Mathematical Excalibur

Volume 11, Number 2 *April 2006 – May 2006*

Olympiad Corner

Below was the Find Round of the 36th Austrian Math Olympiad 2005.

Part 1 (May 30, 2005)

Problem 1. Show that an infinite number of multiples of 2005 exist, in which each of the 10 digits 0,1,2,…,9 occurs the same number of times, not counting leading zeros.

Problem 2. For how many integer values of a with $|a| \le 2005$ does the system of equations $x^2 = y + a$, $y^2 = x + a$ have integer solutions?

Problem 3. We are given real numbers a, b and c and define s_n as the sum $s_n = a^n + b^n + c^n$ of their n-th powers for non-negative integers n. It is known that $s_1 = 2$, $s_2 = 6$ and $s_3 = 14$ hold. Show that

$$|\, s_n^2 - s_{n-1} \cdot s_{n+1} \,| = 8$$

holds for all integers $n > 1$.

Problem 4. We are given two equilateral triangles ABC and PQR with parallel sides, "one pointing up" and "one pointing down." The common area of the triangles' interior is a hexagon. Show that the lines joining opposite corners of this hexagon are concurrent.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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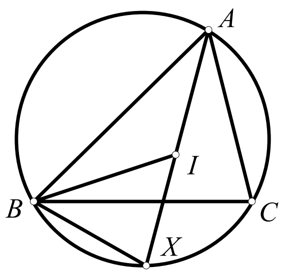
Angle Bisectors Bisect Arcs

Kin Y. Li

In general, angle bisectors of a triangle do not bisect the sides opposite the angles. However, ***angle bisectors always bisect the arcs opposite the angles on the circumcircle of the triangle!*** In math competitions, this fact is very useful for problems concerning angle bisectors or incenters of a triangle involving the circumcircle. Recall that the incenter of a triangle is the point where the three angle bisectors concur.

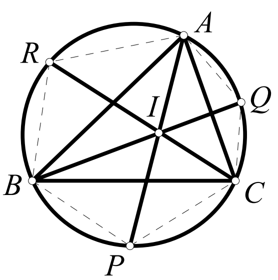
Theorem. Suppose the angle bisector of $\angle BAC$ intersect the circumcircle of ΔABC at $X \ne A$. Let I be a point on the line segment AX. Then I is the incenter of ΔABC if and only if $XI = XB = XC$.



Proof. Note $\angle BAX = \angle CAX = \angle CBX$. So $XB = XC$. Then

$$\begin{aligned} & I \text{ is the incenter of } \Delta ABC \\ \Leftrightarrow\ & \angle CBI = \angle ABI \\ \Leftrightarrow\ & \angle IBX - \angle CBX = \angle BIX - \angle BAX \\ \Leftrightarrow\ & \angle IBX = \angle BIX \\ \Leftrightarrow\ & XI = XB = XC. \end{aligned}$$

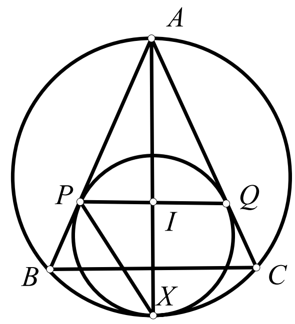
Example 1. (*1982 Australian Math Olympiad*) Let ABC be a triangle, and let the internal bisector of the angle A meet the circumcircle again at P. Define Q and R similarly. Prove that $AP + BQ + CR > AB + BC + CA$.



Solution. Let I be the incenter of ΔABC. By the theorem, we have $2IR = AR + BR > AB$ and similarly $2IP > BC$, $2IQ > CA$. Also $AI + BI > AB$, $BI + CI > BC$ and $CI + AI > CA$. Adding all these inequalities together, we get

$$2(AP + BQ + CR) > 2(AB + BC + CA).$$

Example 2. (*1978 IMO*) In ABC, $AB = AC$. A circle is tangent internally to the circumcircle of ABC and also to the sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of ΔABC.



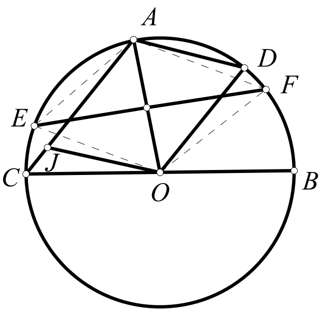
Solution. Let I be the midpoint of line segment PQ and X be the intersection of the angle bisector of $\angle BAC$ with the arc BC not containing A.

By symmetry, AX is a diameter of the circumcircle of ΔABC and X is the midpoint of the arc PXQ on the inside circle, which implies PX bisects $\angle QPB$. Now $\angle ABX = 90^\circ = \angle PIX$ so that X, I, P, B are concyclic. Then

$$\angle IBX = \angle IPX = \angle BPX = \angle BIX.$$

So $XI = XB$. By the theorem, I is the incenter of ΔABC.

Example 3. (*2002 IMO*) Let BC be a diameter of the circle Γ with center O. Let A be a point on Γ such that $0^\circ < \angle AOB < 120^\circ$. Let D be the midpoint of the arc AB not containing C. The line through O parallel to DA meets the line AC at J. The perpendicular bisector of OA meets Γ at E and at F. Prove that J is the incenter of the triangle CEF.

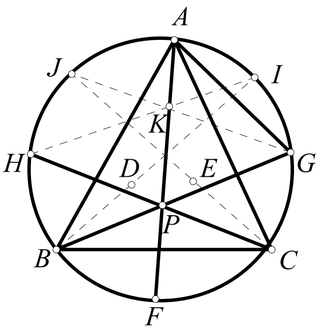


Solution. The condition $\angle AOB$ < 120˚ ensures I is inside ΔCEF (when $\angle AOB$ increases to 120˚, I will coincide with C). Now radius OA and chord EF are perpendicular and bisect each other. So $EOFA$ is a rhombus. Hence A is the midpoint of arc EAF. Then CA bisects $\angle ECF$. Since $OA = OC$, $\angle AOD = 1/2\angle AOB = \angle OAC$. Then DO is parallel to AJ. Hence $ODAJ$ is a parallelogram. Then $AJ = DO = EO = AE$. By the theorem, J is the incenter of ΔCEF.

Example 4. (*1996 IMO*) Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC respectively. Show that AP, BD and CE meet at a point.



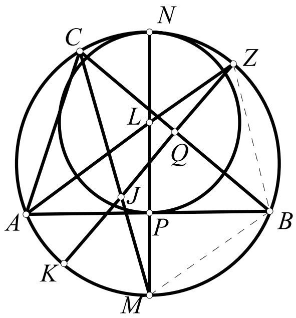
Solution. Let lines AP, BP, CP intersect the circumcircle of ΔABC again at F, G, H respectively. Now

$$\begin{aligned}\angle APB - \angle ACB &= \angle FPG - \angle AGB \\ &= \angle FAG.\end{aligned}$$

Similarly, $\angle APC - \angle ABC = \angle FAH$. So AF bisects $\angle HAG$. Let K be the incenter of ΔHAG. Then K is on AF and lines HK, GK pass through the midpoints I, J of minor arcs AG, AH respectively. Note lines BD, CE also pass through I, J as they bisect $\angle ABP$, $\angle ACP$ respectively.

Applying Pascal's theorem (see *vol.*10, *no*. 3 of Math Excalibur) to B, G, J, C, H, I on the circumcircle, we see that $P=BG\cap CH$, $K=GJ\cap HI$ and $BI\cap CJ= BD\cap CE$ are collinear. Hence, $BD\cap CE$ is on line PK, which is the same as line AP.

Example 5. (*2006 APMO*) Let A, B be two distinct points on a given circle O and let P be the midpoint of line segment AB. Let O_1 be the circle tangent to the line AB at P and tangent to the circle O. Let ℓ be the tangent line, different from the line AB, to O_1 passing through A. Let C be the intersection point, different from A, of ℓ and O. Let Q be the midpoint of the line segment BC and O_2 be the circle tangent to the line BC at Q and tangent to the line segment AC. Prove that the circle O_2 is tangent to the circle O.



Solution. Let the perpendicular to AB through P intersect circle O at N and M with N and C on the same side of line AB. By symmetry, segment NP is a diameter of the circle of O_1 and its midpoint L is the center of O_1. Let line AL intersect circle O again at Z. Let line ZQ intersect line CM at J and circle O again at K.

Since AB and AC are tangent to circle O_1, AL bisects $\angle CAB$ so that Z is the midpoint of arc BC. Since Q is the midpoint of segment BC, $\angle ZQB$ = 90˚ = $\angle LPA$ and $\angle JQC$ = 90˚ = $\angle MPB$. Next

$$\angle ZBQ = \angle ZBC = \angle ZAC = \angle LAP.$$

So ΔZQB, ΔLPA are similar. Since M is the midpoint of arc AMB,

$$\angle JCQ = \angle MCB = \angle MCA = \angle MBP.$$

So ΔJQC, ΔMPB are similar.

By the intersecting chord theorem, $AP\cdot BP = NP\cdot MP = 2LP\cdot MP$. Using the similar triangles above, we have

$$\frac{1}{2} = \frac{LP\cdot MP}{AP\cdot BP} = \frac{ZQ\cdot JQ}{BQ\cdot CQ}.$$

By the intersecting chord theorem, $KQ\cdot ZQ = BQ\cdot CQ$ so that

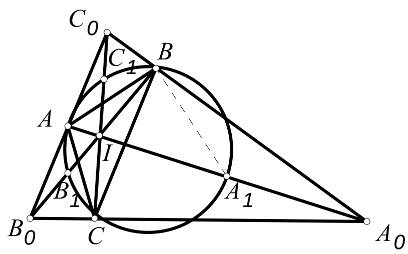
$$KQ = (BQ\cdot CQ)/ZQ = 2JQ.$$

This implies J is the midpoint of KQ. Hence the circle with center J and diameter KQ is tangent to circle O at K and tangent to BC at Q. Since J is on the bisector of $\angle BCA$, this circle is also tangent to AC. So this circle is O_2.

Example 6. (*1989 IMO*) In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that:

(i) the area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$,

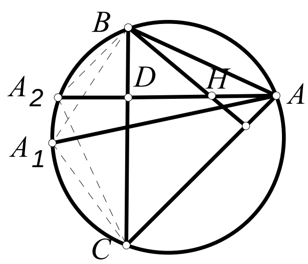
(ii) the area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.



Solution. (i) Let I be the incenter of ΔABC. Since internal angle bisector and external angle bisector are perpendicular, we have $\angle B_0BA_0$ = 90˚. By the theorem, $A_1I = A_1B$. So A_1 must be the midpoint of the hypotenuse A_0I of right triangle IBA_0. So the area of ΔBIA_0 is twice the area of ΔBIA_1.

Cutting the hexagon $AC_1BA_1CB_1$ into six triangles with common vertex I and applying a similar area fact like the last statement to each of the six triangles, we get the conclusion of (i).

(ii) Using (i), we only need to show the area of hexagon $AC_1BA_1CB_1$ is at least twice the area of ΔABC.



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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***August 16, 2006.***

Problem 251. Determine with proof the largest number x such that a cubical gift of side x can be wrapped completely by folding a unit square of wrapping paper (without cutting).

Problem 252. Find all polynomials $f(x)$ with integer coefficients such that for every positive integer n, $2^n - 1$ is divisible by $f(n)$.

Problem 253. Suppose the bisector of $\angle BAC$ intersect the arc opposite the angle on the circumcircle of ΔABC at A_1. Let B_1 and C_1 be defined similarly. Prove that the area of $\Delta A_1B_1C_1$ is at least the area of ΔABC.

Problem 254. Prove that if $a, b, c > 0$, then

$$\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})+(a+b+c)^2 \geq 4\sqrt{3abc\,(a+b+c)}.$$

Problem 255. Twelve drama groups are to do a series of performances (with some groups possibly making repeated performances) in seven days. Each group is to see every other group's performance at least once in one of its day-offs.

Find with proof the minimum total number of performances by these groups.

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Solutions

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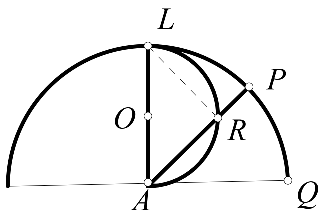
Problem 246. A spy plane is flying at the speed of 1000 kilometers per hour along a circle with center A and radius 10 kilometers. A rocket is fired from A at the same speed as the spy plane such that it is always on the radius from A to the spy plane. Prove such a path for the rocket exists and find how long it takes for the rocket to hit the spy plane.

(*Source: 1965 Soviet Union Math Olympiad*)

Solution. **Jeff CHEN** (Virginia, USA), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher), **G.R.A. 20 Math Problem Group** (Roma, Italy) and **Alex O Kin-Chit** (STFA Cheng Yu Tung Secondary School).



Let the spy plane be at Q when the rocket was fired. Let L be the point on the circle obtained by rotating Q by 90° in the forward direction of motion with respect to the center A. Consider the semicircle with diameter AL on the same side of line AL as Q. We will show the path from A to L along the semicircle satisfies the conditions.

For any point P on the arc QL, let the radius AP intersect the semicircle at R. Let O be the midpoint of AL. Since

$$\angle QAP = \angle RLA = 1/2\angle ROA$$

and $AL = 2AO$, the length of arc AR is the same as the length of arc QP. So the conditions are satisfied.
Finally, the rocket will hit the spy plane at L after $5\pi/1000$ hour it was fired.

<u>*Comments:*</u> One solver guessed the path should be a curve and decided to try a circular arc to start the problem. The other solvers derived the equation of the path by a differential equation as follows: using <u>*polar coordinates*</u>, since the spy plane has a constant angular velocity of 1000/10 = 100 rad/sec, so at time t, the spy plane is at $(10, 100t)$ and the rocket is at $(r(t), \theta(t))$. Since the rocket and the spy plane are on the same radius, so $\theta(t) = 100t$. Now they have the same speed, so

$$(r'(t))^2 + (r(t)\theta'(t))^2 = 10^6.$$

Then

$$\frac{r'(t)}{\sqrt{100 - r(t)^2}} = 100.$$

Integrating both sides from 0 to t, we get the equation $r = 10\sin(100t) = 10\sin\theta$, which describes the path above.

Problem 247. (a) Find all possible positive integers $k \geq 3$ such that there are k positive integers, every two of them are not relatively prime, but every three of them are relatively prime.

(b) Determine with proof if there exists an infinite sequence of positive integers satisfying the conditions in (a) above.

(*Source: 2003 Belarussian Math Olympiad*)

Solution. **G.R.A. 20 Math Problem Group** (Roma, Italy) and **YUNG Fai**.

(a) We shall prove by induction that the conditions are true for every positive integer $k \geq 3$.

For $k = 3$, the numbers 6, 10, 15 satisfy the conditions. Assume it is true for some $k \geq 3$ with the numbers being $a_1, a_2, \ldots, a_k$. Let $p_1, p_2, \ldots, p_k$ be distinct prime numbers such that each p_i is greater than $a_1a_2\ldots a_k$. For $I = 1$ to k, let $b_i = a_ip_i$ and let $b_{k+1} = p_1p_2\ldots p_k$. Then

$gcd(b_i, b_j) = gcd(a_i, a_j) > 1$ for $1 \leq i < j \leq k$,

$gcd(b_i, b_{k+1}) = p_i > 1$ for $1 \leq i \leq k$,

$gcd(b_h, b_i, b_j) = gcd(a_h, a_i, a_j) = 1$
for $1 \leq h \leq i < j \leq k$ and

$gcd(b_i, b_j, b_{k+1}) = 1$ for $1 \leq i < j \leq k$,

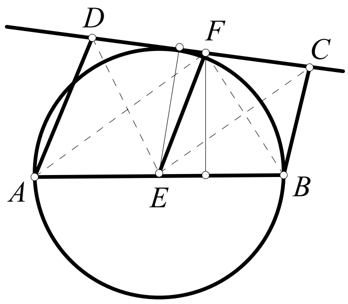
completing the induction.

(b) Assume there are infinitely many positive integers $a_1, a_2, a_3, \ldots$ satisfying the conditions in (a). Let a_1 have exactly m prime divisors. For $i = 2$ to $m + 2$, since each of the $m + 1$ numbers $gcd(a_1, a_i)$ is divisible by one of these m primes, by the pigeonhole principle, there are i, j with $2 \leq i < j \leq m + 2$ such that $gcd(a_1, a_i)$ and $gcd(a_1, a_j)$ are divisible by the same prime. Then $gcd(a_1, a_i, a_j) > 1$, a contradiction.

Commended solvers: **CHAN Nga Yi** (Carmel Divine Grace Foundation Secondary School, Form 6) and **CHAN Yat Sing** (Carmel Divine Grace Foundation Secondary School, Form 6).

Problem 248. Let $ABCD$ be a convex quadrilateral such that line CD is tangent to the circle with side AB as diameter. Prove that line AB is tangent to the circle with side CD as diameter if and only if lines BC and AD are parallel.

Solution. **Jeff CHEN** (Virginia, USA) and **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher).



Let E be the midpoints of AB. Since CD is tangent to the circle, the distance from E to line CD is $h_1 = AB/2$. Let F be the midpoint of CD and let h_2 be the distance from F to line AB. Observe that the areas of ΔCEF and ΔDEF = $CD{\cdot}AB/8$. Now

line AB is tangent to the circle with side CD as diameter
$\Leftrightarrow$ $h_2 = CD/2$
$\Leftrightarrow$ areas of ΔAEF, ΔBEF, ΔCEF and ΔDEF are equal to $AB{\cdot}CD/8$
$\Leftrightarrow$ $AD \| EF$, $BC \| EF$
$\Leftrightarrow$ $AD \| BC$.

Problem 249. For a positive integer n, if $a_1,\cdots, a_n$, b_1, $\cdots$, b_n are in [1,2] and $a_1^2+\cdots+a_n^2 = b_1^2+\cdots+b_n^2$, then prove that

$$\frac{a_1^3}{b_1}+\cdots+\frac{a_n^3}{b_n} \le \frac{17}{10}(a_1^2+\cdots+a_n^2).$$

Solution. **Jeff CHEN** (Virginia, USA).

For x, y in [1,2], we have

$$\begin{aligned} & 1/2 \le x/y \le 2 \\ \Leftrightarrow\ & y/2 \le x \le 2y \\ \Leftrightarrow\ & (y/2 - x)(2y - x) \le 0 \\ \Leftrightarrow\ & x^2 + y^2 \le 5xy/2. \end{aligned}$$

Let $x = a_i$ and $y = b_i$, then $a_i^2 + b_i^2 \le 5a_ib_i/2$. Summing and manipulating, we get

$$-\sum_{i=1}^n a_ib_i \le -\frac{2}{5}\sum_{i=1}^n (a_i^2+b_i^2) = -\frac{4}{5}\sum_{i=1}^n a_i^2.$$

Let $x = (a_i^3/b_i)^{1/2}$ and $y = (a_ib_i)^{1/2}$. Then $x/y = a_i/b_i$ in [1,2]. So $a_i^3/b_i + a_ib_i \le 5a_i^2/2$.
Summing, we get

$$\sum_{i=1}^n \frac{a_i^3}{b_i} + \sum_{i=1}^n a_ib_i \le \frac{5}{2}\sum_{i=1}^n a_i^2.$$

Adding the two displayed inequalities, we get

$$\frac{a_1^3}{b_1}+\cdots+\frac{a_n^3}{b_n} \le \frac{17}{10}(a_1^2+\cdots+a_n^2).$$

Problem 250. Prove that every region with a convex polygon boundary cannot be dissected into finitely many regions with nonconvex quadrilateral boundaries.

Solution. **YUNG Fai.**

Assume the contrary that there is a dissection of the region into nonconvex quadrilateral R_1, R_2, …, R_n. For a nonconvex quadrilateral R_i, there is a vertex where the angle is $\theta_i > 180^\circ$, which we refer to as the *large* vertex of the quadrilateral. The three other vertices, where the angles are less than 180° will be referred to as *small* vertices.

Since the boundary of the region is a convex polygon, all the large vertices are in the interior of the region. At a large vertex, one angle is $\theta_i > 180^\circ$, while the remaining angles are angles of small vertices of some of the quadrilaterals and add up to $360^\circ - \theta_i$. Now

$$\sum_{i=1}^n (360^\circ - \theta_i)$$

accounts for all the angles associated with all the small vertices. This is a contradiction since this will leave no more angles from the quadrilaterals to form the angles of the region.

Olympiad Corner

(continued from page 1)

Part 2, Day 1 (June 8, 2005)

Problem 1. Determine all triples of positive integers (a,b,c), such that $a + b + c$ is the least common multiple of a, b and c.

Problem 2. Let a, b, c, d be positive real numbers. Prove

$$\frac{a+b+c+d}{abcd} \le \frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3}.$$

Problem 3. In an acute-angled triangle ABC, circle k_1 with diameter AC and k_2 with diameter BC are drawn. Let E be the foot of B on AC and F be the foot of A on BC. Furthermore, let L and N be the points in which the line BE intersects with k_1 (with L lying on the segment BE) and K and M be the points in which the line AF intersects with k_2 (with K on the segment AF). Prove that $KLMN$ is a cyclic quadrilateral.

Part 2, Day 2 (June 9, 2005)

Problem 4. The function f is defined for all integers {0, 1, 2, …, 2005}, assuming non-negative integer values in each case. Furthermore, the following conditions are fulfilled for all values of x for which the function is defined:

$$f(2x+1) = f(2x),\quad f(3x+1) = f(3x) \\ \text{and}\quad f(5x+1) = f(5x).$$

How many different values can the function assume at most?

Problem 5. Determine all sextuples (a,b,c,d,e,f) of real numbers, such that the following system of equations is fulfilled:

$$4a=(b+c+d+e)^4,\ 4b=(c+d+e+f)^4, \\ 4c=(d+e+f+a)^4,\ 4d=(e+f+a+b)^4, \\ 4e=(f+a+b+c)^4,\ 4f=(a+b+c+d)^4.$$

Problem 6. Let Q be a point in the interior of a cube. Prove that an infinite number of lines passing through Q exists, such that Q is the mid-point of the line-segment joining the two points P and R in which the line and the cube intersect.

Angle Bisectors Bisect Arcs

(continued from page 2)

Let H be the orthocenter of ΔABC. Let line AH intersect BC at D and the circumcircle of ΔABC again at A_2. Note

$$\begin{aligned} \angle A_2BC &= \angle A_2AC \\ &= \angle DAC \\ &= 90^\circ - \angle ACD \\ &= \angle HBC. \end{aligned}$$

Similarly, we have $\angle A_2CB = \angle HCB$. Then $\Delta BA_2C \cong \Delta BHC$. Since A_1 is the midpoint of arc BA_1C, it is at least as far from chord BC as A_2. So the area of $\Delta\, BA_1C$ is at least the area of $\Delta\, BA_2C$. Then the area of quadrilateral BA_1CH is at least twice the area of ΔBHC.

Cutting hexagon $AC_1BA_1CB_1$ into three quadrilaterals with common vertex H and comparing with cutting ΔABC into three triangles with common vertex H in terms of areas, we get the conclusion of (ii).

Remarks. In the solution of (ii), we saw the orthocenter H of ΔABC has the property that $\Delta BA_2C \cong \Delta BHC$ (hence, also $HD = A_2D$). These are useful facts for problems related to the orthocenters involving the circumcircles.