period of time (called the *lookback period*). Lookback options can be broadly classified into two types: *fixed strike* and *floating strike*. Let $m_{T_0}^T (M_{T_0}^T)$ denote the realized minimum (maximum) asset price over the lookback period $[T_0, T]$. A floating strike lookback call gives the holder the right to buy at the lowest realized price while a floating strike lookback put allows the holder to sell at the highest realized price over the lookback period. Their terminal payoffs are given by $S_T - m_{T_0}^T$ and $M_{T_0}^T - S_T$, respectively, where S_T is the terminal asset price. Floating strike lookback options are in a sense not options since

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they are always exercised at expiration. A fixed strike lookback call (put) is a call (put) option on the maximum (minimum) realized price. The terminal payoffs for the fixed strike lookback call and put are $\max(M_{T_0}^T - X, 0)$ and $\max(X - m_{T_0}^T, 0)$, respectively, where X is the strike price. Lookback options guarantee "no-regrets" outcomes for the holders. However, one expects that high premiums are charged for these lookback options, and this high cost nature somewhat limits their wide spread usage.

The earlier works on the derivation of pricing formulas for lookback options were reported by Goldman *et al.* [6], Conze and Viswanathan [4] and Garman [5]. They followed the discounted expectation approach in their derivation procedures and their results are limited to European-style lookback options. For American-style lookback options, Conze and Viswanathan [4] obtained upper bounds on the option values using the technique of Snell envelopes. However, Barraquand and Pudet [2] showed from their numerical experiments that the upper bounds on the American lookback option prices obtained from Snell envelopes are quite loose.

Most of the pricing formulas derived for lookback option models are based on the continuous monitoring for the extremum of the asset price process. However, most financial contracts in real markets are settled by reference to discrete monitoring of the price process at regular fixings. It is a well known fact that the discrete monitoring feature has profound effect on the prices of lookback options. Semi-closed form analytical formulas for discretely monitored European lookback options were obtained by Heynen and Kat [7]. Cheuk and Vorst [3] proposed an one-state variable version of the binomial scheme for numerical valuation of both continuously and discretely monitored floating strike lookback options. AitSahlia and Lai [1] applied the duality property of random walk to develop a numerical method for the valuation of discretely monitored European lookback options. Using an ingenious choice of similarity variables, we illustrate that the dimensionality of the governing differential equation for the floating strike American lookback option models can be reduced. However, similar success of dimension reduction cannot be achieved for the fixed strike American lookback options.

In this paper, we would like to examine the early exercise policies of American lookback options. We formulate the pricing models of American lookback options using the partial differential equation formulation, and then devise numerical algorithms to solve the pricing models. This paper is organized as follows. In the next section, we present the partial differential equation formulation of both floating strike and fixed strike American lookback option models. In Section 3, we discuss the different numerical algorithms for pricing floating strike and fixed strike American lookback options. In Section 4, we present the numerical results on the critical asset price of various types of American lookback options. The characterization of the optimal exercise policies of the American lookback options is discussed. The paper is ended with conclusive remarks in the last section.

2 Partial differential equation formulation

Let T_0 and T denote the starting and expiration dates of a lookback option. We assume that the lookback period is taken to be the whole life of the option so that $[T_0, T]$ is the lookback period. Let t denote the current time. We denote the realized minimum and maximum asset prices from T_0 to t $(T_0 \leq t \leq T)$ by

$$m_{T_0}^t = \min_{T_0 \leq u \leq t} S_u \quad \text{and} \quad M_{T_0}^t = \max_{T_0 \leq u \leq t} S_u, \tag{1}$$

respectively. The above formulation refers to the continuous monitoring of the asset price process S_u. We take the usual Black-Scholes assumptions in our lookback option models. In the risk neutral world, the underlying asset price is assumed to follow the lognormal diffusion process

$$\frac{dS}{S} = (r - q)dt + \sigma dZ, \tag{2}$$

where r, q and σ are the constant riskless interest rate, continuous dividend yield and volatility, respectively. We consider the differential equation formulation of floating strike and fixed strike American lookback options as follows:

American floating strike lookback options

First, we let $P(S, M_{T_0}^t, t)$ denote the price of an American floating strike lookback put option. It can be shown that the governing equation for the lookback put value is given by [9]

$$\frac{\partial P}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 P}{\partial S^2} + (r - q) S \frac{\partial P}{\partial S} - rP = 0, \quad S^* < S < M_{T_0}^t, T_0 < t < T, \tag{3}$$

where $S^* = S^*(t; M_{T_0}^t)$ is the critical asset price at which the American lookback put option would be optimally exercised. The variable $M_{T_0}^t$ does not appear in the differential equation, though it does appear as a parameter in the auxiliary conditions. The final condition in the pricing model is the terminal payoff function, namely,

$$P(S_T, M_{T_0}^T, T) = M_{T_0}^T - S_T. \tag{4a}$$

Using the fact that the lookback put price is insensitive to the current realized maxima when $S = M^t_{T_0}$, the boundary condition at the right end is given by

$$\frac{\partial P}{\partial M^t_{T_0}} = 0 \quad \text{at} \quad S = M^t_{T_0}. \tag{4b}$$

Since the American lookback put is exercised at $S = S^*$, the corresponding left-end boundary condition is given by

$$P(S^*, M^t_{T_0}, t) = M^t_{T_0} - S^* \tag{4c}$$

From the prescription of the auxiliary conditions, it is natural to use the asset price as the numeraire to achieve the reduction of dimensionality of the pricing model. Suppose we choose the following similarity variables:

$$x = \ln \frac{M^t_{T_0}}{S}, \tag{5a}$$

and

$$V(x, \tau) = P(S, M^t_{T_0}, t)/S, \qquad \tau = T - t, \tag{5b}$$

the corresponding governing equation becomes

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \mu \frac{\partial V}{\partial x} - qV, \quad 0 < x < x^*, 0 < \tau < T, \tag{6}$$

where $\mu = q - r - \frac{\sigma^2}{2}$ and $x^* = \ln \frac{M^t_{T_0}}{S^*}$. Since the asset price is used as the numeraire, we expect that the discount rate for V should be equal to q, while the drift rate for x becomes $q - r - \frac{\sigma^2}{2}$. The auxiliary conditions then become

$$V(x, 0) = e^x - 1, \quad \frac{\partial V}{\partial x}(0, \tau) = 0 \quad \text{and} \quad V(x^*, \tau) = e^{x^*} - 1. \tag{7}$$

The above differential equation formulation of the American lookback put option resembles that of an American vanilla put option, except that the boundary condition at $x = 0$ is of Neumann type. While it is a relatively easy task to express the early exercise premium in an integral form for American vanilla options [8], unfortunately, this change of the nature of boundary condition causes the derivation of the integral representation too cumbersome for American lookback options. The evaluation of the early exercise premium based on the corresponding integral representation would become too tedious, so that such approach has little practical significance.

American fixed strike lookback options

The success of the above similarity transformation of variables which reduces the dimensionality of the floating strike lookback option models relies on the non-occurrence of any quantity which is in terms of actual dollars. However, this situation does not come about in American fixed strike lookback option models where the realized maxima $M^t_{T_0}$ or realized minima $m^t_{T_0}$ and the strike price X both occur in the exercised payoff function.

We consider an American fixed strike lookback put option where the terminal payoff is given by $\max(X - m^T_{T_0}, 0)$. Now, the appropriate similarity variables are

$$x = \ln \frac{S}{m^t_{T_0}} \qquad \text{and} \qquad y = \ln \frac{X}{m^t_{T_0}}. \tag{8}$$

The governing equation for the American put value $P(x, y, \tau)$ is given by

$$\frac{\partial P}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial x^2} + (r - q - \frac{\sigma^2}{2})\frac{\partial P}{\partial x} - rP, \quad 0 < x < x^*, 0 < y < \infty, 0 < \tau < T, \tag{9}$$

where $x^* = \ln \frac{S^*}{m^t_{T_0}}$. When $m^t_{T_0} \geq X$, the American fixed strike lookback put option will never be exercised since the exercised payoff becomes zero. In this case, it reduces to its European counterpart. The early exercise premium associated with the fixed strike put exits only when $m^t_{T_0} < X$. Accordingly, the domain of definition of the fixed strike put option model is only restricted to $0 < y < \infty$. The corresponding auxiliary conditions become

$$P(x, y, 0) = X \max(1 - e^{-y}, 0), \tag{10a}$$

$$\frac{\partial P}{\partial x}(0, y, \tau) + \frac{\partial P}{\partial y}(0, y, \tau) = 0, \tag{10b}$$

$$P(x^*, y, \tau) = X(1 - e^{-y}). \tag{10c}$$

Note that the independent variable y only appears in the auxiliary conditions but not in the differential equation.

Discrete monitoring feature

The above formulation assumes continuous monitoring of the extremum of the asset price process. In reality, the extremum of the price can only be monitored discretely at predetermined fixing dates. Between two successive fixing dates, the extremum price will not be recorded even a new extremum occurs, that is, the extremum price stays at the value recorded at the last monitoring date prior to the current time.

We consider a discretely monitored American floating strike lookback call option. Let L be the total number of monitoring dates, and m_ℓ denote the recorded minima at the ℓ^{th} monitoring date. When the current time does not fall on one of the monitoring dates, the American floating strike lookback call option behaves like an American vanilla call option. The strike price is a known quantity, which is the recorded minima on the last monitoring date prior to the current time. Hence, the governing equation of the American call price between consecutive monitoring dates is

$$\frac{\partial C}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2}+(r-q)S\frac{\partial C}{\partial S}-rC=0, \quad 0<S<S^*, t_\ell<t<t_{\ell+1}, \tag{11}$$

where $S^* = S^*(t; m_\ell)$ is the critical asset price. On the other hand, when the current time happens to be on a monitoring date, accordingly an updated asset price minima is recorded. The above governing equation remains valid except that the interval of definition of S is changed, that is,

$$\frac{\partial C}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2}+(r-q)S\frac{\partial C}{\partial S}-rC=0, \quad m_\ell<S<S^*, t=t_\ell, l=1,2,\cdots,L. \tag{12}$$

The left-end Neumann boundary condition

$$\frac{\partial C}{\partial m_\ell}=0 \quad \text{at} \quad S=m_\ell \tag{13}$$

is applied only on the monitoring dates. Over the time interval $[t_\ell, t_{\ell+1})$, we adopt the following set of similarity variables:

$$x=\ln\frac{S}{m_\ell}, \quad \text{and} \quad V=C(S,m_\ell,t)/S. \tag{14}$$

For American lookback option models with discrete monitoring feature, the governing equation takes the same form as that of the continuous monitoring counterparts. However, the interval of definition for x changes accordingly to whether the current time is on a monitoring date or not. When the current time is not on a monitoring date, the interval of definition is $-\infty < x < x^*$, where $x^* = \ln\frac{S^*}{m_\ell}$. However when the current time is on a monitoring date, the interval becomes $0 < x < x^*$; and the corresponding left-end boundary condition is given by

$$\frac{\partial V}{\partial x}(0,\tau_\ell)=0, \quad \tau_\ell=T-t_\ell. \tag{15}$$

3 Construction of the numerical algorithms

The differential equation formulation of a continuously monitored floating strike American lookback option resembles closely to that of the American vanilla option, except that the near field boundary condition becomes the

Neumann type. For the numerical solution of the pricing model, one can then apply the standard finite difference scheme incorporated with the dynamic programming procedure of taking the maximum of the continuation value and exercised payoff at each lattice node.

For the discretely monitored floating strike American lookback option, we note that the interval of definition of x equals $(-\infty, \infty)$ when the current time is not falling on one of the monitoring dates, but becomes $[0, \infty)$ otherwise. However, the governing equation of the lookback option remains the same in both cases. For convenience, we extend the interval of definition of the option model from $[0, \infty)$ to $(-\infty, \infty)$ when the current time hits a monitoring date. The option value in the extended interval $(-\infty, 0)$ is given by

$$V(x, t_l^-) = V(0, t_l), \quad x < 0. \tag{16}$$

Here, t_l is one of the monitoring dates and t_l^- is the time just right before t_l. The following argument is used to justify the above claim. When the asset price at time t_l^- falls below the recorded minimum at the earlier monitoring date t_{l-1}, this asset value will become the new minimum on the monitoring date t_l. The corresponding value of x exactly on the monitoring date t_l then becomes 0.

The finite difference calculations for the discretely monitored American floating strike lookback put option resemble those for the American vanilla counterpart, except that some modifications are required at those time levels that correspond to the monitoring dates. The interval of the computation domain now extends to $[-J\Delta x, J\Delta x]$ at all time levels for some sufficiently large J. At those time levels corresponding to monitoring dates, the option values at $j = -1, -2, \cdots, -J$ are all set to have the same option value at $j = 0$, rather than finding the option values using the finite difference scheme. One should be cautioned that the similarity variable x in the floating strike lookback call option is $x = \ln \frac{S}{m_\ell}$, and m_ℓ changes only when the time marching moves across a monitoring date.

It is interesting to note that the governing equation for the fixed strike American lookback put option involves only the independent variable x (see Eq. (9)), while the auxiliary conditions involve only the independent variable y (see Eqs. (10a-c)). The domain of the differential equation formulation is the quarter plane. Since the finite difference solution must be solved within a finite rectangular domain, we need to prescribe some artificial numerical bounday conditions along the three sides of the numerical solution domain $\{(x, y) : 0 \leq x \leq L_x, 0 < y \leq L_y\}$:

(i) Along the side $x = L_x, 0 < y \leq L_y$, the American fixed strike lookback put is sure to be exercised so that

$$P(L_x, y, \tau) = X(1 - e^{-y}), \qquad 0 < y \leq L_y. \tag{17}$$

(ii) Along the side $y = L_y, 0 \leq x \leq L_x$, the realised minimum $m_{T_0}^t$ assumes infinitesimal values. The American lookback put is deep-in-the-money, so it will be exercised optimally. The asymptotic boundary values are then given by

$$P(x, L_y, \tau) = X, \qquad 0 \leq x \leq L_x. \tag{18}$$

(iii) Along the side $y = 0, 0 \leq x \leq L_x$, the option holder will never choose to exercise the option. The American lookback put becomes an European lookback put option, whose analytical price formula is available [4].

4 Characterization of the early exercise policies

In the semi-infinite domain $\{(x, \tau) : x \geq 0, 0 \leq \tau \leq T\}$ of the American floating strike lookback put option model, the American put is alive when $x < x^*$ and becomes dead when $x \geq x^*$. The boundary which divides the continuation region (option remains alive) and the stopping region (option becomes dead) is time dependent, that is, x^* is a function of t. The above observation leads to the conclusion that

$$\frac{S^*(t; M_{T_0}^t)}{M_{T_0}^t} = F(t), \tag{19}$$

for some function $F(t)$. Similar to the usual arguments for American vanilla put options that the critical exercise price should be a monotonically increasing function of time, we also expect $F(t)$ to be a monotonically increasing function of time. As deduced from Eq. (6), we observe that at a given time t, the critical exercise price $S^*(t; M_{T_0}^t)$ increases linearly with $M_{T_0}^t$. Since both $F(t)$ and $M_{T_0}^t$ are increasing functions of time, and so $S^*(t; M_{T_0}^t)$ increases as time is approaching expiration.

Similarly, the critical asset price for the American floating strike lookback call option observes the relation:

$$\frac{S^*(t; m_{T_0}^t)}{m_{T_0}^t} = G(t), \tag{20}$$

where $G(t)$ is a monotonically decreasing function of time. The plots of $G(t)$ against time t with varying interest rate are shown in Figure 1. Since lower value of interest rate causes the loss of the time value of the strike price to be smaller when the American call option is exercised prematurely, so the critical asset price decreases when the interest rate assumes lower value.

Limiting behaviors of the critical asset prices

The limiting behaviors at times close to maturity of the critical asset prices for the American floating strike lookback put and call options are respectively given by

$$\lim_{t\to T^-} S^*(t; M_{T_0}^t) = \min\left(\frac{r}{q}, 1\right) \lim_{t\to T^-} M_{T_0}^t \tag{21}$$

and

$$\lim_{t\to T^-} S^*(t; m_{T_0}^t) = \max\left(\frac{r}{q}, 1\right) \lim_{t\to T^-} m_{T_0}^t. \tag{22}$$

Since the strike prices of the American lookback put and call options are set to be $M_{T_0}^T$ and $m_{T_0}^T$, respectively, the usual argument of analysis of limiting behaviors can be applied in a similar manner like that for ordinary American vanilla options [8], and thus leads to the above results. The asymptotic result stated in Eq. (22) is clearly revealed by the plots in Figure 1.

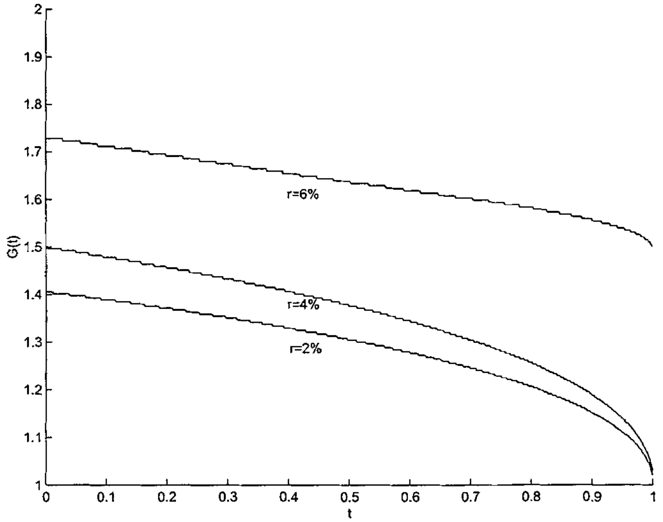

Figure 1 Plot of $G(t)$ against time t for a continuously monitored American floating strike lookback call option with varying interest rate. The parameter values of the option model are: $\sigma = 20\%, q = 4\%, m_{T_0}^t = 75$ and r is taken to be 2%, 4% and 6%, successively.

Discrete monitoring

The formulation of the discrete monitoring model is almost identical to that of the continuous counterpart, except that the Neumann boundary condition is applied only on the monitoring dates (see Eq. (13)). Therefore, the same argument for the analysis of the critical asset prices in the continuous monitoring model can be applied, in particular, the relations between the critical asset prices and historical maximum (minimum) values as depicted in Eqs. (19-22) can be obtained in a similar manner. However, one should be aware that the similarity variable used in the discretely monitored models is modified whenever a monitoring instant is crossed, say, from $x = \ln \frac{S}{m_{\ell-1}}$ to $x = \ln \frac{S}{m_\ell}$ as time t passes through the value t_ℓ. Therefore, though the time dependent function, $F(t)$ or $G(t)$, remains continuous, the critical asset price itself, $S^*(t; M_\ell)$ or $S^*(t; m_\ell)$, exhibits discrete jumps across the monitoring instants. The plots of $G(t)$ against t for discretely monitored American floating strike lookback call options with different monitoring frequencies are shown in Figure 2. The convergence of $G(t)$ of the discretely monitored option model at high monitoring frequency to that of the continuously monitored counterpart is illustrated in the figure.

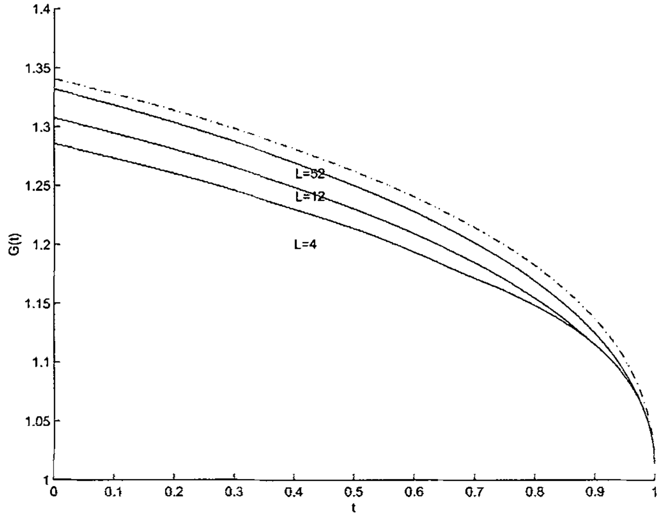

Figure 2 Plot of $G(t)$ against time t for a discretely monitored American floating strike lookback call option with varying monitoring frequencies, $L = 4, 12, 52$. The dotted curve corresponds to the plot of $G(t)$ for the continuously monitored counterpart. The parameter values of the option model are: $\sigma = 20\%, q = 6\%, r = 2\%$ and $T = 1$.

The functional dependence of $S^*(t; m_{T_0}^t)$ of an American fixed strike lookback put option is more complicated compared to the floating strike counterpart. Now, the normalized critical asset price $\frac{S^*(t;m_{T_0}^t)}{m_{T_0}^t}$ is no longer a function of time only. The three-dimensional plot of $\frac{S^*(t;m_{T_0}^t)}{m_{T_0}^t}$ against varying values of

$m^t_{T_0}$ and t of an American fixed strike lookback put is shown in Figure 3. The American fixed strike put is never optimally exercised when $m^t_{T_0} \geq X$ since the exercised payoff becomes negative. The plot agrees with the financial intuition that $\frac{S^*(t;m^t_{T_0})}{m^t_{T_0}}$ stays slightly above one at low values of $m^t_{T_0}$, then it increases with increasing value of $m^t_{T_0}$ and tends to infinity as $m^t_{T_0}$ tends to X from below. Also, for a fixed value of $m^t_{T_0}$, $\frac{S^*(t;m^t_{T_0})}{m^t_{T_0}}$ is a decreasing function of time. The normalized critical asset price always attains the value one at maturity, independent of the ratio r/q. This is expected since the exercised payoff involves actual cash settlement, not cash for asset. This is distinctive from the limiting behavior at times close to maturity of the critical asset price for the floating strike counterparts.

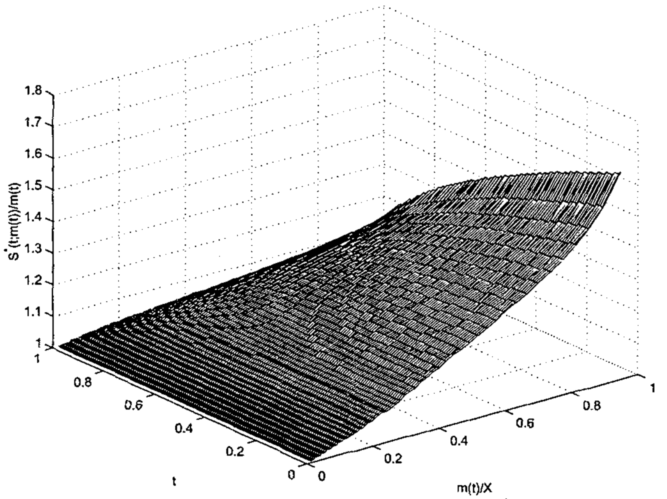

Figure 3 Plot of the normalized critical asset price $\frac{S^*(t;m^t_{T_0})}{m^t_{T_0}}$ against varying values of $m^t_{T_0}$ and t for an American fixed strike lookback put option. The parameter values of the option model are: $X = 100$, $\sigma = 20\%, r = 6\%, q = 3\%$ and $T = 1$.

5 Conclusions

In this paper, the early exercise policies of the floating strike and fixed strike American lookback options are examined. The monitoring process for the extremum of the asset prices can be done either continuously or at predetermined discrete instants. For floating strike American lookback options, since the payoff functions do not involve actual dollar numeration, it becomes natural to use the asset price as the numeraire in the pricing models. The critical asset price S^* at which a floating strike American lookback option should be optimally exercised is seen to be the product of the recorded realized extremum at the current time and a time dependent function. The recorded realized extremum of the asset price at the current time refers to either the realized extremum at the last monitoring date prior to the current time for discrete monitoring or the current realized extremum for continuous monitoring. The properties of the time dependent normalized critical asset price of the floating strike American lookback options resemble those for the critical asset price of the American vanilla counterparts. The convergence of the time dependent normalized critical asset price curves for the discretely monitored models at high monitoring frequencies to the normalized critical asset price for the continuously monitored counterpart is verified through numerical experiments.

For fixed strike American lookback options, since the payoff functions depend on actual dollar numeration, the pricing models involve two stochastic state variables, namely, the asset price process and the process for the extremum of the asset prices. The functional dependence of the critical asset price on the current realized extremum and time becomes more complicated for the fixed strike lookback option models. One important observation is that it is never optimal to exercise when the realized minima (maxima) is above (below) the strike price for a fixed strike American lookback put (call) option.

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