Olympiad Corner

Following are the problems of 2004 Estonian IMO team selection contest.

Problem 1. Let \( k > 1 \) be a fixed natural number. Find all polynomials \( P(x) \) satisfying the condition \( P(k^x) = (P(x))^k \) for all real number \( x \).

Problem 2. Let \( O \) be the circumcentre of the acute triangle \( ABC \) and let lines \( AO \) and \( BC \) intersect at a point \( K \). On sides \( AB \) and \( AC \), points \( L \) and \( M \) are chosen such that \( KL = KB \) and \( KM = KC \). Prove that segments \( LM \) and \( BC \) are parallel.

Problem 3. For which natural number \( n \) is it possible to draw \( n \) line segments between vertices of a regular \( 2n \)-gon so that every vertex is an endpoint for exactly one segment and these segments have pairwise different lengths?

Problem 4. Denote

\[
f(m) = \sum_{k=1}^{m} (-1)^k \cos \frac{k\pi}{m+1}.
\]

For which positive integers \( m \) is \( f(m) \) rational?

(continued on page 4)

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數益得，滿足題意。


即最多可選出 666 個，即使其中任意兩數之和不能被這兩數之差整除。

例 8 設自然數 n 有以下性質：從 1, 2, …, n 中任取 50 個不同的取，這 50 個數中必有兩個數之差等於 7，這樣的 n 最大的一個是多少？

解 對 n 的最大值是 98，說明如下：
(1) 一方面當自然數從 1, 2, …, 98 中任取 50 個不同的取，必有两个數之差等於 7，這是因為：

首先將自然數 1, 2, …, 98 分成 7 組：(1, 2, 3, 4, 5, 6, 7, 8, 9), (10, 11, 12, 13, 14), (15, 16, 17, 18, 19, 20, 21), (22, 23, 24, 25, 26, 27, 28), …, (85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98).

考慮取出的數中不出現某兩個數之差等於 7 的情形：由於各組中含有的數之差等於 7，這樣從上述 7 組中最多只能取出 7×7=49 個數。

根據抽屜原理，知從 1, 2, …, 98 中任取 50 個不同的取，必有两个數之差等於 7，且規則：
(2) 另一方面當自然數從 1, 2, …, 99 中任取 50 個不同的取，不能保證必有兩數之差等於 7，這是因為：

首先將自然數 1, 2, …, 99 分成 8 組：(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14), (15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28), …, (85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98).

比如此，取出前 7 組中的前 7 個數，第 8 組的 99 個數，就不必有兩個數之差等於 7。


例 10 某校組織了 20 次天文觀測活動，每次都 5 名學生參加，任何 2 名學生都至少參加過一次觀測。證明：至少有 21 名學生参加過這些觀測活動。

證法 1 （反證法）假設至多有 20 名學生參加過這些觀測活動。

每次觀測活動中，5 名學生中有 C_5^2=5×4=10 個 2 小組，且又由題意知 20 次觀測中 2 小組各不相同，所以 20 次觀測中有 20 個小組。

而另一方面：20 名學生中 2 小組最多有 C_{20}^2=20×19=190 個.

兩者自相矛盾，故至少有 21 名學生參加過這些觀測活動。

由題意知：(1) 有 20 次觀測次；(2) 最多有 C_{20}^2=190 個次。

不放設 k≥n≥h，考慮 A 中選手第二輪總分之和 S，設 A 中選手第一輪總分之和 T，

另一方面， fulfil A 中選手和 B 中選手或 B 中選手的 k−1 次比賽中，所調總分之和為 kh，充其量至多為 A 中選手總分，

則 S≤C_{k}^{2}+kh，如 A 中選手第二輪總分之和為 S’，那麼 S=S’+kh≥k，

因為 h≥n，所以 S≥C_{h}^{2}+kh≥C_{h}^{2}，從而得 h≥n，即 h=n，並且以上等式均為等式

所以 A 中每個選手第二輪總分均比第一輪總分多 n 分，B 中每個選手第一輪總分均有 n 分，因此，原命題成立。

(to be continued)
Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong. The deadline for submitting solutions is May 7, 2005.

Problem 221. (Due to Alfred Eckstein, Arad, Romania) The Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Prove that $7F_{n+2}^3 - F_n^3 - F_{n+1}^3$ is divisible by $F_{n+3}$.

Problem 222. All vertices of a convex quadrilateral $ABCD$ lie on a circle $\omega$. The rays $AD$, $BC$ intersect in point $K$ and the rays $AB$, $DC$ intersect in point $L$.

Prove that the circumcircle of triangle $AKL$ is tangent to $\omega$ if and only if the circumcircle of triangle $CKL$ is tangent to $\omega$.

(Source: 2001-2002 Estonian Math Olympiad, Final Round)

Problem 223. Let $n \geq 3$ be an integer and $x$ be a real number such that the numbers $x$, $x^2$, and $x^n$ have the same fractional parts. Prove that $x$ is an integer.

Problem 224. (Due to Abderrahim Ouardini) Let $a$, $b$, $c$ be the sides of triangle $ABC$ and $I$ be the incenter of the triangle.

Prove that

$$IA \cdot IB \cdot IC \leq \frac{abc}{3\sqrt{3}}$$

and determine when equality occurs.

Problem 225. A luminous point is in space. Is it possible to prevent its luminosity with a finite number of disjoint spheres of the same size? (Source: 2003-2004 Iranian Math Olympiad, Second Round)

********************** Solutions **********************

Problem 226. (Due to Alfred Eckstein, Arad, Romania) Solve the equation

$$4x^6 - 6x^3 + 2\sqrt{2} = 0.$$

Solution. Kwok Sze CHAI Charles (HKU, Math Major, Year 1), CHAN Tsz Lung, HUDREA Mihail (High School “Tiberiu Popoviciu” Cluj-Napoca Romania), MA HOI Sing (Shun Lee Catholic Secondary School, Form 5), Achilleas P. PORFYRIADIS (American College of Thessaloniki “Anatolia”, Thessaloniki, Greece), Anna YING PUN (STFA Leung Kau Kui College, Form 6), Badr SBAI (Morocco), TAM Yat Fung (Valtorta College, Form 5), WANG Wei Hua and WONG Kwok Cheung (Carmel Alison Lam Foundation Secondary School, Form 6).

We have $8x^6 - 12x^2 + 4\sqrt{2} = 0$.

Let $t = 2x^2$. We get

$$0 = t^3 - 6t + 4\sqrt{2} = t^3 - (\sqrt{2})^3 - (6t - 6\sqrt{2}) = (t - \sqrt{2})(t^2 + \sqrt{2}t - 4) = (t - \sqrt{2})(t - 2\sqrt{2})(t + 2\sqrt{2}).$$

Solving $2x^2 = \sqrt{2}$ and $2x^2 = -2\sqrt{2}$, we get $x = \pm 1/\sqrt{2}$ or $\pm i\sqrt{2}$.

Other commended solvers: CHAN Pak Woon (Wah Yan College, Kowloon, Form 7), Kin-Chit O (STFA Cheng Yu Tung Secondary School) and WONG Sze Wai (True Light Girls’ College, Form 4).

Problem 217. Prove that there exist infinitely many positive integers which cannot be represented in the form

$$x^3 + x^5 + x^7 + x^9 + x^{11},$$

where $x_1, x_2, x_3, x_4, x_5$ are positive integers. (Source: 2002 Belarusian Mathematical Olympiad, Final Round)

Solution. Achilleas P. PORFYRIADIS (American College of Thessaloniki “Anatolia”, Thessaloniki, Greece) and Tak Wai Alan WONG (Markham, ON, Canada).

On the interval $[1, n]$, if there is such an integer, then

$$x \leq [n^{1/3}], x_2 \leq [n^{1/5}], \ldots, x_5 \leq [n^{1/11}].$$

So the number of integers in $[1, n]$ of the required form is at most $n^{1/3}n^{1/5}n^{1/7}n^{1/9}n^{1/11} = n^{10/43}3^{46}$. Those not of the form is at least $n - n^{10/43}3^{46}$, which goes to infinity as $n$ goes to infinity.

Problem 218. Let $O$ and $P$ be distinct points on a plane. Let $ABCD$ be a parallelogram on the same plane such that its diagonals intersect at $O$. Suppose $P$ is not on the reflection of line $AB$ with respect to line $CD$. Let $M$ and $N$ be the midpoints of segments $AP$ and $BP$ respectively. Let $Q$ be the intersection of lines $MC$ and $ND$. Prove that $P$, $Q$, $O$ are collinear and the point $Q$ does not depend on the choice of parallelogram $ABCD$. (Source: 2004 National Math Olympiad in Slovenia, First Round)

Solution. HUDREA Mihail (High School “Tiberiu Popoviciu” Cluj-Napoca Romania) and Achilleas P. PORFYRIADIS (American College of Thessaloniki “Anatolia”, Thessaloniki, Greece).

Let $G_1$ be the intersection of $OP$ and $MC$. Since $OP$ and $MC$ are medians of triangle $APC$, $G_1$ is the centroid of triangle $APC$. Hence $OG_1 = \frac{2}{3}OP$. Similarly, let $G_2$ be the intersection of $OP$ and $ND$. Since $OP$ and $ND$ are medians of triangle $BDP$, $G_2$ is the centroid of triangle $BDP$. Hence $OG_2 = \frac{2}{3}OP$. So $G_1 = G_2$ and it is on both $MC$ and $ND$. Hence it is $Q$. This implies $P$, $Q$, $O$ are collinear and $Q$ is the unique point such that $OQ = \frac{2}{3}OP$, which does not depend on the choice of the parallelogram $ABCD$.

Other commended solvers: CHAN Pak Woon (Wah Yan College, Kowloon, Form 7) and CHAN Tsz Lung, Anna YING PUN (STFA Leung Kau Kui College, Form 6) and WONG Tsun Yu (St. Mark’s School, Form 5).

Problem 219. (Due to Dorin Mărghidanu, Coleg. Nat. “A.I. Cuza”, Corabia, Romania) The sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are defined as follows: $a_0, b_0 > 0$ and

$$a_{n+1} = a_n + \frac{1}{2b_n}, \quad b_{n+1} = b_n + \frac{1}{2a_n}$$

for $n = 1, 2, 3, \ldots$. Prove that

$$\max\{a_{2004}, b_{2004}\} > \sqrt{2005}.$$

Solution. CHAN Tsz Lung, Kin-Chit
Define the distance between least 3 of the solutions.

Then the distance between any \((a_i, a_j)\) with \(a_0 = 0\) or 1 and every \(A_i\) has at least 3 coordinates equal 1.

For \(k = 4\), we have \(n + 1 \leq 2\). Next, suppose \(k > 4\) and the inequality is true for the case \(k-1\).

In column \(k\) of the table, there are at least \([(n + 2)/2]\) of the numbers which are the same (all 0’s or all 1’s). Next we keep only \([(n+2)/2]\) rows whose \(k\)-th coordinates are the same and we remove column \(k\). The condition \(|A_i-A_j| \geq 3\) still holds for these new ordered \((k-1)\)-tuples.

By the case \(k-1\), we get \([(n + 2)/2]\) + 1 \(\leq 2^k\). Since \((n + 1)/2 < [(n + 2)/2] + 1\), we get \(n + 1 \leq 2^k - 3\) and case \(k\) is true.

### Generalization of Problem 203

**Naoki Sato**

We prove the following generalization of problem 203:

Let \(a_1, a_2, \ldots, a_n\) be real numbers, and let \(s_i\) be the sum of the products of the \(a_i\) taken \(i\) at a time. If \(s_i = 0\), then the equation

\[
s_i x^{s_{i-1}} + 2 s_2 x^{s_{i-2}} + \cdots + n s_n = 0
\]

has only real roots.

**Proof.** Let

\[
f(x) = s_1 x^{s_{i-1}} + 2 s_2 x^{s_{i-2}} + \cdots + n s_n.
\]

We can assume that none of the \(a_i\) are equal to 0, for if some of the \(a_i\) are equal to 0, then rearrange them so that \(a_1, \ldots, a_k\) are nonzero and \(a_{k+1}, \ldots, a_n\) are 0. Then \(s_{k+1} = s_{k+2} = \cdots = s_n = 0\), so

\[
f(x) = s_1 x^{s_{i-1}} + 2 s_2 x^{s_{i-2}} + \cdots + n s_n
\]

where \(s_i = s_{i+1} = \cdots = s_n = 0\), so

\[
= s_1 x^{s_{i-1}} + 2 s_2 x^{s_{i-2}} + \cdots + n s_n
\]

Thus, the problem reduces to proving the same result on the numbers \(a_1, a_2, \ldots, a_n\).

We claim that the roots of \(g(x) = 0\) are clearly real, namely \(-1/a_1, -1/a_2, \ldots, -1/a_n\). We claim that the roots of \(g(x) = 0\) are all real.

Suppose the roots of \(g(x) = 0\) are distinct. Let \(r_1 < r_2 < \cdots < r_n\) be these roots. Then by Rolle’s theorem, the equation \(g'(x) = 0\) has a root in each of the intervals \((r_1, r_2), (r_2, r_3), \ldots, (r_{n-1}, r_n)\), so it has \(n - 1\) real roots.

Now, suppose the equation \(g(x) = 0\) has \(j\) distinct roots \(r_1 < r_2 < \cdots < r_j\) and root \(r_k\) has multiplicity \(m_j\). Suppose \(r_1 < r_2 < \cdots < r_j\) and \(r_k\) are roots of \(g(x) = 0\) having multiplicity \(m_j\). In addition, again by Rolle’s theorem, the equation has a root in each of the intervals \((r_1, r_2), (r_2, r_3), \ldots, (r_{j-1}, r_j)\), so the equation \(g(x) = 0\) has the requisite

\[(m_1-1)(m_2-1)+\cdots+(m_j-1)+j-1=n-1\]

real roots.

Expanding, we have that

\[
g(x) = (a_i x^{s_{i+1}}) (a_j x^{s_{j+1}}) \cdots (a_n x^{s_n})
\]

So \(g'(x) = n s_1 x^{s_{i-1}} + n s_2 x^{s_{i-2}} + \cdots + n s_n\).

Since \(s_i \neq 0\), \(0\) is not a root of \(g'(x) = 0\). Finally, we get that the polynomial

\[
x^{s_{i-1}} g'(x) = s_1 x^{s_{i-1}} + 2 s_2 x^{s_{i-2}} + \cdots + n s_n
\]

has all real roots.

### Olympiad Corner

(continued from page 1)

#### Problem 5

Find all natural numbers \(n\) for which the number of all positive divisors of the number \(\text{lcm}(1,2,\ldots,n)\) is equal to \(2^k\) for some non-negative integer \(k\).

#### Problem 6

Call a convex polyhedron a footballoid if it has the following properties.

1. Any face is either a regular pentagon or a regular hexagon.
2. All neighbours of a pentagonal face are hexagonal (a neighbour of a face is a face that has a common edge with it).

Find all possibilities for the number of a pentagonal and hexagonal faces of a footballoid.