Mathematical Excalibur

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Olympiad Corner

Below was Slovenia's Selection Examinations for the IMO 2005.

First Selection Examination

Problem 1. Let M be the intersection of diagonals AC and BD of the convex quadrilateral $ABCD$. The bisector of angle ACD meets the ray BA at the point K. Prove that if $MA{\cdot}MC + MA{\cdot}CD = MB{\cdot}MD$, then $\angle BKC = \angle BDC$.

Problem 2. Let R_+ be the set of all positive real numbers. Find all functions $f\colon R_+\to R_+$ such that $x^2(f(x)+f(y)) = (x+y)f(f(x)y)$ holds for any positive real numbers x and y.

Problem 3. Find all pairs of positive integers (m, n) such that the numbers m^2-4n and n^2-4m are perfect squares.

Second Selection Examination

Problem 1. How many sequences of 2005 terms are there such that the following three conditions hold:

(a) no sequence has three consecutive terms equal to each other,
(b) every term of every sequence is equal to 1 or −1, and

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 16, 2006***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Muirhead's Inequality

Lau Chi Hin

Muirhead's inequality is an important generalization of the AM-GM inequality. It is a powerful tool for solving inequality problem. First we give a definition which is a generalization of arithmetic and geometric means.

Definition. Let $x_1, x_2, \ldots, x_n$ be positive real numbers and $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$. The <u>*p-mean*</u> of $x_1, x_2, \ldots, x_n$ is defined by

$$[p] = \frac{1}{n!}\sum_{\sigma\in S_n} x_{\sigma(1)}^{p_1} x_{\sigma(2)}^{p_2}\cdots x_{\sigma(n)}^{p_n},$$

where S_n is the set of all permutations of $\{1,2,\ldots, n\}$. (The summation sign means to sum $n!$ terms, one term for each permutation σ in S_n.)

For example, $[(1,0,\ldots,0)] = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the arithmetic mean of $x_1, x_2, \ldots, x_n$ and $[(1/n,1/n,\ldots,1/n)] = x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n}$ is their geometric mean.

Next we introduce the concept of majorization in $\mathbb{R}^n$. Let $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$ satisfy conditions

1. $p_1 \ge p_2 \ge \cdots \ge p_n$ and $q_1 \ge q_2 \ge \cdots \ge q_n$,
2. $p_1 \ge q_1$, $p_1+p_2 \ge q_1+q_2$, … , $p_1+p_2+\cdots+p_{n-1} \ge q_1+q_2+\cdots+q_{n-1}$ and
3. $p_1+p_2+\cdots+p_n = q_1+q_2+\cdots+q_n$.

Then we say $(p_1, p_2, \ldots, p_n)$ <u>*majorizes*</u> $(q_1, q_2, \ldots, q_n)$ and write

$$(p_1, p_2, \ldots, p_n) \succ (q_1, q_2, \ldots, q_n).$$

Theorem (Muirhead's Inequality). Let $x_1, x_2, \ldots, x_n$ be positive real numbers and $p, q \in \mathbb{R}^n$. If $p \succ q$, then $[p] \ge [q]$. Furthermore, for $p \ne q$, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Since $(1,0,\ldots,0) \succ (1/n,1/n,\ldots,1/n)$, AM-GM inequality is a consequence.

Example 1. For any $a, b, c > 0$, prove that

$$(a+b)(b+c)(c+a) \ge 8abc.$$

Solution. Expanding both sides, the desired inequality is

$$a^2b+a^2c+b^2c+b^2a+c^2a+c^2b \ge 6abc.$$

This is equivalent to $[(2,1,0)] \ge [(1,1,1)]$, which is true by Muirhead's inequality since $(2,1,0)\succ(1,1,1)$.

For the next example, we would like to point out a useful trick. When the product of $x_1, x_2, \ldots, x_n$ is 1, we have

$$[(p_1, p_2, \ldots, p_n)] = [(p_1-r, p_2-r, \ldots, p_n-r)]$$

for any real number r.

Example 2. (*IMO 1995*) For any $a, b, c > 0$ with $abc = 1$, prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \ge \frac{3}{2}.$$

Solution. Multiplying by the common denominator and expanding both sides, the desired inequality is

$$\begin{aligned}&2(a^4b^4+b^4c^4+c^4a^4)\\&+2(a^4b^3c+a^4c^3b+b^4c^3a+b^4a^3c+c^4a^3b\\&+c^4b^3a)+2(a^3b^3c^2+b^3c^3a^2+c^3a^3b^2)\\&\ge 3(a^5b^4c^3+a^5c^4b^3+b^5c^4a^3+b^5a^4c^3\\&+c^5a^4b^3+c^5b^4a^3)+6a^4b^4c^4.\end{aligned}$$

This is equivalent to $[(4,4,0)]+2[(4,3,1)] + [(3,3,2)] \ge 3[(5,4,3)] + [(4,4,4)]$. Note $4+4+0 = 4+3+1 = 3+3+2 = 8$, but $5+4+3 = 4+4+4 = 12$. So we can set $r = 4/3$ and use the trick above to get $[(5,4,3)] = [(11/3,8/3,5/3)]$ and also $[(4,4,4)] = [(8/3,8/3,8/3)]$.

Observe that $(4,4,0) \succ (11/3,8/3,5/3)$, $(4,3,1) \succ (11/3,8/3,5/3)$ and $(3,3,2) \succ (8/3,8/3,8/3)$. So applying Muirhead's inequality to these three majorizations and adding the inequalities, we get the desired inequality.

Example 3. (*1990 IMO Shortlisted Problem*) For any $x, y, z > 0$ with $xyz = 1$, prove that

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)}\ge\frac{3}{4}.$$

Solution. Multiplying by the common denominator and expanding both sides, the desired inequality is

$$4(x^4+y^4+z^4+x^3+y^3+z^3)$$
$$\ge 3(1+x+y+z+xy+yz+zx+xyz).$$

This is equivalent to $4[(4,0,0)] + 4[(3,0,0)] \ge [(0,0,0)] + 3[(1,0,0)] + 3[(1,1,0)] + [(1,1,1)]$.

For this, we apply Muirhead's inequality and the trick as follow:

$[(4,0,0)] \ge [(4/3,4/3,4/3)] = [(0,0,0)]$,
$3[(4,0,0)] \ge 3[(2,1,1)] = 3[(1,0,0)]$,
$3[(3,0,0)] \ge 3[(4/3,4/3,1/3)] = 3[(1,1,0)]$
and $[(3,0,0)] \ge [(1,1,1)]$.

Adding these, we get the desired inequality.

Remark. For the following example, we will modify the trick above. In case $xyz \ge 1$, we have

$$[(p_1, p_2, p_3)] \ge [(p_1–r, p_2–r, p_3–r)]$$

for every $r \ge 0$. Also, we will use the following

Fact. For $p, q \in \mathbb{R}^n$, we have

$$\frac{[p]+[q]}{2}\ge\left[\frac{p+q}{2}\right].$$

This is because by the AM-GM inequality,

$$\frac{x_{\sigma(1)}^{p_1}\cdots x_{\sigma(n)}^{p_n}+x_{\sigma(1)}^{q_1}\cdots x_{\sigma(n)}^{q_n}}{2}$$
$$\ge x_{\sigma(1)}^{(p_1+q_1)/2}\cdots x_{\sigma(n)}^{(p_n+q_n)/2}.$$

Summing over $\sigma\in S_n$ and dividing by $n!$, we get the inequality.

Example 4. (*2005 IMO*) For any $x, y, z > 0$ with $xyz \ge 1$, prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}\ge 0.$$

Solution. Multiplying by the common denominator and expanding both sides, the desired inequality is equivalent to $[(9,0,0)]+4[(7,5,0)]+[(5,2,2)]+[(5,5,5)] \ge [(6,0,0)] + [(5,5,2)] + 2[(5,4,0)] + 2[(4,2,0)] + [(2,2,2)]$.

To prove this, we note that

(1) $[(9,0,0)] \ge [(7,1,1)] \ge [(6,0,0)]$

(2) $[(7,5,0)] \ge [(5,5,2)]$

(3) $2[(7,5,0)] \ge 2[(6,5,1)] \ge 2[(5,4,0)]$

(4) $[(7,5,0)] + [(5,2,2)] \ge 2[(6,7/2,1)]$
$\ge 2[(9/2,2,–1/2)] \ge 2[(4,2,0)]$

(5) $[(5,5,5)] \ge [(2,2,2)]$,

where (1) and (3) are by Muirhead's inequality and the remark, (2) is by Muirhead's inequality, (4) is by the fact, Muirhead's inequality and the remark and (5) is by the remark.

Considering the sum of the leftmost parts of these inequalities is greater than or equal to the sum of the rightmost parts of these inequalities, we get the desired inequalities.

Alternate Solution. Since

$$\frac{x^5-x^2}{x^5+y^2+z^2}-\frac{x^5-x^2}{x^3(x^2+y^2+z^2)}$$
$$=\frac{(x^3-1)^2(y^2+z^2)}{x(x^2+y^2+z^2)(x^5+y^2+z^2)}\ge 0,$$

we have

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}$$
$$\ge\frac{x^5-x^2}{x^3(x^2+y^2+z^2)}+\frac{y^5-y^2}{y^3(y^2+z^2+x^2)}+\frac{z^5-z^2}{z^3(z^2+x^2+y^2)}$$
$$\ge\frac{1}{x^2+y^2+z^2}\left(x^2-\frac{1}{x}+y^2-\frac{1}{y}+z^2-\frac{1}{z}\right)$$
$$\ge\frac{1}{x^2+y^2+z^2}(x^2+y^2+z^2-yz-zx-xy)$$
$$=\frac{(x-y)^2+(y-z)^2+(z-x)^2}{2(x^2+y^2+z^2)}\ge 0.$$

Proofs of Muirhead's Inequality

Kin Yin Li

Let $p \succ q$ and $p \ne q$. From $i = 1$ to n, the first nonzero $p_i - q_i$ is positive by condition 2 of majorization. Then there is a negative $p_i - q_i$ later by condition 3. It follows that there are $j < k$ such that $p_j > q_j$, $p_k < q_k$ and $p_i = q_i$ for any possible i between j, k.

Let $b = (p_j+p_k)/2$, $d = (p_j–p_k)/2$ so that $[b–d,b+d] = [p_k, p_j] \supset [q_k, q_j]$. Let c be the maximum of $|q_j–b|$ and $|q_k–b|$, then $0 \le c < d$. Let $r = (r_1,\ldots,r_n)$ be defined by $r_i = p_i$ except $r_j = b + c$ and $r_k = b - c$. By the definition of c, either $r_j = q_j$ or $r_k=q_k$. Also, by the definitions of b, c, d, we get $p \succ r, p \ne r$ and $r \succ q$. Now

$$n!([p]-[r])=\sum_{\sigma\in S_n}x_\sigma\left(x_{\sigma(j)}^{p_j}x_{\sigma(k)}^{p_k}-x_{\sigma(j)}^{r_j}x_{\sigma(k)}^{r_k}\right)$$
$$=\sum_{\sigma\in S_n}x_\sigma\left(u^{b+d}v^{b-d}-u^{b+c}v^{b-c}\right),$$

where x_σ is the product of $x_{\sigma(i)}^{p_i}$ for $i \ne j, k$ and $u = x_{\sigma(j)}$, $v = x_{\sigma(k)}$. For each permutation σ, there is a permutation ρ such that $\sigma(i) = \rho(i)$ for $i \ne j, k$ and $\sigma(j) = \rho(k)$, $\sigma(k) = \rho(j)$. In the above sum, if we pair the terms for σ and ρ, then $x_\sigma = x_\rho$ and combining the parenthetical factors for the σ and ρ terms, we have

$$(u^{b+d}v^{b-d}-u^{b+c}v^{b-c})+(v^{b+d}u^{b-d}-v^{b+c}u^{b-c})$$
$$=u^{b-d}v^{b-d}(u^{d+c}-v^{d+c})(u^{d-c}-v^{d-c})\ge 0.$$

So the above sum is nonnegative. Then $[p] \ge [r]$. Equality holds if and only if $u = v$ for all pairs of σ and ρ, which yields $x_1 = x_2 = \cdots = x_n$. Finally we recall r has at least one more coordinate in agreement with q than p. So repeating this process finitely many times, we will eventually get the case $r = q$. Then we are done.

Next, <u>*for the advanced readers*</u>, we will outline a longer proof, which tells more of the story. It is consisted of two steps. The first step is to observe that if $c_1, c_2, \ldots, c_k \ge 0$ with sum equals 1 and $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$, then

$$\sum_{i=1}^{k}c_i[v_i]\ge\left[\sum_{i=1}^{k}c_iv_i\right].$$

This follows by using the weighted AM-GM inequality instead in the proof of the fact above. (For the statement of the weighted AM-GM inequality, see *Mathematical Excalibur*, *vol.* 5, *no.* 4, *p.* 2, remark in column 1).

The second step is the difficult step of showing $p \succ q$ implies there exist nonnegative numbers $c_1, c_2, \ldots, c_{n!}$ with sum equals 1 such that

$$q=\sum_{i=1}^{n!}c_iP_i,$$

where $P_1, P_2, \ldots, P_{n!} \in \mathbb{R}^n$ whose coordinates are the $n!$ permutations of the coordinates of p. Muirhead's inequality follows immediately by applying the first step and observing that $[P_i]=[p]$ for $i=1,2,\ldots, n!$.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***April 16, 2006.***

Problem 246. A spy plane is flying at the speed of 1000 kilometers per hour along a circle with center A and radius 10 kilometers. A rocket is fired from A at the same speed as the spy plane such that it is always on the radius from A to the spy plane. Prove such a path for the rocket exists and find how long it takes for the rocket to hit the spy plane. (*Source: 1965 Soviet Union Math Olympiad*)

Problem 247. (*a*) Find all possible positive integers $k \ge 3$ such that there are k positive integers, every two of them are not relatively prime, but every three of them are relatively prime.

(*b*) Determine with proof if there exists an infinite sequence of positive integers satisfying the conditions in (*a*) above.
(*Source: 2003 Belarussian Math Olympiad*)

Problem 248. Let $ABCD$ be a convex quadrilateral such that line CD is tangent to the circle with side AB as diameter. Prove that line AB is tangent to the circle with side CD as diameter if and only if lines BC and AD are parallel.

Problem 249. For a positive integer n, if $a_1,\cdots, a_n, b_1, \cdots, b_n$ are in [1,2] and $a_1^2+\cdots+a_n^2=b_1^2+\cdots+b_n^2$, then prove that

$$\frac{a_1^3}{b_1}+\cdots+\frac{a_n^3}{b_n}\le\frac{17}{10}(a_1^2+\cdots+a_n^2).$$

Problem 250. Prove that every region with a convex polygon boundary cannot be dissected into finitely many regions with nonconvex quadrilateral boundaries.

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Solutions

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Problem 241. Determine the smallest possible value of

$$S = a_1\cdot a_2\cdot a_3+b_1\cdot b_2\cdot b_3+c_1\cdot c_2\cdot c_3,$$

if a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 is a permutation of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9. (*Source: 2002 Belarussian Math. Olympiad*)

Solution. **CHAN Ka Lok** (STFA Leung Kau Kui College), **CHAN Tsz Lung** (HKU Math PG Year 1), **G.R.A. 20 Math Problem Group** (Roma, Italy), **D. Kipp JOHNSON** (Valley Catholic School, Beaverton, OR, USA, teacher), **KWOK Lo Yan** (Carmel Divine Grace Foundation Secondary School, Form 6), **Problem Solving Group @ Miniforum** and **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School).

The idea is to get the 3 terms as close as possible. We have 214 = 70 + 72 + 72 = $2\cdot5\cdot7 + 1\cdot8\cdot9 + 3\cdot4\cdot6$. By the AM-GM inequality, $S \ge 3(9!)^{1/3}$. Now $9! = 70\cdot72\cdot72 > 70\cdot73\cdot71 > 71^3$. So $S > 3\cdot71 = 213$. Therefore, 214 is the answer.

Problem 242. Prove that for every positive integer n, 7 is a divisor of 3^n+n^3 if and only if 7 is a divisor of 3^nn^3+1. (*Source: 1995 Bulgarian Winter Math Competition*)

Solution. **CHAN Tsz Lung** (HKU Math PG Year 1), **G.R.A. 20 Math Problem Group** (Roma, Italy), **D. Kipp JOHNSON** (Valley Catholic School, Beaverton, OR, USA, teacher), **KWOK Lo Yan** (Carmel Divine Grace Foundation Secondary School, Form 6), **Problem Solving Group @ Miniforum**, **Tak Wai Alan WONG** (Markham, ON, Canada) and **YUNG Fai**.

Note $3^n \not\equiv 0 \pmod 7$. If $n \not\equiv 0 \pmod 7$, then $n^3 \equiv 1$ or $-1 \pmod 7$. So 7 is a divisor of 3^n+n^3 if and only if $-3^n \equiv n^3 \equiv 1 \pmod 7$ or $-3^n \equiv n^3 \equiv -1 \pmod 7$ if and only if 7 is a divisor of 3^nn^3+1.

Commended solvers: **CHAN Ka Lok** (STFA Leung Kau Kui College), **LAM Shek Kin** (TWGHs Lui Yun Choy Memorial College) and **WONG Kai Cheuk** (Carmel Divine Grace Foundation Secondary School, Form 6).

Problem 243. Let R^+ be the set of all positive real numbers. Prove that there is no function $f: R^+ \to R^+$ such that

$$(f(x))^2 \ge f(x+y)(f(x)+y)$$

for arbitrary positive real numbers x and y. (*Source: 1998 Bulgarian Math Olympiad*)

Solution. **José Luis DíAZ-BARRERO**, (Universitat Politècnica de Catalunya, Barcelona, Spain).

Assume there is such a function. We rewrite the inequality as

$$f(x)-f(x+y)\ge\frac{f(x)y}{f(x)+y}.$$

Note the right side is positive. This implies $f(x)$ is a strictly decreasing.

First we prove that $f(x) - f(x+1) \ge 1/2$ for $x > 0$. Fix $x > 0$ and choose a natural number n such that $n \ge 1 / f(x+1)$. When $k = 0, 1, \ldots, n-1$, we obtain

$$f(x+\frac{k}{n})-f(x+\frac{k+1}{n})$$

$$\ge\frac{f(x+\frac{k}{n})\frac{1}{n}}{f(x+\frac{k}{n})+\frac{1}{n}}\ge\frac{1}{2n}.$$

Adding the above inequalities, we get $f(x) - f(x+1) \ge 1/2$.

Let m be a positive integer such that $m \ge 2f(x)$. Then

$$f(x)-f(x+m)=\sum_{i=1}^{m}(f(x+i-1)-f(x+i))$$

$$\ge m/2 \ge f(x).$$

So $f(x+m) \le 0$, a contradiction.

Commended solvers: **Problem Solving Group @ Miniforum**.

Problem 244. An infinite set S of coplanar points is given, such that every three of them are not collinear and every two of them are not nearer than 1cm from each other. Does there exist any division of S into two disjoint infinite subsets R and B such that inside every triangle with vertices in R is at least one point of B and inside every triangle with vertices in B is at least one point of R? Give a proof to your answer. (*Source: 2002 Albanian Math Olympiad*)

Solution.(Official Solution)

Assume that such a division exists and let M_1 be a point of R. Then take four points M_2, M_3, M_4, M_5 different from M_1, which are the nearest points to M_1 in R. Let r be the largest distance between M_1 and each of these four points. Let H be the convex hull of these five points. Then the interior of H lies inside the circle of radius r centered at M_1, but all other points of R is outside or on the circle. Hence the interior of H does not contain any other point of R.

Below we will say two triangles are <u>*disjoint*</u> if their interiors do not intersect. There are 3 possible cases:

(*a*) H is a pentagon. Then H may be divided into three disjoint triangles with vertices in R, each of them containing a point of B inside. The triangle with these points of B as vertices would contain another point of R, which would be in H. This is impossible.

(*b*) H is a quadrilateral. Then one of the M_i is inside H and the other M_j, M_k, M_l, M_m are at its vertices, say clockwise. The four disjoint triangles $M_iM_jM_k$, $M_iM_kM_l$, $M_iM_lM_m$, $M_iM_mM_i$ induce four points of B, which can be used to form two disjoint triangles with vertices in B which would contain two points in R. So H would then contain another point of R inside, other than M_i, which is impossible.

(*c*) H is a triangle. Then it contains inside it two points M_i, M_j. One of the three disjoint triangles $M_iM_kM_l$, $M_iM_lM_m$, $M_iM_mM_k$ will contain M_j. Then we can break that triangle into three smaller triangles using M_j. This makes five disjoint triangles with vertices in R, each having one point of B inside. With these five points of B, three disjoint triangles with vertices in B can be made so that each one of them having one point of R. Then H contains another point of R, different from M_1, M_2, M_3, M_4, M_5, which is impossible.

Problem 245. $ABCD$ is a concave quadrilateral such that $\angle BAD = \angle ABC = \angle CDA = 45^\circ$. Prove that $AC = BD$.

Solution. **CHAN Tsz Lung** (HKU Math PG Year 1), **KWOK Lo Yan** (Carmel Divine Grace Foundation Secondary School, Form 6), **Problem Solving Group @ Miniforum**, **WONG Kai Cheuk** (Carmel Divine Grace Foundation Secondary School, Form 6), **WONG Man Kit** (Carmel Divine Grace Foundation Secondary School, Form 6) and **WONG Tsun Yu** (St. Mark's School, Form 6).

Let line BC meet AD at E, then $\angle BEA = 180^\circ - \angle ABC - \angle BAD = 90^\circ$. Note ΔAEB and ΔCED are 45°-90°-45° triangles. So $AE = BE$ and $CE = DE$. Then $\Delta AEC \cong \Delta BED$. So $AC = BD$.

Commended solvers: **CHAN Ka Lok** (STFA Leung Kau Kui College), **CHAN Pak Woon** (HKU Math UG Year 1), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 7) and **YUEN Wah Kong** (St. Joan of Arc Secondary School).

Olympiad Corner

(continued from page 1)

Problem 1. (Cont.)

(c) the sum of all terms of every sequence is at least 666?

Problem 2. Let O be the center of the circumcircle of the acute-angled triangle ABC, for which $\angle CBA < \angle ACB$ holds. The line AO intersects the side BC at the point D. Denote by E and F the centers of the circumcircles of triangles ABD and ACD respectively. Let G and H be two points on the rays BA and CA such that $AG=AC$ and $AH=AB$, and the point A lies between B and G as well as between C and H. Prove the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60^\circ$.

Problem 3. Let a, b and c be positive numbers such that $ab + bc + ca = 1$. Prove the inequality

$$3\sqrt[3]{\frac{1}{abc} + 6(a+b+c)} \le \frac{\sqrt[3]{3}}{abc}.$$

Proofs of Muirhead's Inequality

(continued from page 2)

For the proof of the second step, we follow the approach in J. Michael Steele's book <u>*The Cauchy-Schwarz Master Class*</u>, MAA-Cambridge, 2004. For a $n\times n$ matrix M, we will denote its entry in the j-th row, k-th column by M_{jk}. Let us introduce the term <u>*permutation matrix*</u> for $\sigma \in S_n$ to mean the $n\times n$ matrix $M(\sigma)$ with $M(\sigma)_{jk} = 1$ if $\sigma(j)=k$ and $M(\sigma)_{jk} = 0$ otherwise. Also, introduce the term <u>*doubly stochastic matrix*</u> to mean a square matrix whose entries are nonnegative real numbers and the sum of the entries in every row and every column is equal to one. The proof of the second step follows from two results:

<u>***Hardy-Littlewood-Polya's Theorem.***</u> If $p \succ q$, then there is a $n\times n$ doubly stochastic matrix D such that $q = Dp$, where we write p and q as column matrices.

<u>***Birkhoff's Theorem.***</u> For every doubly stochastic matrix D, there exist nonnegative numbers $c(\sigma)$ with sum equals 1 such that

$$D = \sum_{\sigma\in S_n} c(\sigma) M(\sigma).$$

Granting these results, for P_i's in the second step, we can just let $P_i = M(\sigma_i)p$.

Hardy-Littlewood-Polya's theorem can be proved by introducing r as in the first proof. Following the idea of Hardy-Littlewood-Polya, we take T to be the matrix with

$$T_{jj} = \frac{d+c}{2d} = T_{kk},\ \ T_{jk} = \frac{d-c}{2d} = T_{kj},$$

all other entries on the main diagonal equal 1 and all other entries of the matrix equal 0. We can check T is doubly stochastic and $r = Tp$. Then we repeat until $r = q$.

Birkhoff's theorem can be proved by induction on the number N of positive entries of D using Hall's theorem (see *Mathematical Excalibur*, *vol.* 1, *no.* 5, *p.* 2). Note $N \ge n$. If $N = n$, then the positive entries are all 1's and D is a permutation matrix already. Next for $N > n$, suppose the result is true for all doubly stochastic matrices with less than N positive entries. Let D have exactly N positive entries. For $j = 1,\ldots, n$, let W_j be the set of k such that $D_{jk} > 0$. We need a <u>*system of distinct representatives*</u> (*SDR*) for $W_1,\ldots,W_n$. To get this, we check the condition in Hall's theorem. For every collection $W_{j_1},\ldots,W_{j_m}$, note m is the sum of all positive entries in column $j_1,\ldots,j_m$ of D. This is less than or equal to the sum of all positive entries in those columns that have at least one positive entry among row $j_1,\ldots,j_m$. This latter sum is the number of such columns and is also the number of elements in the union of $W_{j_1},\ldots,W_{j_m}$.

So the condition in Hall's theorem is satisfied and there is a *SDR* for $W_1,\ldots, W_n$. Let $\sigma(i)$ be the representative in W_i, then $\sigma \in S_n$. Let $c(\sigma)$ be the minimum of $D_{1\sigma(1)},\ldots,D_{n\sigma(n)}$. If $c(\sigma) = 1$, then D is a permutation matrix. Otherwise, let

$$D' = (1 - c(\sigma))^{-1}(D - c(\sigma)\, M(\sigma)).$$

Then $D = c(\sigma)\, M(\sigma) + (1 - c(\sigma))\, D'$ and D' is a double stochastic matrix with at least one less positive entries than D. So we may apply the cases less than N to D' and thus, D has the required sum.