Pricing Algorithms for Options with Exotic Path-Dependence

YUE KUEN KWOK AND KA WO LAU

YUE KUEN KWOK
is an associate professor of mathematics at the Hong Kong University of Science and Technology in Hong Kong.

KA WO LAU
is a Ph.D. student in the Department of Computer Science at the Hong Kong University of Science and Technology.

Pricing algorithms for options with exotic path-dependence using the forward shooting grid approach are characterized by the augmentation of an auxiliary state vector at each grid node on a lattice tree that simulates the discrete asset price process. The state vector is used to capture the specific path-dependent feature of the option contract.

We demonstrate the versatility of the forward shooting grid algorithms by generalizing the approach to price various types of Parisian options, options with reset features, and alpha quantile options. The convergence behaviors of the numerical results obtained by the forward shooting grid algorithms are also examined.

The advantage of the forward shooting grid approach over the finite-difference approach becomes more apparent when the governing differential equation for the option value cannot be expressed in a simple form.

Among the three popular classes of option pricing algorithms, lattice-based methods (binomial and trinomial schemes) continue to enjoy great popularity due to their pedagogical appeal and ease of construction. A variant of the lattice-based method, called the *forward shooting grid* (FSG) method, has been successfully applied to price a wide variety of path-dependent options, like lookback options and Asian options.

The FSG approach is characterized by an auxiliary state vector at each node on the lattice tree. The state vector is used to capture the path-dependent feature of the option contract, like the extreme value of the asset price achieved so far or the average value (geometric or arithmetic) of the asset prices.

In construction of the FSG algorithm, unlike the finite-difference algorithms, it is not necessary to deal with the corresponding governing differential equation for the value of the exotic path-dependent option. For some types of path-dependent options, like the window Parisian option and the alpha quantile option considered here, it is not quite straightforward to explicitly derive the partial differential equation for the option value. In these cases, the FSG approach has an advantage over the finite-difference approach in construction of the option pricing algorithms.

I. FORWARD SHOOTING GRID ALGORITHM

The FSG approach was pioneered by Hull and White [1993] and Ritchken, Sankarasubramanian, and Vijh [1993] for the pricing of American- and European-style Asian and lookback options. A more systematic framework of the FSG method is presented by Barraquand and Pudet [1996]. Forsyth, Vetzal and Zvan [1999] study the convergence of the FSG algorithm in the pricing of Asian options and show that convergence of the numerical solutions depends in an important way on the method of interpolation of the average asset values between the lattice nodes.

Consider an exotic path-dependent option. Let F denote the path-dependent variable associated with the option contract, say, the extreme value or the average value of the asset price S. Let G denote the function that describes the correlated evolution of F with S over the time interval Δt; that is:

$$F_{t+\Delta t} = G(F_t, S_{t+\Delta t}) \tag{1}$$

Let $V[m, j; k]$ denote the numerical option value of the exotic path-dependent option at the m-th time level (m time steps from inception) and j upward jumps from the initial asset value, and k the numbering index for the various possible values of F at the (m, j)–th node on the trinomial tree. The trinomial tree is constructed to have uniform time step Δt and step width Δx, where $x = \ln S$. The probabilities of upward, zero, and downward moves of the asset price are denoted by p_u, p_0, and p_d, respectively. These probability values are given by

$$\begin{aligned} p_u &= \frac{\mu + c}{2} \\ p_d &= \frac{\mu - c}{2} \\ p_0 &= 1 - \mu \end{aligned} \tag{2}$$

where $\mu = \sigma^2 \Delta t/\Delta x^2$ and $c = (r - q - \sigma^2/2(\Delta t/\Delta x)$. Here, σ, q, and r are the volatility, dividend yield, and riskless interest rate, respectively.

Let $g(k, j)$ denote the grid function that is the discrete equivalence of the evolution function G. A trinomial version of the FSG algorithm can be succinctly represented as

$$\begin{aligned} V[m-1, j; k] = \{ & p_u V[m, j+1; g(k, j+1)] + \\ & p_0 V[m, j; g(k, j)] + \\ & p_d V[m, j-1; g(k, j-1)]\} e^{-r\Delta t} \end{aligned} \tag{3}$$

where $e^{-r\Delta t}$ is the discount factor (at the riskless interest rate) over Δt.

The algorithm design for pricing a specific exotic path-dependent option amounts to determination of the grid function $g(k, j)$. In some cases, like arithmetically averaged Asian options, the value of the evolution function G may not fall onto the set of discrete asset values on the trinomial tree. It is then necessary to interpolate the value for G between the grid values that simulate the asset price.

We demonstrate how to choose $g(k, j)$ for various types of Parisian options, options with reset features, and alpha quantile options.

II. PARISIAN OPTIONS

Options with barrier clauses are becoming so popular that they are no longer considered exotic options. They are attractive because buyers do not pay a premium for scenarios they perceive as unlikely. Buyers of barrier options always have to strike a balance between premium reduction (or increment) and the risk of being knocked out (or the chance of being knocked in).

Compared to their plain vanilla counterparts, barrier options provide more flexibility for investors in hedging and speculation operations by allowing them to bet on their views on the future market direction. The one-time breaching of barriers, however, is well known to have the undesirable effect of terminating the option if the price spikes, no matter how briefly the breaching occurs. This is reflected in the discontinuity of the delta and infinite gamma at the barrier.

Hedging becomes difficult for option writers when the asset price is very close to the barrier. This may increase market volatility around popular barrier levels, particularly in the foreign exchange markets, due to manipulation of the underlying asset price to activate knock-out.

To circumvent the spiking effect and short-period price manipulation, several modifications of the one-touch knock-out (or knock-in) provisions have been practiced in the market. In the *Parisian* variant of knock-out, knock-out is activated only when the underlying asset price breaches the barrier for a prespecified period of time. The cumulative Parisian feature counts the cumulative time that the price spends beyond the barrier throughout the whole life of the option. These knock-out provisions based on the duration of breaching may also be applied to the convertibility and callability features in convertible bonds.

As is the practice for the usual barrier features, the breaching is commonly monitored at discrete instants (such as at the end of every trading day) rather than continuously. In the discretely monitored models, the number of breaches at the monitoring instants is counted, rather than the length of the breaching period as in the continuously monitored counterparts.

Several researchers have addressed the pricing of Parisian knock-out options using either the quasi-analytic approach or the numerical approach. For continuously monitored Parisian options, Chesney, Jeanblanc-Picque, and Yor [1997] define the option value in terms of an integral expressed as an inverse Laplace transform using the theory of Brownian excursions. The valuation procedures of the Laplace inversion are outlined in Chesney et al. [1997]. Closed-form analytical price formulas for continuously monitored cumulative Parisian options have been obtained by Hugonnier [1999].

Haber, Schönbucher, and Wilmott [1999] develop a partial differential equation formulation of both consecutive and cumulative continuously monitored Parisian options, and propose finite-difference algorithms for the valuation of Parisian options. Their method performs time discretization along the characteristic associated with the time variable and the excursion time variable.

Using a similar approach, Vetzal and Forsyth [1999] construct a finite-difference scheme by direct discretization of the governing differential equation (the usual Black-Scholes equation with an additional term associated with the excursion time variable). Avellaneda and Wu [1999] develop a modified trinomial scheme to incorporate the Parisian feature using the density function of the first-passage time at which the asset price first reaches the barrier. The programming logic of their method has less pedagogical appeal, and the analytic expressions for the first-passage time density are quite complex.

We describe how numerical schemes are constructed using the FSG approach to price both consecutive and cumulative Parisian options. The formulation is also extended to price Parisian options with the window feature. The performance of these numerical schemes, in particular the convergence behavior and rate of convergence of the numerical option values, is examined. The evolution function G for Parisian options is a new time variable measuring the excursion time in the knock-out region. This is different from lookback options and Asian options, where the corresponding evolution function G in these options assumes values of the asset price.

Cumulative Parisian Options

Let N be the prespecified number of breaches counted throughout the whole life of the option that is required to activate knock-out, and K be the integer variable that counts the number of breaches so far. $V(S, t; N, K)$ denotes the option value of a cumulative Parisian option, S is the asset price, and t is the time. It is obvious that

$$V(S,t;\ n-k,\ 0) = V(S,t;\ n,\ k) \quad \text{for } k < n \tag{4}$$

By adjusting the number of breaches required to activate knock-out, it is seen that it is necessary to consider only algorithms that compute option values corresponding to $K = 0$.

Let $V_{cum}[m, j; k]$ denote the option value of a cumulative Parisian option at the (m, j)-th node on a trinomial tree. Let B denote the down barrier associated with the Parisian option. The augmented path-dependent state variable at each node is the number of breaches at which the asset price S falls on or below B. The index k counts the number of breaches that have occurred so far. The value of k is not changed except at the time steps corresponding to a monitoring instant.

Suppose $m\Delta t \neq t_l^*$, where t_l^* is the l-th monitoring instant for some l. The trinomial calculations proceed as those for the plain vanilla counterparts. Let x_j denote the value of x (recall that $x = \ln S$) corresponds to j upward moves on the trinomial tree. When $m\Delta t$ happens to be a monitoring instant, the index k increases its value by 1 if the asset price S falls on or below the barrier B; that is: $x_j \leq \ln B$.

To accommodate the cumulative Parisian feature, the appropriate choice of the grid evolution function $g_{cum}(k, j)$ should be

$$g_{cum}(k,j) = k + 1_{\{x_j \leq \ln B\}} \tag{5-A}$$

where $1\{x_{j \leq \ln B}\}$ is the indicator function whose value is defined by

$$1_{\{x_j \leq \ln B\}} = \begin{cases} 1 & \text{if } x_j \leq \ln B \\ 0 & \text{if } x_j > \ln B \end{cases} \tag{5-B}$$

Now, the corresponding FSG algorithm for pricing the cumulative Parisian option can be represented by

$$V_{cum}[m-1,j;k] = \begin{cases} \{p_u V_{cum}[m,j+1;k] \\ + p_0 V_{cum}[m,j;k] \\ + p_d V_{cum}[m,j-1;k]\} e^{-r\Delta t} & \text{if } m\Delta t \neq t_l^* \\ \{p_u V_{cum}[m,j+1; g_{cum}(k,j+1)] \\ + p_0 V_{cum}[m,j; g_{cum}(k,j)] \\ + p_d V_{cum}[m,j-1; g_{cum}(k,j-1)]\} e^{-r\Delta t} & \text{if } m\Delta t = t_l^* \end{cases} \tag{6}$$

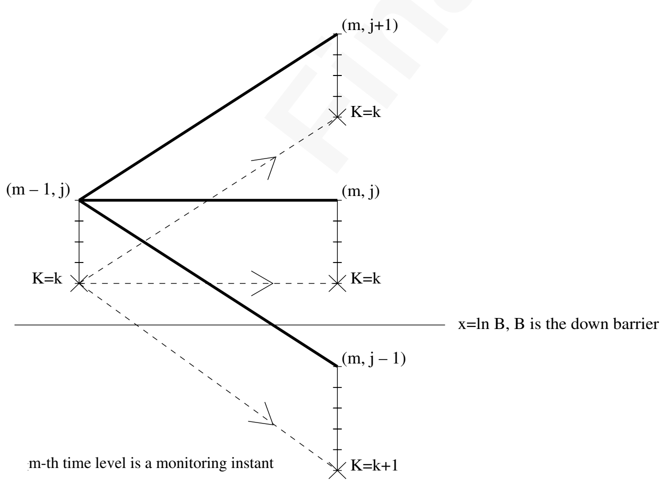
The schematic diagram that illustrates the numerical scheme depicted in Equation (6) is shown in Exhibit 1.

At maturity of the option, the terminal payoff of a Parisian call option is equal to $\max(e^{x_j} - X, 0)$, where X is the strike price, provided that $k < N$. To initiate the calculations, we start at $V_{cum}[m, j; N-1]$, and then $V_{cum}[m, j; N-2]$, and proceed down in the value of index k until k hits 0. We compute $V_{cum}[m, j; N-1]$ by setting $k = N-1$ in Equation (6) and observing that $V_{cum}[m, j; N] = 0$ for all m and j. Indeed, $V_{cum}[m, j; N-1]$ is the option value of the one-touch down-and-out call option at the same node.

Normally, the amount of computational effort necessary to price a cumulative Parisian option requiring N breaches to knock out is about N times that of an one-touch knock-out barrier option.

1. The pricing of continuously monitored Parisian options can be obtained by setting all time steps to be monitoring instants.

EXHIBIT 1
Forward Shooting Grid Algorithm for Pricing a Cumulative Parisian Option



2. The best numerical accuracy can be achieved if the barrier is placed between two horizontal rows of nodes for the discretely monitored barrier options and exactly on a horizontal row of nodes for the continuously monitored counterparts (see Boyle and Tian [1998]). The same rule of thumb is applied here for Parisian options.

Consecutive Parisian Options

The numerical scheme for valuation of consecutive Parisian options can be developed by a slight modification of the grid evolution function $g(k, j)$. Now, let K be the integer variable that counts the number of consecutive breaches at previous monitoring instants on or before the current time. When the current time happens to be a monitoring instant, K is reset to zero whenever the asset price does not stay in the knock-out region, and it increases its value by 1 if the asset price stays in the knock-out region.

Let $V_{con}[m, j; k]$ denote the option value of a consecutive Parisian option at the (m, j)-th node and with k consecutive breaches at previous monitoring instants on or before the current time. The index k will not be changed except at those time levels corresponding to a monitoring instant. When the m–th time level is not one of the monitoring instants, we have the usual trinomial calculations as those for plain vanilla options. When $m\Delta t$ is at a monitoring instant, the index k would increase its value by 1 if $x_j \leq B$ and reset to 0 if $x_j > \ln B$.

The FSG algorithm for pricing the discretely monitored consecutive Parisian option takes exactly the same form as that in Equation (6), except that the grid function has to be modified as

$$g_{con}(k,j) = (k+1)\mathbf{1}_{\{x_j \leq \ln B\}} \tag{7}$$

It is necessary to make available the value of $V_{con}[m, j; 0]$ for $x_j > \ln B$ in order to proceed to the calculations beyond a monitoring instant, since the index k would drop to zero when $x_j > \ln B$. Therefore, the necessary strategy of the numerical procedure is to compute $V_{con}[m, j; k]$ for all index values k, $k = N-1, N-2, ..., 0$, before we move to a time level corresponding to a monitoring instant. At a monitoring instant, it is observed that $V_{con}[m, j; k]$ is not defined for $k > 0$ if $x_j > \ln B$; nor for $k = 0$ if $x_j \leq \ln B$. Similar to the cumulative Parisian counterpart, we have $V_{con}[m, j; N] = 0$ for all m and j.

It takes almost the same amount of computational effort to price the consecutive Parisian option as to price the cumulative counterpart.

Window Parisian Options

We also consider a hybrid variant of the cumulative and consecutive Parisian options, defining a moving window with N_w consecutive monitoring instants on and before the current time. The option will be knocked out if the asset price falls within the knock-out region exactly $N(N \leq N_w)$ times within the window of N_w previous consecutive monitoring instants. If the moving window covers the whole life of the option from inception to the current time, the window Parisian option reduces to the cumulative Parisian option. Or, if N_w is set equal to N, it becomes the consecutive Parisian option.

First, we define a binary string $A = a_1 a_2, \ldots, a_{N_w}$ of size N_w to represent the history of the asset price path falling inside or outside the knock-out region at the previous N_w consecutive monitoring instants prior to the current time. By notation, the value of a_p is set to be 1 if the asset price falls on or below the down barrier B at the p–th monitoring instant counting backward from the current time, and is set to be 0 if otherwise.

There are altogether 2^{N_w} different strings to represent all possible breaching history of asset price paths at the previous N_w monitoring instants. The number of states that have to be recorded is $C_0^{N_w} + C_1^{N_w} + \ldots + C_{N-1}^{N_w}$, where $C_i^{N_w}$ denotes the combination of N_w strings taken i at a time. This is because the window Parisian option value becomes zero when the number of breaches reaches N, so those states with N or more "1" in the string are irrelevant.

The window Parisian option has two distinctive features. First, the associated path-dependent state vector has elements that are binary strings rather than scalars. Second, the differential equation for the option value cannot be written in a simple form. Even without the governing differential equation, the FSG method remains a viable approach for pricing window Parisian options. This demonstrates the advantage of the FSG approach over the finite-difference approach in the construction of pricing algorithms.

Let $V_{win}[m, j; A]$ denote the value of a window Parisian option at the (m, j)–th node, and with the asset price path history represented by the binary string A. The binary string A has to be modified according to the event of either breaching or no breaching at a monitoring instant.

The corresponding numerical scheme can be succinctly represented by

$$V_{win}[m-1, j; A] = \begin{cases} \{p_u V_{win}[m, j+1; A] \\ + p_0 V_{win}[m, j; A] \\ + p_d V_{win}[m, j-1; A]\} e^{-r\Delta t} & \text{if } m\Delta t \neq t_l^* \\ \{p_u V_{win}[m, j+1; g_{win}(A, j+1)] \\ + p_0 V_{win}[m, j; g_{win}(A, j)] \\ + p_d V_{win}[m, j-1; g_{win}(A, j-1)]\} e^{-r\Delta t} & \text{if } m\Delta t = t_l^* \end{cases} \tag{8-A}$$

where

$$g_{win}(A, j) = \begin{cases} 1a_1 a_2 \cdots a_{N_w - 1} & \text{if } x_j \leq \ln B \\ 0a_1 a_2 \cdots a_{N_w - 1} & \text{if } x_j > \ln B \end{cases} \tag{8-B}$$

As in the numerical procedure for the consecutive Parisian option, it is necessary to compute $V_{win}[m, j; A]$ for those states of A with $N - 1$ or less "1" in the string before we move to a time level corresponding to a monitoring instant. Note that $V_{win}[m, j; A] = 0$ at a monitoring instant when the string A has N or more "1." Due to the higher level of path-dependence exhibited by the window feature, the operation counts of the window Parisian option calculations are about $C_0^{N_w} + C_1^{N_w} + \ldots + C_{N-1}^{N_w}$ times of those of plain vanilla option calculations.

Numerical Experiments and Comparison of Their Performance

It is commonly known that quite a large number of time steps are required to attain sufficient accuracy in numerical calculations of path-dependent options using the FSG approach. Barraquand and Pudet [1996] report that their lookback option calculations reveal only a square root rate of convergence; that is, the number of time steps has to be quadrupled in order to reduce the numerical errors by half. Also, it is quite common to produce numerical option values incorrect by more than 10% even when more than 1,000 time steps are used.

First, we consider the numerical valuation of the continuously monitored cumulative Parisian call option with a down barrier B; that is, the option is knocked out only when the asset price stays below B cumulatively for

EXHIBIT 2
Numerical Option Values Obtained Using Equation (6) for a Cumulative Parisian Call Option

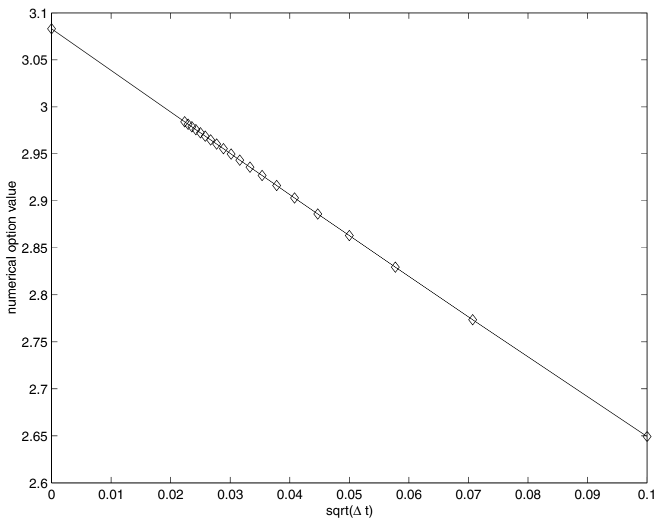


EXHIBIT 3
Option Values of Cumulative Parisian Call Option Obtained Three Ways

d	Analytic Price Formula[a]	Equation (6) $M = 500$[b]	Equation (6) M = 1,000[b]	Extrapolation to $\Delta t = 0$[c]
0.25	4.88453	4.71818	4.76807	4.88851
0.50	3.08308	2.88602	2.94335	3.08175
0.75	0.98758	0.85923	0.89504	0.98149

[a] *Analytic price formula (Hugonnier [1999]).*

[b] *Equation (6) with time steps M = 500 and M =1,000.*

[c] *Shanks transformation to extrapolate to the infinitesimal time step.*

a sufficient amount of time. There are analytic price formulas for cumulative Parisian options (see Hugonnier [1999]).

In Exhibit 2, we plot the numerical option values obtained using Equation (6) for a cumulative Parisian call option against the square root of the time step, $\sqrt{\Delta t}$. The parameter values used in the calculations are: $S = 95$, $X = 100$, $B = 110$, $\sigma = 20\%$, $T = 1$, $r = 5\%$, $q = 2\%$, and $d = 0.5$, where d is the minimum cumulative excursion time to stay above the barrier to avoid knock-out. Note that the option remains alive even though the initial asset price stays below the barrier. The option value at vanishing Δt is obtained from the analytic price formula, found to be 3.083 by Hugonnier [1999]. The plot clearly reveals the square root rate of convergence of the numerical option values.

One may apply a non-linear extrapolation technique to hasten the rate of convergence of numerical option values obtained using a varying number of time steps. The commonly used extrapolation techniques for improving the rate of convergence of a slowly convergent sequence include the Shanks transformation, the Richardson extrapolation, and the Pade approximants (a good exposition of these extrapolation techniques is found in Bender and Orszag [1978]).

The success of extrapolation to an infinitesimal time step depends critically on smooth convergence behavior of the numerical option values. The smoothness of convergence may be hurt by discontinuity of the first-order derivative in the terminal payoff of the option. To avoid oscillatory convergence behavior, one may use either the Black-Scholes adjustment method (using the Black-Scholes option price formula to compute option values at the nodes on the last monitoring instant) or convolution smoothing of the terminal payoff function (see Heston and Zhou [2000]).

To illustrate the effectiveness of applying extrapolation, we compute the option values of a cumulative Parisian call option using Equation (6) with a varying number of time steps. We then apply the Shanks extrapolation scheme to obtain the best estimates of the option values at an infinitesimal time step.

Exhibit 3 lists the option values corresponding to varying values of d. The other parameter values for the cumulative Parisian call option are the same as those used in generating the plot in Exhibit 2. The extrapolated option values agree much better with the option values obtained using the analytic price formula (considered to be exact) compared to those obtained using either 500 or 1,000 time steps.

The consecutive and cumulative Parisian call option values are seen to be increasing functions of N, where N is the number of breaches required to activate knock-out. When $N = 1$, both consecutive and cumulative Parisian call options reduce to the usual one-touch barrier call option. When N tends to infinity, both types of Parisian options become the usual plain vanilla option. When $1 < N < \infty$, the consecutive Parisian option is more expensive than its cumulative counterpart since knock-out can be activated easier for the cumulative Parisian option.

All these properties of Parisian option values are revealed in Exhibit 4. The parameter values used in the calculations are: $S = 100$, $X = 95$, $B = 80$, $T = 1$, $\sigma = 25\%$, $r = 5\%$, and $q = 0$. The number of time steps is 1,000, and the number of monitoring instants is 200.

EXHIBIT 4
Option Values of Consecutive Parisian Call Options (*) and Cumulative Parisian Call Options (+)

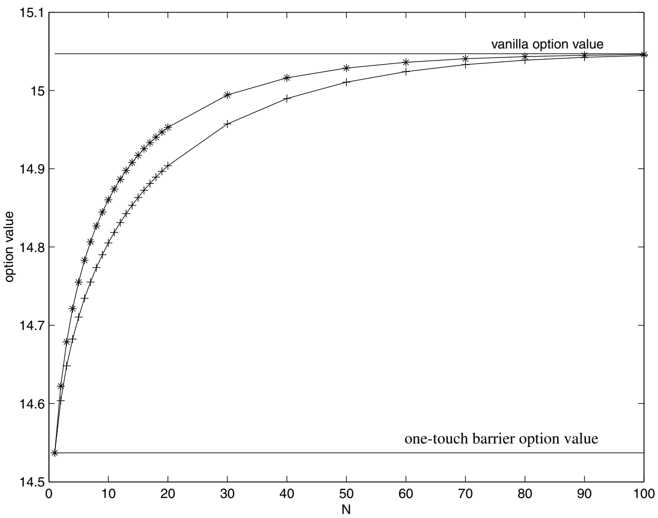
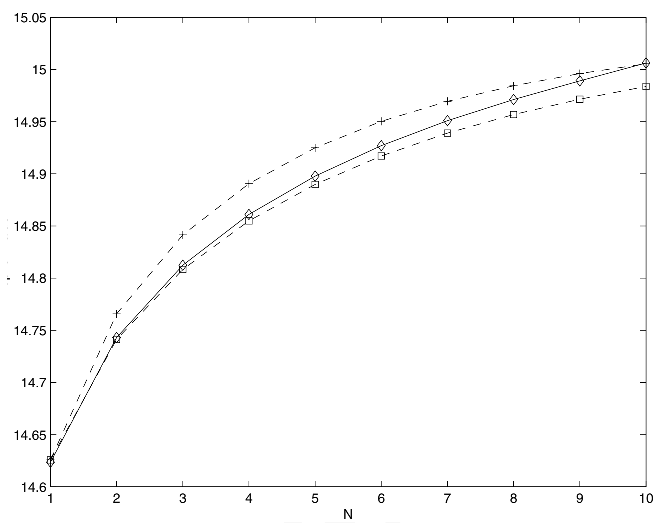


EXHIBIT 5
Option Values of Window Parisian Call Options



◇ *Window Parisian option.*
\+ *Consecutive Parisian option.*
□ *Cumulative Parisian option.*

In Exhibit 5, we plot the option values of a discretely monitored window Parisian call option against N, where N is the required number of breaches within the moving window to activate knock-out. The moving window is taken to consist of ten previous monitoring instants, that is, $N_w = 10$. Correspondingly, N may assume an integer value that ranges from 1 to 10. The parameter values used in the option calculations are the same as those for Exhibit 4, except that the total number of monitoring instants is set to be 50.

The option values of the window Parisian call options are seen to be bounded above by the corresponding consecutive Parisian call option values and bounded below by the corresponding cumulative Parisian call option values. When $N = 1$, the three types of Parisian call options all reduce to the corresponding one-touch barrier call option. When $N = 10$, the window Parisian call option becomes the consecutive Parisian call option.

III. OPTIONS WITH RESET FEATURES

A reset feature embedded in an option contract serves as a sweetener to the buyer of the option. A discussion of options with reset feature occurs in Cheng and Zhang [2000]. An example is a reset call option where the strike price of the call is reset to the prevailing asset price on a predetermined reset date if the option is out of the money on that date. Now, the strike price of the call option at expiration is not fixed, but will depend on the actual realization of the asset price path (in particular, the asset values on those predetermined reset dates).

Let $\hat{t}_l$, $l = 1, 2, \ldots, M$ be the reset dates throughout the life of the option, and let X_l be the strike price of the call option chosen on the reset date $\hat{t}_l$. We write $S(\hat{t}_l)$ as the asset value realized by the asset price path at time $\hat{t}_l$. The reset feature then dictates the strike price X_l to be given by

$$\begin{aligned} X_l &= \min[X, S(\hat{t}_1), S(\hat{t}_2), \ldots, S(\hat{t}_l)] \\ &= \min[X, X_{l-1}, S(\hat{t}_l)] \end{aligned} \tag{9}$$

where X is the original strike price set at initiation of the option contract. The strike price used for the valuation of the terminal payoff is then given by $X_M = \min[X, S(\hat{t}_1), \ldots, S(\hat{t}_M)]$.

If we apply the backward induction procedure in a trinomial calculation for pricing the reset option, we encounter difficulty in evaluating the terminal payoff since the strike price is not yet known. The difficulty arises because the strike price adopted in the payoff depends on realization of the asset price on the trinomial tree.

Let m_l denote the number of time steps counting from the top node of the trinomial tree to the l–th reset date. The number of possible strike prices at those time levels between the l–th and $(l + 1)$–th reset dates is $2m_l + 2$, $l = 0, 1, \ldots, M$. Here, the 0–th reset date and the

(M +1)–th reset date are taken to be the inception time and the expiration date, respectively. We have $(2m_l + 2)$ possible strike prices, since there are $2m_l + 1$ possible asset values at the time level that is m_l time steps from the top node of the trinomial tree, and the one additional possible strike price is the original strike price X set at initiation of the option contract.

When we follow the backward induction procedure in the reset option calculation, we first compute the terminal payoff values for all possible strike prices ($2m_M + 2$ of them). Now, the augmented state vector at each lattice node in the FSG algorithm includes all possible strike prices. As we proceed backward, in particular at a time level corresponding to a reset date, the vector of strike prices will be adjusted according to the rule stated in Equation (9).

Let k denote the index relating to the (log) strike price x_k(recall that $x_k = \ln S + k\Delta x$, where S is the asset value at the top of the trinomial tree), and write $V_{res}[m, j; k]$ as the numerical value of the reset option at the (m, j)–th node with (log) strike price x_k. Let the original strike price X be related to the index value k_0 by $x_{k_0} = \ln X = \ln S + k_0\Delta x$.

The construction of the FSG algorithm for pricing the reset call option gives

$$V_{res}[m-1,j;k] = \begin{cases} \{p_u V_{res}[m,j+1;k] \\ + p_0 V_{res}[m,j;k] \\ + p_d V_{res}[m,j-1;k]\}e^{-r\Delta t} & \text{if } m\Delta t \neq \hat{t}_l \\ \{p_u V_{res}[m,j+1;g_{res}(k,j+1)] \\ + p_0 V_{res}[m,j;g_{res}(k,j)] \\ + p_d V_{res}[m,j-1;g_{res}(k,j-1)]\}e^{-r\Delta t} & \text{if } m\Delta t = \hat{t}_l \end{cases} \tag{10–A}$$

where the grid function is given by

$$g_{res}(k, j) = \min(k, j, k_0) \tag{10–B}$$

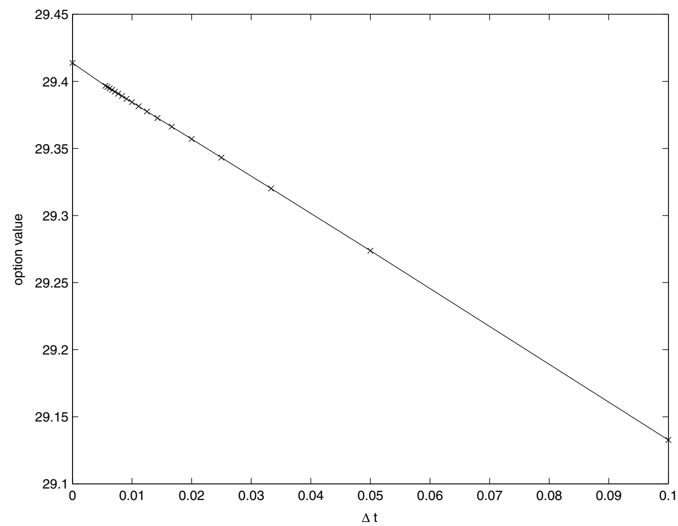
At maturity (say, M_T time steps from the current time on the trinomial tree), the terminal payoff is given by

$$V_{res}[M_T, j; k] = \max(e^{x_j} - e^{x_k}, 0) \tag{11}$$

for $-M_T \leq j \leq M_T$ and $-m_M \leq k \leq m_M$

EXHIBIT 6

Numerical Option Values of Reset Call Option



Computed by Equations (10-A) and (10-B) with varying time step Δt.

The size of the augmented state vector containing the possible strike prices shrinks whenever we march backward on the trinomial tree past a time step corresponding to a reset date. The order of complexity of the trinomial calculations for pricing a reset option is seen to be $O(M_T^3)$.

We apply the FSG algorithm to price a reset call option [see Equations (10–A) and (10 – B)]. The parameter values of the reset call option are: $S = 100$, $X = 100$, $r = 5\%$, $\sigma = 20\%$, $T = 4$, and the reset dates are 1, 2, and 3.

In Exhibit 6, we plot the numerical option values of the reset call option against the time step Δt. The evaluation of the analytic pricing formula for this reset call option is 29.4138 (see Cheng and Zhang [2000]). The plot clearly reveals the linear rate of convergence in Δt of the numerical option values to the value obtained from the analytic price formula.

Note that the convergence behavior of the numerical option values depends significantly on the method of placing the current asset price and the strike price on the lattice tree. In our calculations, both the current asset price and the strike price are placed on a lattice grid. Our numerical experiments show that erratic convergence behavior of the numerical option values may result if other methods of positioning of the lattice nodes are adopted.

IV. ALPHA QUANTILE OPTIONS

Usually barrier options prescribe fixed barrier levels. The alpha quantile option takes the barrier level to be a stochastic variable that defines the terminal payoff. As an illustration, consider the terminal payoff of a call option as given by $\max(S_{median} - X, 0)$, where X is the strike price, and S_{median} is the median of the asset price process over the monitoring period. The quantity S_{median} is a stochastic variable that defines the *barrier level*, whereby the asset price is below the median over exactly half of the monitoring period.

In general, the terminal payoff of an alpha quantile option depends on S_α, where S_α is the barrier level so that the asset price is below S_α over exactly α% of the monitoring period, $0 \le \alpha \le 1.0$. When $\alpha = 0.5$, $S_{0.5}$ becomes S_{median}; and when $\alpha = 1.0$, $S_{1.0}$ is the realized maximum of the asset price path over the monitoring period. Despite the highly exotic nature of the alpha quantile options, their analytic price formulas have been obtained by Akahori [1995] and Dassios [1995].

Let S_t denote the asset price process indexed by t. For a given percentile α, $0 \le \alpha \le 1.0$, we define the α percentile of $\{S_t\}_{t\in[0,T]}$ as

$$B_{inf}(T;\alpha) = \inf\left\{B : \frac{1}{T}\int_0^T \mathbf{1}_{\{S_t \le B\}}dt > \alpha\right\} \quad (12)$$

The terminal payoff of the alpha quantile option (with parameters T and α) is defined to be $\max(B_{inf}(T;\alpha) - X, 0)$, where X is the strike price. Here, B_{inf} is a stochastic state variable whose value depends on the realization of the asset price path over the period $[0, T]$. The quantity B_{inf} corresponds to the infimum of all possible barrier levels where the percentage of excursion time that the asset price stays on or below the barrier level is greater than a given percentile α. Note that the excursion time decreases as the barrier level is lowered. To find B_{inf}, we gradually lower the barrier level B until the requirement of the percentage of excursion time greater than α is satisfied critically.

Assume that there are M time steps for the whole monitoring period $[0, T]$, and let S_j^M, $j = -M, \ldots, 0, 1, \ldots, M$ denote the discrete terminal asset prices at maturity (the last time step). In the discrete world of the trinomial tree, the possible values taken by the stochastic variable B_{inf} are limited to S_j^M, $j = -M, \ldots, 0, 1, \ldots, M$.

The numerical approximate value of the continuously monitored alpha quantile call option is given by

$$V_{alp} = e^{-rT}\sum_{j=-M}^{M} P[B_{inf} = S_j^M]\max(S_j^M - X, 0) \quad (13)$$

where $P[B_{inf} = S_j^M]$ is the probability that the stochastic barrier B_{inf} assumes the value S_j^M, $j = -M, \ldots, 0, 1, \ldots, M$. It is observed that $P[B_{inf} = S_j^M]$ times the discount factor e^{-rT} can be approximated by the difference of the prices of two cumulative Parisian binary options (see the appendix).

The cumulative Parisian binary option has the same property of excursion time counting as that to activate knock-out for a usual cumulative Parisian option. Its terminal payoff is equal to unity conditional on no knock-out during the life of the option. Let $V_{cum}^{bin}[d, B]$ denote the price of the continuously monitored cumulative Parisian binary option with down barrier B, and d be the minimum cumulative time staying above the down barrier to avoid knock-out.

It is shown in the appendix that

$$\begin{aligned} e^{-rT}P[B_{inf} = S_j^M] &\approx V_{cum}^{bin}[(1-\alpha)T,\ S_{j-1}^M] - \\ &\quad V_{cum}^{bin}[(1-\alpha)T,\ S_j^M] \\ &\quad \text{for } j = -M, \ldots, 0, 1, \ldots, M \end{aligned} \quad (14)$$

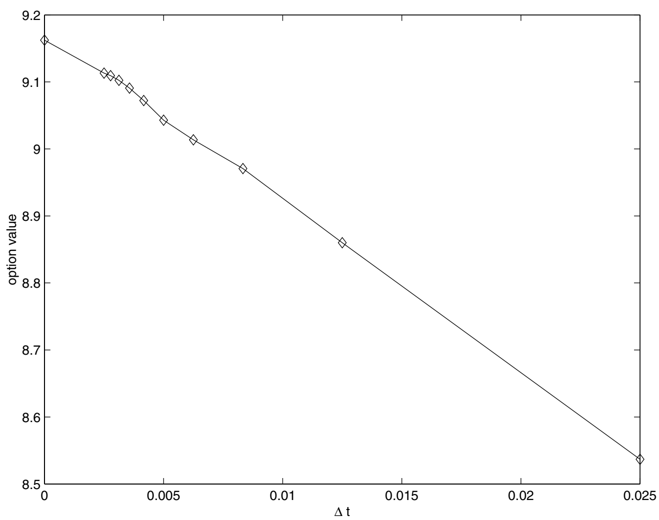
For notational convenience, we take $V_{cum}^{bin}[(1-\alpha)T, S_{-(M+1)}^M] = e^{-rT}$. Combining Equations (13) and (14), we obtain

$$\begin{aligned} V_{alp} = \sum_{j=-M}^{M} &\max(S_j^M - X, 0) \times \\ &\left\{V_{cum}^{bin}[(1-\alpha)T,\ S_{j-1}^M] - V_{cum}^{bin}[(1-\alpha)T,\ S_j^M]\right\} \end{aligned} \quad (15)$$

Interestingly, the pricing of an alpha quantile call option is closely related to that of the cumulative Parisian binary options. The operation counts of the numerical calculation of an alpha quantile option are about $2M + 1$ times that of the cumulative Parisian option.

We compute the numerical option values of an alpha quantile call option with varying time step using Equation (15). The numerical option values are plotted against Δt in Exhibit 7. The extrapolated option value at

EXHIBIT 7
Numerical Option Values of Continuously Monitored Alpha Quantile Call Option



Computed by Equation (15).

an infinitesimal time step is obtained by numerical valuation of the analytic price formula obtained by Akahori [1995]. The parameter values of the alpha quantile call option are: $\alpha = 80\%$, $S = 100$, $X = 95$, $r = 5\%$, $q = 0$, $\sigma = 20\%$, and $T = 0.25$. The plot illustrates the apparent linear rate of convergence of the numerical option values to the analytic solution.

V. CONCLUSION

We have developed forward shooting grid algorithms for pricing Parisian options, options with a reset feature, and alpha quantile options, and have also examined the convergence behavior of the numerical option values. Although the Parisian options, the reset options, and the alpha quantile options look quite different, their pricing algorithms closely resemble each other. The pricing of an exotic path-dependent option using the forward shooting grid approach requires simply determination of the appropriate discrete evolution function of the path-dependent feature (including some special consideration for initiating the trinomial calculations). These algorithms can be expressed elegantly in succinct forms, so they are pedagogically appealing to practitioners.

The window Parisian options appear to exhibit a higher level of path-dependence, but the forward shooting grid approach can be generalized to handle the window feature effectively. This is done by storing the memory of the asset price path at the previous monitoring instants in a binary string. Even in the absence of a governing partial differential equation for the option value of an alpha quantile option, we can develop a numerical algorithm for effective pricing of the alpha quantile option. We also show the close resemblance between the pricing of an alpha quantile option and cumulative Parisian binary options.

When numerical option values reveal a clear pattern of convergence, it is demonstrated that the effective use of an extrapolation technique may hasten the rate of convergence.

APPENDIX
Proof of Equation (14)

Let $P[B_{inf} > S_j^M]$ denote the path probability that B_{inf} assumes a value that is higher than S_j^M on the trinomial tree and $V_{cum}^{bin}[d, B]$ denote the numerical value of a cumulative Parisian binary option with parameter d and down barrier B as defined in the text. From the definition of B_{inf} and the property of cumulative Parisian binary options, we have:

$$e^{-rT} P[B_{inf} > S_{j-1}^M] \approx V_{cum}^{bin}[(1-\alpha)T,\ B_{j-1}],$$

$$e^{-rT} P[B_{inf} > S_j^M] \approx V_{cum}^{bin}[(1-\alpha)T,\ B_j].$$

In the discrete world of the trinomial tree, we obtain

$$\begin{aligned} e^{-rT} P[B_{inf} = S_j^M] &= e^{-rT} \left\{ P[B_{inf} > S_{j-1}^M] - P[B_{inf} > S_j^M] \right\} \\ &\approx V_{cum}^{bin}[(1-\alpha)T,\ B_{j-1}] - \\ &\quad V_{cum}^{bin}[(1-\alpha)T,\ B_j] \end{aligned}$$

REFERENCES

Akahori, J. "Some Formulae for a New Type of Path-Dependent Option." *Annals of Applied Probability*, 5 (1995), pp. 383-388.

Avellaneda, M., and L. Wu. "Pricing Parisian-Style Options with a Lattice Method." *International Journal of Theoretical and Applied Finance*, 2 (1999), pp. 1-16.

Barraquand, J., and T. Pudet. "Pricing of American Path-Dependent Contingent Claims." *Mathematical Finance*, 6 (1996), pp. 17-51.

Bender, C.M., and S.A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. New York: McGraw-Hill, 1978.

Boyle, P., and Y. Tian. "An Explicit Finite Difference Approach to the Pricing of Barrier Options." *Applied Mathematical Finance*, 5 (1998), pp. 6-14.

Cheng, W.Y., and S. Zhang. "The Analytics of Reset Options." *The Journal of Derivatives,* 8 (2000), pp. 59-71.

Chesney, M., J. Cornwall, M. Jeanblanc-Picque, G. Kentwell, and M. Yor. "Parisian Pricing." *Risk*, 10 (January 1997), pp. 77-79.

Chesney, M., M. Jeanblanc-Picque, and M. Yor. "Brownian Excursions and Parisian Barrier Options." *Advances in Applied Probability*, 29 (1997), pp. 165-184.

Dassios, A. "The Distribution of the Quantile of a Brownian Motion with Drift and the Pricing of Related Path-Dependent Options." *Annals of Applied Probability*, 5 (1995), pp. 389-398.

Forsyth, P.A., K.R. Vetzal, and R. Zvan. "Convergence of Lattice and PDE Methods for Pricing Asian Options." Proceedings of the 9th Annual Derivatives Securities Conference, Boston, 1999.

Haber, R.J., P.J. Schönbucher, and P. Wilmott. "Pricing Parisian Options." *The Journal of Derivatives*, 6 (1999), pp. 71-79.

Heston, S., and G. Zhou. "On the Rate of Convergence of Discrete-Time Contingent Claims." *Mathematical Finance*, 10 (2000), pp. 53-75.

Hugonnier, J. "The Feynman-Kac Formula and Pricing Occupation Time Derivatives." *International Journal of Theoretical and Applied Finance*, 2 (1999), pp. 153-178.

Hull, J., and A. White. "Efficient Procedures for Valuing European and American Path-Dependent Options." *The Journal of Derivatives*, 1 (1993), pp. 21-31.

Ritchken, P., L. Sankarasubramanian, and A.M. Vijh. "The Valuation of Path Dependent Contracts on the Average." *Management Science*, 39 (1993), pp. 1202-1213.

Vetzal, K.R., and P.A. Forsyth. "Discrete Parisian and Delayed Barrier Options: A General Numerical Approach." *Advances in Futures and Options Research*, 10 (1999), pp. 1-15.