Mathematical Excalibur

Volume 10, Number 3 August 2005 – September 2005

Olympiad Corner

The 2005 International Mathematical Olympiad was held in Merida, Mexico on July 13 and 14. Below are the problems.

Problem 1. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC; B_1, B_2 on CA; C_1, C_2 on AB. These points are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

Problem 2. Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for each positive integer n, the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders on division by n. Prove that each integer occurs exactly once in the sequence.

Problem 3. Let x, y and z be positive real numbers such that $xyz \geq 1$. Prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}\geq 0.$$

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 30, 2005***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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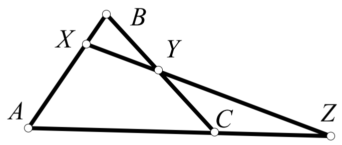
Famous Geometry Theorems

Kin Y. Li

There are many famous geometry theorems. We will look at some of them and some of their applications. Below we will write $P = WX \cap YZ$ to denote P is the point of intersection of lines WX and YZ. If points A, B, C are collinear, we will introduce the <u>*sign convention*</u>: $AB/BC = \overrightarrow{AB}/\overrightarrow{BC}$ (so if B is between A and C, then $AB/BC \geq 0$, *otherwise* $AB/BC \leq 0$).

<u>Menelaus' Theorem</u> Points X, Y, Z are taken from lines AB, BC, CA (which are the sides of $\triangle ABC$ extended) respectively. If there is a line passing through X, Y, Z, then

$$\frac{AX}{XB}\cdot\frac{BY}{YC}\cdot\frac{CZ}{ZA}=-1.$$



<u>Proof</u> Let L be a line perpendicular to the line through X, Y, Z and intersect it at O. Let A', B', C' be the feet of the perpendiculars from A, B, C to L respectively. Then

$$\frac{AX}{XB}=\frac{A'O}{OB'},\frac{BY}{YC}=\frac{B'O}{OC'},\frac{CZ}{ZA}=\frac{C'O}{OA'}.$$

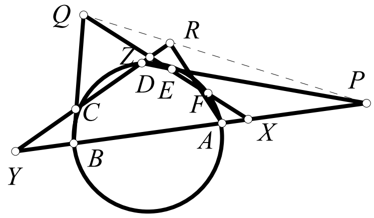
Multiplying these equations together, we get the result.

The converse of Menelaus' Theorem is also true. To see this, let $Z'=XY\cap CA$. Then applying Menelaus theorem to the line through X, Y, Z' and comparing with the equation above, we get $CZ/ZA=CZ'/Z'A$. It follows $Z=Z'$.

<u>Pascal's Theorem</u> Let A, B, C, D, E, F be points on a circle (which are not necessarily in cyclic order). Let

$$P=AB\cap DE,\ Q=BC\cap EF,\ R=CD\cap FA.$$

Then P,Q,R are collinear.



<u>Proof</u> Let $X = EF \cap AB$, $Y = AB \cap CD$, $Z = CD \cap EF$. Applying Menelaus' Theorem respectively to lines BC, DE, FA cutting $\triangle XYZ$ extended , we have

$$\frac{ZQ}{QX}\cdot\frac{XB}{BY}\cdot\frac{YC}{CZ}=-1,$$
$$\frac{XP}{PY}\cdot\frac{YD}{DZ}\cdot\frac{ZE}{EX}=-1,$$
$$\frac{YR}{RZ}\cdot\frac{ZF}{FX}\cdot\frac{XA}{AY}=-1.$$

Multiplying these three equations together, then using the intersecting chord theorem (see *vol* 4, *no*. 3, *p*. 2 of *Mathematical Excalibur*) to get $XA\cdot XB = XE\cdot XF$, $YC\cdot YD = YA\cdot YB$, $ZE\cdot ZF = ZC\cdot ZD$, we arrive at the equation

$$\frac{ZQ}{QX}\cdot\frac{XP}{PY}\cdot\frac{YR}{RZ}=-1.$$

By the converse of Menelaus' Theorem, this implies P, Q, R are collinear.

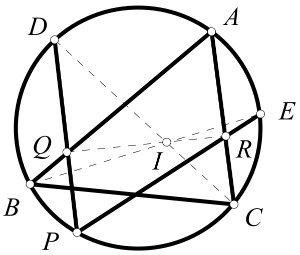
We remark that there are limiting cases of Pascal's Theorem. For example, we may move A to approach B. In the limit, A and B will coincide and the line AB will become the tangent line at B.

Below we will give some examples of using Pascal's Theorem in geometry problems.

<u>Example 1</u> (*2001 Macedonian Math Olympiad*) For the circumcircle of $\triangle ABC$, let D be the intersection of the tangent line at A with line BC, E be the intersection of the tangent line at B with line CA and F be the intersection of the tangent line at C with line AB. Prove that points D, E, F are collinear.

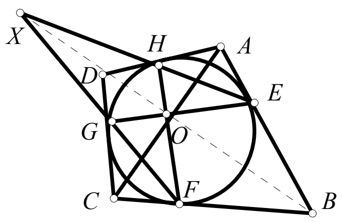
Solution Applying Pascal's Theorem to *A*, *A*, *B*, *B*, *C*, *C* on the circumcircle, we easily get *D*, *E*, *F* are collinear.

Example 2 Let *D* and *E* be the midpoints of the minor arcs *AB* and *AC* on the circumcircle of △ *ABC*, respectively. Let *P* be on the minor arc *BC*, *Q* = *DP* ∩ *BA* and *R* = *PE* ∩ *AC*. Prove that line *QR* passes through the incenter *I* of △*ABC*.



Solution Since *D* is the midpoint of arc *AB*, line *CD* bisects ∠ *ACB*. Similarly, line *EB* bisects ∠*ABC*. So *I* = *CD* ∩ *EB*. Applying Pascal's Theorem to *C*, *D*, *P*, *E*, *B*, *A*, we get *I*, *Q*, *R* are collinear.

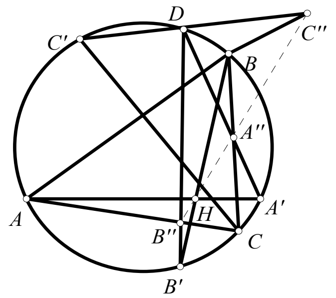
Newton's Theorem A circle is inscribed in a quadrilateral *ABCD* with sides *AB*, *BC*, *CD*, *DA* touch the circle at points *E*, *F*, *G*, *H* respectively. Then lines *AC*, *EG*, *BD*, *FH* are concurrent.



Proof. Let *O* = *EG*∩*FH* and *X* = *EH*∩*FG*. Since *D* is the intersection of the tangent lines at *G* and at *H* to the circle, applying Pascal's Theorem to *E,G,G,F,H,H*, we get *O*, *D*, *X* are collinear. Similarly, applying Pascal's Theorem to *E*, *E*, *H*, *F*, *F*, *G*, we get *B*, *X*, *O* are collinear.

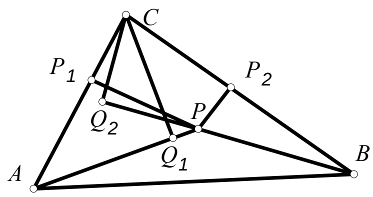
Then *B,O,D* are collinear and so lines *EG*, *BD*, *FH* are concurrent at *O*. Similarly, we can also obtain lines *AC*, *EG*, *FH* are concurrent at *O*. Then Newton's Theorem follows.

Example 3 (*2001 Australian Math Olympiad*) Let *A*, *B*, *C*, *A'*, *B'*, *C'* be points on a circle such that *AA'* is perpendicular to *BC*, *BB'* is perpendicular to *CA*, *CC'* is perpendicular to *AB*. Further, let *D* be a point on that circle and let *DA'* intersect *BC* in *A"*, *DB'* intersect *CA* in *B"*, and *DC'* intersect *AB* in *C"*, all segments being extended where required. Prove that *A"*, *B"*, *C"* and the orthocenter of triangle *ABC* are collinear.



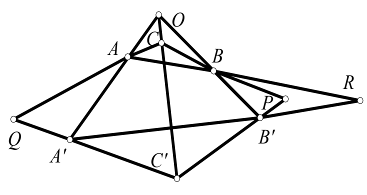
Solution Let *H* be the orthocenter of △ *ABC*. Applying Pascal's theorem to *A, A', D, C', C, B*, we see *H, A", C"* are collinear. Similarly, applying Pascal's theorem to *B', D, C', C, A, B*, we see *B", C", H* are collinear. So *A", B", C", H* are collinear.

Example 4 (*1991 IMO unused problem*) Let *ABC* be any triangle and *P* any point in its interior. Let P_1, P_2 be the feet of the perpendiculars from *P* to the two sides *AC* and *BC*. Draw *AP* and *BP* and from *C* drop perpendiculars to *AP* and *BP*. Let Q_1 and Q_2 be the feet of these perpendiculars. If $Q_2 \neq P_1$ and $Q_1 \neq P_2$, then prove that the lines P_1Q_2, Q_1P_2 and *AB* are concurrent.



Solution Since $\angle CP_1P$, $\angle CP_2P$, $\angle CQ_2P$, $\angle CQ_1P$ are all right angles, we see that the points *C*, Q_1, P_1, *P*, P_2, Q_2 lie on a circle with *CP* as diameter. Note $A = CP_1 \cap PQ_1$ and $B = Q_2P \cap P_2C$. Applying Pascal's theorem to *C*, P_1, Q_2, *P*, Q_1, P_2, we see $X = P_1Q_2 \cap Q_1P_2$ is on line *AB*.

Desargues' Theorem For △*ABC* and △ *A'B'C'*, if lines *AA'*, *BB'*, *CC'* concur at a point *O*, then points *P*, *Q*, *R* are collinear, where *P* = *BC* ∩ *B'C'*, *Q* = *CA* ∩ *C'A'*, *R* = *AB* ∩ *A'B'*.



Proof Applying Menelaus' Theorem respectively to line *A'B'* cutting △*OAB* extended, line *B'C'* cutting △ *OBC* extended and the line *C'A'* cutting △*OCA* extended, we have

$$\frac{OA'}{A'A}\cdot\frac{AR}{RB}\cdot\frac{BB'}{B'O}=-1,$$

$$\frac{OB'}{B'B}\cdot\frac{BP}{PC}\cdot\frac{CC'}{C'O}=-1,$$

$$\frac{AA'}{A'O}\cdot\frac{OC'}{C'C}\cdot\frac{CQ}{QA}=-1.$$

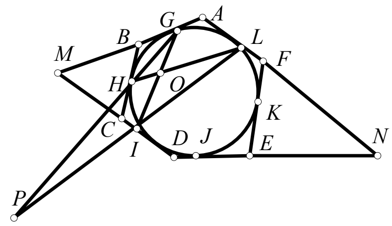
Multiplying these three equations,

$$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=-1.$$

By the converse of Menelaus' Theorem, this implies *P*, *Q*, *R* are collinear.

We remark that the converse of Desargues' Theorem is also true. We can prove it as follow: let *O* = *BB'* ∩ *CC'*. Consider △*RBB'* and △*QCC'*. Since lines *RQ*, *BC*, *B'C'* concur at *P*, and *A* = *RB* ∩ *QC*, *O* = *BB'* ∩ *CC'*, *A'* = *BR'* ∩ *C'Q*, by Desargues' Theorem, we have *A,O,A'* are collinear. Therefore, lines *AA'*, *BB'*, *CC'* concur at *O*.

Brianchon's Theorem Lines *AB*, *BC*, *CD*, *DE*, *EF*, *FA* are tangent to a circle at points *G*, *H*, *I*, *J*, *K*, *L* (not necessarily in cyclic order). Then lines *AD*, *BE*, *CF* are concurrent.



Proof Let *M* = *AB* ∩ *CD*, *N* = *DE* ∩ *FA*. Applying Newton's Theorem to quadrilateral *AMDN*, we see lines *AD*, *IL*, *GJ* concur at a point *A'*. Similarly, lines *BE*, *HK*, *GJ* concur at a point *B'* and lines *CF*, *HK*, *IL* concur at a point *C'*. Note line *IL* coincides with line *A'C'*. Next we apply Pascal's Theorem to *G*, *G*, *I*, *L*, *L*, *H* and get points *A*, *O*, *P* are collinear, where *O* = *GI* ∩ *LH* and *P* = *IL* ∩ *HG*. Applying Pascal's Theorem again to *H*, *H*, *L*, *I*, *I*, *G*, we get *C*, *O*, *P* are collinear. Hence *A*, *C*, *P* are collinear.

Now *G* = *AB* ∩ *A'B'*, *H* = *BC* ∩ *B'C'*, *P* = *CA* ∩ *IL* = *CA* ∩ *C'A'*. Applying the converse of Desargues' Theorem to △*ABC* and △*A'B'C'*, we get lines *AA'* = *AD*, *BB'* = *BE*, *CC'* = *CF* are concurrent.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***October 30, 2005.***

Problem 231. On each planet of a star system, there is an astronomer observing the nearest planet. The number of planets is odd, and pairwise distances between them are different. Prove that at least one planet is not under observation.
(*Source*: *1966 Soviet Union Math Olympiad*)

Problem 232. B and C are points on the segment AD. If $AB = CD$, prove that $PA + PD \geq PB + PC$ for any point P.
(*Source*: *1966 Soviet Union Math Olympiad*)

Problem 233. Prove that every positive integer not exceeding $n!$ can be expressed as the sum of at most n distinct positive integers each of which is a divisor of $n!$.

Problem 234. Determine all polynomials $P(x)$ of the smallest possible degree with the following properties:

a) The coefficient of the highest power is 200.
b) The coefficient of the lowest power for which it is not equal to zero is 2.
c) The sum of all its coefficients is 4.
d) $P(-1) = 0$, $P(2) = 6$ and $P(3) = 8$.

(*Source: 2002 Austrian National Competition*)

Problem 235. Forty-nine students solve a set of three problems. The score for each problem is an integer from 0 to 7. Prove that there exist two students A and B such that, for each problem, A will score at least as many points as B.

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Solutions

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Problem 226. Let $z_1, z_2, \ldots, z_n$ be complex numbers satisfying

$$|z_1| + |z_2| + \cdots + |z_n| = 1.$$

Prove that there is a nonempty subset of $\{z_1, z_2, \ldots, z_n\}$ the sum of whose elements has modulus at least 1/4.

Solution. **LEE Kai Seng** (HKUST).

Let $z_k = a_k + b_k i$ with a_k, b_k real. Then $|z_k| \leq |a_k| + |b_k|$. So

$$1 = \sum_{k=1}^{n} |z_k| \leq \sum_{k=1}^{n} |a_k| + \sum_{k=1}^{n} |b_k|$$

$$= \sum_{a_k \geq 0} a_k + \sum_{a_k < 0} (-a_k) + \sum_{b_k \geq 0} b_k + \sum_{b_k < 0} (-b_k).$$

Hence, one of the four sums is at least 1/4, say $\sum_{a_k \geq 0} a_k \geq \frac{1}{4}$. Then

$$\left|\sum_{a_k \geq 0} z_k\right| \geq \left|\sum_{a_k \geq 0} a_k\right| \geq \frac{1}{4}.$$

Problem 227. For every integer $n \geq 6$, prove that

$$\sum_{k=1}^{n-1} \frac{n}{n-k} \cdot \frac{1}{2^{k-1}} \leq \frac{16}{5}.$$

Comments. In the original statement of the problem, the displayed inequality was stated incorrectly. The < sign should be an ≤ sign.

Solution. **CHAN Pak Woon** (Wah Yan College, Kowloon, Form 7), **Roger CHAN** (Vancouver, Canada) and **LEE Kai Seng** (HKUST).

For $n = 6, 7, \ldots$, let

$$a_n = \sum_{k=1}^{n-1} \frac{n}{n-k} \cdot \frac{1}{2^{k-1}}.$$

Then $a_6 = 16/5$. For $n \geq 6$, if $a_n \leq 16/5$, then

$$a_{n+1} = \sum_{k=1}^{n} \frac{n+1}{n+1-k} \cdot \frac{1}{2^{k-1}} = \sum_{j=0}^{n-1} \frac{n+1}{n-j} \cdot \frac{1}{2^j}$$

$$= \frac{n+1}{n} + \frac{n+1}{2n} \sum_{j=1}^{n-1} \frac{n}{n-j} \cdot \frac{1}{2^{j-1}}$$

$$= \frac{n+1}{n}\left(1 + \frac{a_n}{2}\right) \leq \frac{7}{6}\left(1 + \frac{8}{5}\right) < \frac{16}{5}.$$

The desired inequality follows by mathematical induction.

Problem 228. In $\triangle ABC$, M is the foot of the perpendicular from A to the angle bisector of $\angle BCA$. N and L are respectively the feet of perpendiculars from A and C to the bisector of $\angle ABC$. Let F be the intersection of lines MN and AC. Let E be the intersection of lines BF and CL. Let D be the intersection of lines BL and AC.

Prove that lines DE and MN are parallel.

Solution. **Roger CHAN** (Vancouver, Canada).

Extend AM to meet BC at G and extend AN to meet BC at I. Then $AM = MG$, $AN = NI$ and so lines MN and BC are parallel.

From $AM = MG$, we get $AF = FC$. Extend CL to meet line AB at J. Then $JL = LC$. So lines LF and AB are parallel.

Let line LF intersect BC at H. Then $BH = HC$. In $\triangle BLC$, segments BE, LH and CD concur at F. By Ceva's theorem (see *vol.* 2, *no.* 5, *pp.* 1-2 of *Mathematical Excalibur*),

$$\frac{BH}{HC} \cdot \frac{CE}{EL} \cdot \frac{LD}{DB} = 1.$$

Since $BH = HC$, we get $CE/EL = DB/LD$, which implies lines DE and BC are parallel. Therefore, lines DE and MN are parallel.

Problem 229. For integer $n \geq 2$, let a_1, a_2, a_3, a_4 be integers satisfying the following two conditions:

(1) for $i = 1, 2, 3, 4$, the greatest common divisor of n and a_i is 1 and
(2) for every $k = 1, 2, \ldots, n-1$, we have

$$(ka_1)_n + (ka_2)_n + (ka_3)_n + (ka_4)_n = 2n,$$

where $(a)_n$ denotes the remainder when a is divided by n.

Prove that $(a_1)_n, (a_2)_n, (a_3)_n, (a_4)_n$ can be divided into two pairs, each pair having sum equals n.
(*Source: 1992 Japanese Math Olympiad*)

Solution. (***Official Solution***)

Since n and a_1 are relatively prime, the remainders $(a_1)_n, (2a_1)_n, \ldots, ((n-1)a_1)_n$ are nonzero and distinct. So there is a k among $1, 2, \ldots, n-1$ such that $(ka_1)_n = 1$. Note that such k is relatively prime to n. If $(ka_1)_n + (ka_j)_n = n$, then $ka_1 + ka_j \equiv 0 \pmod n$ so that $a_1 + a_j \equiv 0 \pmod n$ and $(a_1)_n + (a_j)_n = n$. Thus, to solve the problem, we may replace a_i by $(ka_i)_n$ and assume $1 = a_1 \leq a_2 \leq a_3 \leq$

$a_4 \le n-1$. By condition (2), we have

$$1+a_2+a_3+a_4=2n. \qquad \text{(A)}$$

For $k = 1, 2, \ldots, n-1$, let

$$f_i(k) = [ka_i/n] - [(k-1)a_i/n],$$

then $f_i(k) \le (ka_i/n) + 1 - (k-1)a_i/n = 1 + (a_i/n) < 2$. So $f_i(k) = 0$ or 1. Since $x = [x/n]n + (x)_n$, subtracting the case $x = ka_i$ from the case $x = (k-1)a_i$, then summing $i = 1, 2, 3, 4$, using condition (2) and (A), we get

$$f_1(k) + f_2(k) + f_3(k) + f_4(k) = 2.$$

Since $a_1 = 1$, we see $f_1(k)=0$ and exactly two of $f_2(k), f_3(k), f_4(k)$ equal 1. (B)

Since $a_i<n, f_i(2) = [2a_i/n]$. Since $a_2 \le a_3 \le a_4 < n$, we get $f_2(2) = 0, f_3(2) = f_4(2) = 1$, i.e. $1 = a_1 \le a_2 < n/2 < a_3 \le a_4 \le n-1$.

Let $t_2 = [n/a_2] + 1$, then $f_2(t_2) = [t_2a_2/n] - [(t_2-1)a_2/n] = 1 - 0 = 1$. If $1 \le k < t_2$, then $k < n/a_2$, $f_2(k) = [ka_2/n] - [(k-1)a_2/n] = 0 - 0 = 0$. Next if $f_2(j) = 1$, then $f_2(k) = 0$ for $j < k < j + t_2 - 1$ and exactly one of $f_2(j + t_2 - 1)$ or $f_2(j + t_2) = 1$. (C)

Similarly, for $i = 3, 4$, let $t_i = [n/(n - a_i)] + 1$, then $f_i(t_i) = 0$ and $f_i(k) = 1$ for $1 \le k < t_i$. Also, if $f_i(j) = 0$, then $f_i(k) = 1$ for $j < k < j + t_i - 1$ and exactly one of $f_i(j + t_i - 1)$ or $f_i(j + t_i) = 0$. (D)

Since $f_3(t_3) = 0$, by (B), $f_2(t_3) = 1$. If $k < t_3 \le t_4$, then by (D), $f_3(k) = f_4(k) = 1$. So by (B), $f_2(k) = 0$. Then by (C), $t_2 = t_3$.

Assume $t_4 < n$. Since $n/2 < a_4 < n$, we get $f_4(n-1) = (a_4 - 1) - (a_4 - 2) = 1 \ne 0 = f_4(t_4)$ and so $t_4 \ne n-1$. Also, $f_4(t_4) = 0$ implies $f_2(t_4) = f_3(t_4) = 1$ by (B).

Since $f_3(t_3) = 0 \ne 1 = f_3(t_4)$, $t_3 \ne t_4$. Thus $t_2 = t_3 < t_4$. Let $s < t_4$ be the largest integer such that $f_2(s) = 1$. Since $f_2(t_4) = 1$, we have $t_4 = s + t_2 - 1$ or $t_4 = s + t_2$. Since $f_2(s) = f_4(s) = 1$, we get $f_3(s) = 0$. As $t_2 = t_3$, we have $t_4 = s + t_3 - 1$ or $t_4 = s + t_3$. Since $f_3(s) = 0$ and $f_3(t_4) = 1$, by (D), we get $f_3(t_4 - 1) = 0$ or $f_3(t_4 + 1) = 0$. Since $f_2(s) = 1, f_2(t_4) = 1$ and $t_2 > 2$, by (C), we get $f_2(s + 1) = 0$ and $f_2(t_4 + 1) = 0$. So $s + 1 \ne t_4$, which implies $f_2(t_4 - 1) = 0$ by the definition of s. Then $k = t_4 - 1$ or $t_4 + 1$ contradicts (B).

So $t_4 \ge n$, then $n - a_4 = 1$. We get $a_1 + a_4 = n = a_2 + a_3$.

Problem 230. Let k be a positive integer. On the two sides of a river, there are in total at least 3 cities. From each of these cities, there are exactly k routes, each connecting the city to a distinct city on the other side of the river. Via these routes, people in every city can reach any one of the other cities.

Prove that if any one route is removed, people in every city can still reach any one of the other cities via the remaining routes.
(*Source: 1996 Iranian Math Olympiad, Round 2*)

Solution. **LEE Kai Seng** (HKUST).

Associate each city with a vertex of a graph. Suppose there are X and Y cities to the left and to the right of the river respectively. Then the number of routes (or edges of the graph) in the beginning is $Xk = Yk$ so that $X = Y$. We have $X + Y \ge 3$.

After one route between city A and city B is removed, assume the cities can no longer be connected via the remaining routes. Then each of the other cities can only be connected to exactly one of A or B. Then the original graph decomposes into two connected graphs G_A and G_B, where G_A has A as vertex and G_B has B as vertex.

Let X_A be the number of cities among the X cities on the left sides of the river that can still be connected to A after the route between A and B was removed and similarly for X_B, Y_A, Y_B. Then the number of edges in G_A is $X_Ak-1 = Y_Ak$. Then $(X_A - Y_A)k = 1$. So $k = 1$. Then in the beginning $X = 1$ and $Y = 1$, contradicting $X + Y \ge 3$.

Olympiad Corner

(continued from page 1)

Problem 4. Consider the sequence a_1, a_2, … defined by

$$a_n = 2^n + 3^n + 6^n - 1 \quad (n = 1, 2, \ldots)$$

Determine all positive integers that are relatively prime to every term of the sequence.

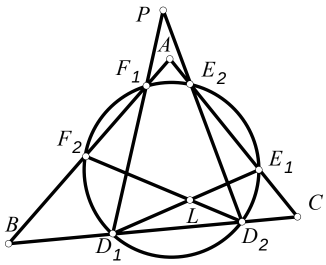
Problem 5. Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

Problem 6. In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than 2/5 of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.

Famous Geometry Theorems

(continued from page 2)

Example 5 (*2005 Chinese Math Olympiad*) A circle meets the three sides BC, CA, AB of triangle ABC at points D_1, D_2; E_1, E_2 and F_1, F_2 in turn. The line segments D_1E_1 and D_2F_2 intersect at point L, line segments E_1F_1 and E_2D_2 intersect at point M, line segments F_1D_1 and F_2E_2 intersect at point N. Prove that the three lines AL, BM and CN are concurrent.



Solution. Let $P = D_1F_1 \cap D_2E_2$, $Q = E_1D_1 \cap E_2F_2$, $R = F_1E_1 \cap F_2D_2$. Applying Pascal's Theorem to E_2, E_1, D_1, F_1, F_2, D_2, we get A, L, P are collinear. Applying Pascal's Theorem to F_2, F_1, E_1, D_1, D_2, E_2, we get B, M, Q are collinear. Applying Pascal's Theorem to D_2, D_1, F_1, E_1, E_2, F_2, we get C, N, R are collinear.

Let $X = E_2E_1 \cap D_1F_2 = CA \cap D_1F_2$, $Y = F_2F_1 \cap E_1D_2 = AB \cap E_1D_2$, $Z = D_2D_1 \cap F_1E_2 = BC \cap F_1E_2$. Applying Pascal's Theorem to D_1, F_1, E_1, E_2, D_2, F_2, we get P, R, X are collinear. Applying Pascal's Theorem to E_1, D_1, F_1, F_2, E_2, D_2, we get Q, P, Y are collinear. Applying Pascal's Theorem to F_1, E_1, D_1, D_2, F_2, E_2, we get R, Q, Z are collinear.

For $\triangle ABC$ and $\triangle PQR$, we have $X = CA \cap RP$, $Y = AB \cap PQ$, $Z = BC \cap QR$. By the converse of Desargues' Theorem, lines $AP = AL$, $BQ = BM$, $CR = CN$ are concurrent.