Mathematical Excalibur

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Olympiad Corner

Following are the problems of 2004 Estonian IMO team selection contest.

Problem 1. Let $k > 1$ be a fixed natural number. Find all polynomials $P(x)$ satisfying the condition $P(x^k) = (P(x))^k$ for all real number x.

Problem 2. Let O be the circumcentre of the acute triangle ABC and let lines AO and BC intersect at a point K. On sides AB and AC, points L and M are chosen such that $KL = KB$ and $KM = KC$. Prove that segments LM and BC are parallel.

Problem 3. For which natural number n is it possible to draw n line segments between vertices of a regular $2n$-gon so that every vertex is an endpoint for exactly one segment and these segments have pairwise different lengths?

Problem 4. Denote

$$f(m) = \sum_{k=1}^{m} (-1)^k \cos \frac{k\pi}{2m+1}.$$

For which positive integers m is $f(m)$ rational?

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 7, 2005***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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例析數學競賽中的計數問題(二)

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3　運用算兩次原理與抽屜原理

算兩次原理，就是把一個量從兩個（或更多）方面去考慮它，然後綜合起來得到一個關係式（可以是等式或不等式），或者導出一個矛盾的結論．具體表示爲三步：“一方面（利用一部分條件）……，另一方面（利用另一部分條件）……，綜合這兩個方面 ……”．義大利數學家富比尼（Fubini）首先應用這個思想方法，因此今天我們也稱它爲富比尼原理．

在解這些問題時，要根據問題的特點選擇一個適當的量，再將這個量用兩（或幾）種不同的方法表達出來．

抽屜原理，德國數學家狄利克雷（Dirichlet）提出．對於這個原理的具體解釋，想必很多同學早就知道了，在此不再贅述．

<u>例 6</u>　有 26 個不同國家的集郵愛好者，想通過互相通信的方法交換各國最新發行的紀念郵票，爲了使這 26 人每人都擁有這 26 個國家的一套最新紀念郵票，他們至少要通多少封信？

<u>解答</u>　不妨設這 26 個集郵愛好者中的某一個人爲組長．

一方面，對於組長，要接收到其他 25 個國家的最新紀念郵票，必須從這 25 個集郵愛好者的手中發出（不管他們是否直接發給組長），至少要通 25 封信；同樣地，其他 25 個集郵愛好者分別要接收到組長的一套紀念郵票，必須由組長發出（不管組長是否直接發給這 25 個集郵愛好者），至少要通 25 封信．總計至少要通 50 封信．另一方面，其餘 25 個集郵愛好者每人將本國的一套最新的紀念郵票 25 份或 26 份發給組長，計 25 封信；組長收到這 25 封信後，再分別給這 25 個集郵愛好者各發去一封信，每封信中含有 25 套郵票（發給某人的信中不含其本國的郵票）或 26 套郵票（發給某人的信中包含其本國的郵票），計 25 封信．總計 50 封信．這就是說通 50 封信可以使這 26 人每人都擁有這 26 個國家的一套最新紀念郵票．
因此他們至少要通 50 封信．

<u>例 7</u>　從 1，2，3，…，1997 這 1997 個數中至多能選出多少個數，使得選出的數中沒有一個是另一個的 19 倍？

<u>解答</u>　因爲 1997÷19=105…2，所以 106，107，…，1997 這 1892 個數中沒有一個是另一個的 19 倍．
又因 106÷19=5…11，故 1，2，3，4，5，106，107，…，1997 這 1897 個數中沒有一個是另一個的 19 倍．
另一方面，從（6，6×19），（7，7×19），…，（105，105×19）這 100 對互異的數中最多可選出 100 個數（每對中至多選 1 個），即滿足題意的數至少剔除 100 個數．

綜上所述，從 1，2，3，…，1997 中至多選出 1897 個數，使得選出的數中沒有一個是另一個的 19 倍．

<u>例 8</u>　在正整數 1，2，3，…，1995，1996，1997 裏，最多能選出多少數，使其中任意兩個數的和不能被這兩個數的差整除．

<u>解答</u>　在所選的數中，不能出現連續自然數、連續奇數或連續偶數，這是由於連續自然數之和必能被其差 1 整除；連續奇數或連續偶數之和是偶數，必能被其差 2 整除．再考慮差值爲 3 的兩數，不能是 3 的倍數，否則其和仍是 3 的倍數，必能被其差 3 整除；而選擇全是 3 除餘 1，或全是 3 除餘 2 的數，注意到各自中任意兩數之和非 3 的倍數，不能被其差 3 的倍

數整除，滿足題意．

另一方面，從(1,2,3)，(4,5,6)，⋯，(1993,1993,1995)，(1996,1997)中，最多可選出 666 個(每組至多可選一個)，否則會出現連續自然數、連續奇數或連續偶數，而不滿足題意．又間隔 4 的所有數的個數較上述滿足題意的所有數的個數少．

綜上可知，1，4，7，⋯，1990，1993，1996（666 個）或 2，5，8，⋯，1991，1994，1997（666 個）均滿足題意．

即最多可選出 666 個，使其中任意兩數之和不能被這兩數之差整除．

例 9 設自然數n有以下性質：從 1，2，⋯，n中任取 50 個不同的數，這 50 個數中必有兩個數之差等於 7，這樣的n最大的一個是多少？

解答 n的最大值是 98．說明如下：

⑴一方面當自然數從 1，2，⋯，98 中任取 50 個不同的數，必有兩個數之差等於 7．這是因爲：

首先將自然數 1，2，⋯，98 分成 7 組：（1，2，3，4，5，6，7，8，9，10，11，12，13，14），（15，16，17，18，19，20，21，22，23，24，25，26，27，28），⋯，（85，86，87，88，89，90，91，92，93，94，95，96，97，98）．

考慮取出的數中不出現某兩個數之差等於 7 的情形：由於每組中含有差爲 7 的兩數，故每組最多可取出 7 個數（即每組中屬於 7 的同一個剩餘類的兩個數只能取其中的任意一個）．並且如果在第 1 組中取出了 m（m=1，2，⋯，14），那麼後面的每組分別取出 m+14n（n=1，2，⋯，6），可使所取數中的任意兩個數之差都不是 7．這樣從上述 7 組數中最多只能取出 7×7=49 個數．

根據抽屜原理，知從 1，2，⋯，98 中任取 50 個不同的數，必有兩個數之差等於 7．

⑵另一方面當自然數從 1，2，⋯，99 中任取 50 個不同的數，不能保證必有兩個數之差等於 7．這是因爲：

首先將自然數 1，2，⋯，99 分成 8 組：（1，2，3，4，5，6，7，8，9，10，11，12，13，14），（15，16，17，18，19，20，21，22，23，24，25，26，27，28），⋯，（85，86，87，88，89，90，91，92，93，94，95，96，97，98），（99）．

比如，取出前 7 組中每組的前 7 個數，第 8 組的 99 這 50 個數，就不含有兩個數之差等於 7．

綜合⑴、⑵，可得 n 的最大值是 98．

例 10 某校組織了 20 次天文觀測活動，每次有 5 名學生參加，任何 2 名學生都至多同時參加過一次觀測．證明：至少有 21 名學生參加過這些觀測活動．

證法 1 （反證法）假設至多有 20 名學生參加過這些觀測活動．

每次觀測活動中的 5 名學生中有 $C_5^2=\frac{5\times4}{2\times1}=10$ 個 2 人小組，又由題意知 20 次觀測中 2 人小組各不相同，所以 20 次觀測中 2 人小組總共有 $20\times10=200$ 個．

而另一方面，20 名學生中的 2 人小組最多有 $C_{20}^2=\frac{20\times19}{2\times1}=190$ 個．

兩者自相矛盾．故至少有 21 名學生參加過這些觀測活動．

稍作簡化，即可證明如下：

證法 2 （反證法）假設至多有 20 名學生參加過這些觀測活動．

由題意知：⑴共有 20 次觀測；⑵最多有 $\frac{C_{20}^2}{C_5^2}=19$ 次觀測．

兩者自相矛盾．故至少有 21 名學生參加過這些觀測活動．

對於低年級學生，還可作出如下證明：

證法 3 設參加觀測活動次數最多的學生A參加了a次觀測，共有x名學生參加過天文觀測活動．

由於有 A 參加的每次觀測活動中，除了 A，其他學生各不相同（這是因爲任何 2 名學生都至多同時參加過一次觀測），故 $x\geq 4a+1$ ．(I)

另一方面，學生 A 參加觀測的次數不小於每名學生平均觀測次數．即

$a\geq\frac{20\times5}{x}$ ．(II)

綜合（I）、（II），得 $x\geq\frac{400}{x}+1$，

$x^2-x-400\geq0$ ．從而 $x\geq 21$ ．

即至少有 21 名學生參加過這些觀測活動．

例 11 $2n$名選手參加象棋循環賽，每一輪中每個選手與其他 $2n-1$ 人各賽一場，勝得 1 分，平各得 $\frac{1}{2}$ 分，負得 0 分．證明：如果每個選手第一輪總分與第二輪總分至少相差n分，那麼每個選手兩輪總分恰好相差n分．

證明 令集A={第二輪總分>第一輪總分的人}，集B={第二輪總分<第一輪總分的人}，並且|A|=k，|B|=h，k+h=2n．

不妨設 $k\geq n\geq h$．考慮 A 中選手第二輪總分之和 S（若 $h\geq n\geq k$，則考慮 B 中選手第一輪總分之和 T）．另一方面，對於每輪 A 中選手和 B 中選手的 kh 場比賽中，所得總分之和爲 kh，充其量全爲 A 中選手取勝，則 $S\leq C_k^2+kh$．如 A 中選手第一輪總分之和爲 S'，那麼 $S-S'\geq kn$，$C_k^2+kh-kn\geq S-kn\geq S'\geq C_k^2$．從而得 $h\geq n$，所以 n=h=k，並且以上不等式均爲等式．

所以 A 中每個選手第二輪總分恰比第一輪總分多 n 分，B 中每個選手第一輪總分恰比第二輪總分多 n 分．因此，原命題成立．

(to be continued)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***May 7, 2005.***

Problem 221. (*Due to Alfred Eckstein, Arad, Romania*) The Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1}+F_{n-2}$ for $n \geq 2$.

Prove that $7F_{n+2}^3 - F_n^3 - F_{n+1}^3$ is divisible by F_{n+3}.

Problem 222. All vertices of a convex quadrilateral *ABCD* lie on a circle ω. The rays *AD*, *BC* intersect in point *K* and the rays *AB*, *DC* intersect in point *L*.

Prove that the circumcircle of triangle *AKL* is tangent to ω if and only if the circumcircle of triangle *CKL* is tangent to ω.
(*Source: 2001-2002 Estonian Math Olympiad, Final Round*)

Problem 223. Let $n \geq 3$ be an integer and x be a real number such that the numbers x, x^2 and x^n have the same fractional parts. Prove that x is an integer.

Problem 224. (*Due to Abderrahim Ouardini*) Let *a*, *b*, *c* be the sides of triangle *ABC* and *I* be the incenter of the triangle.

Prove that

$$IA \cdot IB \cdot IC \leq \frac{abc}{3\sqrt{3}}$$

and determine when equality occurs.

Problem 225. A luminous point is in space. Is it possible to prevent its luminosity with a finite number of disjoint spheres of the same size?
(*Source: 2003-2004 Iranian Math Olympiad, Second Round*)

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Solutions

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Problem 216. (*Due to Alfred Eckstein, Arad, Romania*) Solve the equation

$$4x^6 - 6x^2 + 2\sqrt{2} = 0.$$

Solution. **Kwok Sze CHAI Charles** (HKU, Math Major, Year 1), **CHAN Tsz Lung**, **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6), **Badr SBAI** (Morocco), **TAM Yat Fung** (Valtorta College, Form 5), **WANG Wei Hua** and **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 6).

We have $8x^6 - 12x^2 + 4\sqrt{2} = 0$. Let $t = 2x^2$. We get

$$\begin{aligned} 0 &= t^3 - 6t + 4\sqrt{2} \\ &= t^3 - (\sqrt{2})^3 - (6t - 6\sqrt{2}) \\ &= (t - \sqrt{2})(t^2 + \sqrt{2}t - 4) \\ &= (t - \sqrt{2})(t - \sqrt{2})(t + 2\sqrt{2}). \end{aligned}$$

Solving $2x^2 = \sqrt{2}$ and $2x^2 = -2\sqrt{2}$, we get $x = \pm 1/\sqrt[4]{2}$ or $\pm i\sqrt[4]{2}$.

Other commended solvers: **CHAN Pak Woon** (Wah Yan College, Kowloon, Form 7), **Kin-Chit O** (STFA Cheng Yu Tung Secondary School) and **WONG Sze Wai** (True Light Girls' College, Form 4).

Problem 217. Prove that there exist infinitely many positive integers which cannot be represented in the form

$$x_1^3 + x_2^5 + x_3^7 + x_4^9 + x_5^{11},$$

where x_1, x_2, x_3, x_4, x_5 are positive integers.
(*Source:* 2002 *Belarussian Mathematical Olympiad, Final Round*)

Solution. **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece) and **Tak Wai Alan WONG** (Markham, ON, Canada).

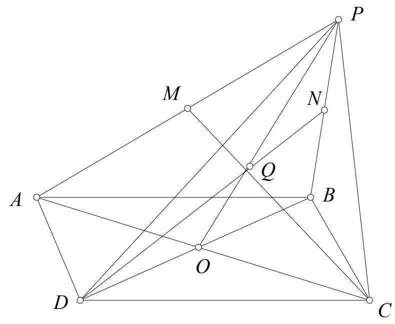
On the interval [1, *n*], if there is such an integer, then

$$x_1 \leq [n^{1/3}], x_2 \leq [n^{1/5}], \cdots, x_5 \leq [n^{1/11}].$$

So the number of integers in [1, *n*] of the required form is at most $n^{1/3}n^{1/5}n^{1/7}n^{1/9}n^{1/11} = n^{3043/3465}$. Those not of the form is at least $n - n^{3043/3465}$, which goes to infinity as *n* goes to infinity.

Problem 218. Let *O* and *P* be distinct points on a plane. Let *ABCD* be a parallelogram on the same plane such that its diagonals intersect at *O*. Suppose *P* is not on the reflection of line *AB* with respect to line *CD*. Let *M* and *N* be the midpoints of segments *AP* and *BP* respectively. Let *Q* be the intersection of lines *MC* and *ND*. Prove that *P*, *Q*, *O* are collinear and the point *Q* does not depend on the choice of parallelogram *ABCD*. (*Source: 2004 National Math Olympiad in Slovenia, First Round*)

Solution. **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania) and **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).



Let G_1 be the intersection of *OP* and *MC*. Since *OP* and *MC* are medians of triangle *APC*, G_1 is the centroid of triangle *APC*. Hence OG_1=⅓*OP*. Similarly, let G_2 be the intersection of *OP* and *ND*. Since *OP* and *ND* are medians of triangle *BPD*, G_2 is the centroid of triangle *BPD*. Hence OG_2=⅓*OP*. So G_1=G_2 and it is on both *MC* and *ND*. Hence it is *Q*. This implies *P*, *Q*, *O* are collinear and *Q* is the unique point such that *OQ*=⅓*OP*, which does not depend on the choice of the parallelogram *ABCD*.

Other commended solvers: **CHAN Pak Woon** (Wah Yan College, Kowloon, Form 7) and **CHAN Tsz Lung**, **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6) and **WONG Tsun Yu** (St. Mark's School, Form 5).

Problem 219. (*Due to Dorin Mărghidanu, Coleg. Nat. "A.I. Cuza", Corabia, Romania)* The sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are defined as follows: $a_0, b_0 > 0$ and

$$a_{n+1} = a_n + \frac{1}{2b_n}, \quad b_{n+1} = b_n + \frac{1}{2a_n}$$

for $n = 1,2,3,\ldots$. Prove that

$$\max\{a_{2004}, b_{2004}\} > \sqrt{2005}.$$

Solution. **CHAN Tsz Lung**, **Kin-Chit**

O (STFA Cheng Yu Tung Secondary School), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).

We have

$$a_{n+1}b_{n+1} = (a_n + \frac{1}{2b_n})(b_n + \frac{1}{2a_n})$$
$$= a_n b_n + \frac{1}{4a_n b_n} + 1$$
$$= a_{n-1}b_{n-1} + \frac{1}{4a_{n-1}b_{n-1}} + \frac{1}{4a_n b_n} + 2$$
$$= \cdots$$
$$= a_0 b_0 + \sum_{i=0}^{n} \frac{1}{4a_i b_i} + n + 1.$$

Then

$$(\max\{a_{2004}, b_{2004}\})^2 \ge a_{2004} \cdot b_{2004}$$
$$> a_0 b_0 + \frac{1}{4a_0 b_0} + 2004$$
$$\ge 2\sqrt{a_0 b_0 \frac{1}{4a_0 b_0}} + 2004$$
$$= 2005$$

and the result follows.

Other commended solvers: **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania).

Problem 220. (*Due to Cheng HAO, The Second High School Attached to Beijing Normal University*) For $i = 1, 2, \ldots, n$, and $k \ge 4$, let $A_i = (a_{i1}, a_{i2}, \ldots, a_{ik})$ with $a_{ij} = 0$ or 1 and every A_i has at least 3 of the k coordinates equal 1. Define the distance between A_i and A_j to be

$$\sum_{m=1}^{k} |a_{im} - a_{jm}|.$$

If the distance between any A_i and A_j ($i \ne j$) is greater than 2, then prove that

$$n \le 2^{k-3} - 1.$$

Solution.

Let $|A_i - A_j|$ denote the distance between A_i and A_j. We add $A_0 = (0,\ldots,0)$ to the n A_m's. Then $|A_i - A_j| \ge 3$ still holds for $A_0, A_1, \ldots, A_n$.

Next we put the coordinates of A_0 to A_n into a $(n + 1) \times k$ table with the coordinates of A_i in the $(i + 1)$-st row.

Note if we take any of the k columns and switch all the 0's to 1's and 1's to 0's, then we get $n + 1$ new ordered k-tuples that still satisfy the condition $|A_i - A_j| \ge 3$. Thus, we may change A_0 to any combination with 0 or 1 coordinates. Then the problem is equivalent to showing $n + 1 \le 2^{k-3}$ for $n + 1$ sets satisfying $|A_i - A_j| \ge 3$, but removing the condition each A_i has at least 3 coordinates equal 1.

For $k = 4$, we have $n + 1 \le 2$. Next, suppose $k > 4$ and the inequality is true for the case $k-1$.

In column k of the table, there are at least $[(n + 2)/2]$ of the numbers which are the same (all 0's or all 1's). Next we keep only $[(n+2)/2]$ rows whose k-th coordinates are the same and we remove column k. The condition $|A_i - A_j| \ge 3$ still holds for these new ordered $(k-1)$-tuples. By the case $k - 1$, we get $[(n + 2)/2] + 1 \le 2^{k-4}$. Since $(n + 1)/2 < [(n + 2)/2] + 1$, we get $n + 1 \le 2^{k-3}$ and case k is true.

Generalization of Problem 203

Naoki Sato

We prove the following generalization of problem 203:

Let $a_1, a_2, \ldots, a_n$ be real numbers, and let s_i be the sum of the products of the a_i taken i at a time. If $s_1 \ne 0$, then the equation

$$s_1 x^{n-1} + 2s_2 x^{n-2} + \cdots + ns_n = 0$$

has only real roots.

<u>*Proof*</u>. Let

$$f(x) = s_1 x^{n-1} + 2s_2 x^{n-2} + \cdots + ns_n.$$

We can assume that none of the a_i are equal to 0, for if some of the a_i are equal to 0, then rearrange them so that $a_1, a_2, \ldots, a_k$ are nonzero and $a_{k+1}, a_{k+2}, \ldots, a_n$ are 0. Then $s_{k+1} = s_{k+2} = \ldots = s_n = 0$, so

$$f(x) = s_1 x^{n-1} + 2s_2 x^{n-2} + \cdots + ns_n$$
$$= s_1 x^{n-1} + 2s_2 x^{n-2} + \cdots + ks_k x^{n-k}$$
$$= x^{n-k}(s_1 x^{k-1} + 2s_2 x^{k-2} + \cdots + ks_k).$$

Thus, the problem reduces to proving the same result on the numbers $a_1, a_2, \ldots, a_k$.

Let $g(x) = (a_1x+1)(a_2x+1)\ldots(a_nx+1)$. The roots of $g(x) = 0$ are clearly real, namely $-1/a_1, -1/a_2, \ldots, -1/a_n$. We claim that the roots of $g'(x)=0$ are all real.

Suppose the roots of $g(x) = 0$ are distinct. Let $r_1 < r_2 < \ldots < r_n$ be these roots. Then by Rolle's theorem, the equation $g'(x) = 0$ has a root in each of the intervals $(r_1,r_2), (r_2,r_3), \ldots, (r_{n-1},r_n)$, so it has $n - 1$ real roots.

Now, suppose the equation $g(x) = 0$ has j distinct roots $r_1 < r_2 < \ldots < r_j$, and root r_i has multiplicity m_i so $m_1+m_2+\ldots+m_j = n$. Then r_i is a root of the equation $g'(x) = 0$ having multiplicity m_i-1. In addition, again by Rolle's theorem, the equation has a root in each of the interval $(r_1,r_2), (r_2,r_3), \ldots, (r_{j-1},r_j)$, so the equation $g'(x) = 0$ has the requisite

$$(m_1-1)+(m_2-1)+\ldots+(m_j-1)+j-1 = n-1$$

real roots.

Expanding, we have that

$$g(x) = (a_1x+1)(a_2x+1)\ldots(a_nx+1)$$
$$= s_nx^n + s_{n-1}x^{n-1} + \ldots + 1,$$

So $g'(x) = ns_nx^{n-1} + (n-1)s_{n-1}x^{n-2} + \ldots + s_1$. Since $s_1 \ne 0$, 0 is not a root of $g'(x) = 0$. Finally, we get that the polynomial

$$x^{n-1}g'(\frac{1}{x}) = s_1x^{n-1} + 2s_2x^{n-1} + \cdots + ns_n$$

has all real roots.

Olympiad Corner

(continued from page 1)

Problem 5. Find all natural numbers n for which the number of all positive divisors of the number lcm$(1,2,\ldots,n)$ is equal to 2^k for some non-negative integer k.

Problem 6. Call a convex polyhedron a *footballoid* if it has the following properties.

(1) Any face is either a regular pentagon or a regular hexagon.

(2) All neighbours of a pentagonal face are hexagonal (a *neighbour* of a face is a face that has a common edge with it).

Find all possibilities for the number of a pentagonal and hexagonal faces of a footballoid.