Olympiads Bisect Arcs

Kin Y. Li

In general, angle bisectors of a triangle do not bisect the sides opposite the angles. However, angle bisectors always bisect the arcs opposite the angles on the circumcircle of the triangle! In math competitions, this fact is very useful for problems concerning angle bisectors or incenters of a triangle involving the circumcircle. Recall that the incenter of a triangle is the point where the three angle bisectors concur.

**Theorem.** Suppose the angle bisector of ∠BAC intersect the circumcircle of ΔABC at X ≠ A. Let I be a point on the line segment AX. Then I is the incenter of ΔABC if and only if XI = XB = XC.

**Proof.** Note ∠BAX = ∠CAX = ∠CBX. So XB = XC.

So XI = XB. Then

I is the incenter of ΔABC

⇔ ∠CBI = ∠ABI
⇔ ∠IBX = ∠CBX = ∠IBX = ∠BAX
⇔ ∠IBX = ∠BIX
⇔ XI = XB = XC.

**Example 1.** (1982 Australian Math Olympiad) Let ABC be a triangle, and let the internal bisector of the angle A meet the circumcircle again at P. Define Q and R similarly. Prove that AP + BQ + CR > AB + BC + CA.

**Solution.** Let I be the incenter of ΔABC. By the theorem, we have 2IR = AR + BR > AB and similarly 2IP > BC, 2IQ > CA. Also AI + BI > AB, BI + CI > BC and CI + AI > CA. Adding all these inequalities together, we get

2(AP + BQ + CR) > 2(AB + BC + CA).

**Example 2.** (1979 IMO) In ΔABC, AB = AC. A circle is tangent internally to the circumcircle of ΔABC and also to the sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of ΔABC.

**Solution.** Let J be the midpoint of line segment PQ and X be the intersection of the arc bisector of ∠BAC with the arc BC not containing A.

By symmetry, AX is a diameter of the circumcircle of ΔABC and X is the midpoint of the arc PXQ on the inside circle, which implies PX bisects ∠QPB. Now ∠ABX = 90° = ∠PIX so that X, I, P, B are concyclic. Then

∠IBX = ∠IPX = ∠BPX = ∠BIX.

So XI = XB. By the theorem, I is the incenter of ΔABC.

**Example 3.** (2002 IMO) Let BC be a diameter of the circle Γ with center O. Let A be a point on Γ such that 0° < ∠AOB < 120°. Let D be the midpoint of the arc AB not containing C. The line through O parallel to DA meets the line AC at J. The perpendicular bisector of OA meets Γ at E and at P. Prove that J is the incenter of the triangle CEF.
Solution. The condition $\angle AOB < 120^\circ$ ensures $I$ is inside $\Delta CEF$ (when $\angle AOB$ increases to $120^\circ$, $I$ will coincide with C). Now radius $OA$ and chord $EF$ are perpendicular and bisect each other. So $EOFA$ is a rhombus. Hence $A$ is the midpoint of arc $EAF$. Then $CA$ bisects $\angle ECF$. Since $OA = OC$, $\angle AOD = 1/2 \angle AOB = \angle OAC$. Then $DO$ is parallel to $AJ$. Hence $ODAJ$ is a parallelogram. Then $AJ = DO = EO = AE$. By the theorem, $J$ is the incenter of $\Delta CEF$.

Example 4. (1996 IMO) Let $P$ be a point inside triangle $ABC$ such that

$$\angle APB = \angle ACB = \angle APC = \angle ABC.$$ 

Let $D$, $E$ be the incenters of triangles $APB$, $APC$ respectively. Show that $AP$, $BD$ and $CE$ meet at a point.

Solution. Let lines $AP$, $BP$, $CP$ intersect the circumcircle of $\Delta ABC$ again at $F$, $G$, $H$ respectively. Now

$$\angle APB = \angle ACB = \angle FPG = \angle AGB = \angle FAG.$$ 

Similarly, $\angle APC = \angle ABC = \angle EAH$. So $AF$ bisects $\angle HAG$. Let $K$ be the incenter of $\Delta HAG$. Then $K$ is on $AF$ and lines $HK$, $GK$ pass through the midpoints $I$, $J$ of minor arcs $AG$, $AH$ respectively. Note lines $BD$, $CE$ also pass through $I$, $J$ as they bisect $\angle ABP$, $\angle ACP$ respectively.

Applying Pascal’s theorem (see vol.10, no. 3 of Math Excalibur) to $B, G, J, C, H, I$ on the circumcircle, we see that $P = BG \cap CH$, $K = GJ \cap HI$ and $B \cap CJ = BD \cap CE$ are collinear. Hence, $BD \cap CE$ is on line $PK$, which is the same as line $AP$.

Example 5. (2006 APMO) Let $A$, $B$ be two distinct points on a given circle $O$ and let $P$ be the midpoint of line segment $AB$. Let $O_1$ be the circle tangent to the line $AB$ at $P$ and tangent to the circle $O$. Let $\ell$ be the tangent line, different from the line $AB$, to $O_1$ passing through $A$. Let $C$ be the intersection point, different from $A$, of $\ell$ and $O_1$. Let $Q$ be the midpoint of the line segment $BC$ and $O_2$ be the circle tangent to the line $BC$ at $Q$ and tangent to the line segment $AC$. Prove that the circle $O_2$ is tangent to the circle $O$.

Solution. Let the perpendicular to $AB$ through $P$ intersect circle $O$ at $N$ and $M$ with $N$ and $C$ on the same side of line $AB$. By symmetry, segment $NP$ is a diameter of the circle of $O_1$ and its midpoint $L$ is the center of $O_1$. Let line $AL$ intersect circle $O$ again at $Z$. Let line $ZQ$ intersect line $CM$ at $J$ and circle $O$ again at $K$.

Since $AB$ and $AC$ are tangent to circle $O_1$, $AL$ bisects $\angle CAB$ so that $Z$ is the midpoint of arc $BC$. Since $Q$ is the midpoint of segment $BC$, $\angle ZQB = 90^\circ = \angle LPA$ and $\angle JQC = 90^\circ = \angle MPB$. Next $\angle ZBQ = \angle ZBC = \angle ZAC = \angle LAP$.

So $\Delta ZQB$, $\Delta LPA$ are similar. Since $M$ is the midpoint of arc $AMB$,$\angle JCQ = \angle MCB = \angle MCA = \angle MPB$.

So $\Delta JQC$, $\Delta MPB$ are similar.

By the intersecting chord theorem, $AP \cdot BP = NP \cdot MP = 2LP \cdot MP$. Using the similar triangles above, we have

$$\frac{1}{2} = \frac{LP \cdot MP}{AP \cdot BP} = \frac{ZQ \cdot JQ}{BQ \cdot CQ}.$$ 

By the intersecting chord theorem, $KQ \cdot ZQ = BQ \cdot CQ$ so that

$$KQ = (BQ \cdot CQ) / ZQ = 2JQ.$$ 

This implies $J$ is the midpoint of $KQ$. Hence the circle with center $J$ and diameter $KQ$ is tangent to circle $O$ at $K$ and tangent to $BC$ at $Q$. Since $J$ is on the bisector of $\angle BCA$, this circle is also tangent to $AC$. So this circle is $O_2$.

Example 6. (1989 IMO) In an acute-angled triangle $ABC$ the internal bisector of angle $A$ meets the circumcircle of the triangle again at $A_1$. Points $B_1$ and $C_1$ are defined similarly. Let $A_2$ be the point of intersection of the line $AA_1$ with the external bisectors of angles $B$ and $C$. Points $B_0$ and $C_0$ are defined similarly. Prove that:

(i) the area of the triangle $A_2B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$,

(ii) the area of the triangle $A_2B_0C_0$ is at least four times the area of the triangle $ABC$.

Solution. (i) Let $I$ be the incenter of $\Delta ABC$. Since internal angle bisector and external angle bisector are perpendicular, we have $\angle B_1BA_0 = 90^\circ$. By the theorem, $\angle AIB_1 = \angle AIB$. So $A_1$ must be the midpoint of the hypotenuse $A_1D$ of right triangle $IBA_0$. So the area of $\Delta BIA_0$ is twice the area of $\Delta BIA_1$.

Cutting the hexagon $AC_1BA_1CB_1$ into six triangles with common vertex $I$ and applying a similar area fact like the last statement to each of the six triangles, we get the conclusion of (i).

(ii) Using (i), we only need to show the area of hexagon $AC_1BA_1CB_1$ is at least twice the area of $\Delta ABC$.

(continued on page 4)
Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong. The deadline for submitting solutions is August 16, 2006.

Problem 251. Determine with proof the largest number \( x \) such that a cubical gift of side \( x \) can be wrapped completely by folding a unit square of wrapping paper (without cutting).

Problem 252. Find all polynomials \( f(x) \) with integer coefficients such that for every positive integer \( n \), \( 2^n - 1 \) is divisible by \( f(n) \).

Problem 253. Suppose the bisector of \( \angle BAC \) intersect the arc opposite the angle on the circumference of \( \Delta ABC \) at \( A_1 \). Let \( B_1 \) and \( C_1 \) be defined similarly. Prove that the area of \( \Delta A_1B_1C_1 \) is at least the area of \( \Delta ABC \).

Problem 254. Prove that if \( a, b, c > 0 \), then
\[
\sqrt{abc}(\sqrt{a} + \sqrt{b} + \sqrt{c}) + (a + b + c) \geq 4\sqrt{3}abc(a + b + c).
\]

Problem 255. Twelve drama groups are to do a series of performances (with some groups possibly making repeated performances) in seven days. Each group is to see every other group’s performance at least once in one of its day-offs. Find with proof the minimum total number of performances by these groups.

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Solutions

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Problem 246. A spy plane is flying at the speed of 1000 kilometers per hour along a circle with center \( A \) and radius 10 kilometers. A rocket is fired from \( A \) at the same speed as the spy plane such that it is always on the radius from \( A \) to the spy plane. Prove such a path for the rocket exists and find how long it takes for the rocket to hit the spy plane. Let the spy plane be at \( Q \) when the rocket was fired. Let \( L \) be the point on the circle obtained by rotating \( Q \) by 90° in the forward direction of motion with respect to the center \( A \). Consider the semicircle with diameter \( AL \) on the same side of line \( AL \) as \( Q \). We will show the path from \( A \) to \( L \) on the semicircle satisfies the conditions.

For any point \( P \) on the arc \( QL \), let the radius \( AP \) intersect the semicircle at \( R \). Let \( O \) be the midpoint of \( AL \). Since
\[
\angle QAP = \angle RLA = 1/2 \angle ROA
\]
and \( AL = 2AO \), the length of arc \( AR \) is the same as the length of arc \( QP \). So the conditions are satisfied.

Finally, the rocket will hit the spy plane at \( L \) after 5\( \pi \)/1000 hour it was fired.

Comments: One solver guessed the path should be a curve and decided to try a circular arc to start the problem. The other solvers derived the equation of the path by a differential equation as follows: using polar coordinates, since the spy plane has a constant angular velocity of 1000/10 = 100 rad/sec, so at time \( t \), the spy plane is at (10, 100t) and the rocket is at \( (r(t), \theta(t)) \). Since the rocket and the spy plane are on the same radius, so \( \theta(t) = 100t \). Now they have the same speed, so
\[
(r'(t))^2 + (r(t)\theta'(t))^2 = 10^6.
\]

Then
\[
\frac{r'(t)}{\sqrt{100 - r(t)^2}} = 100.
\]

Integrating both sides from 0 to \( t \), we get the equation \( r = 10 \sin(100t) = 10 \sin \theta \), which describes the path above.

Problem 247. (a) Find all possible positive integers \( k \geq 3 \) such that there are \( k \) positive integers, every two of them are not relatively prime, but every three of them are relatively prime.

(b) Determine with proof if there exists an infinite sequence of positive integers satisfying the conditions in (a) above.

**Source:** 1965 Soviet Union Math Olympiad

**Solution.** Jeff CHEN (Virginia, USA), Koyrtis G. CHRYSSOSTOMOS (Larissa, Greece, teacher), G.R.A. 20 Math Problem Group (Roma, Italy) and Alex O Kin-Chit (STFA Cheng Yu Tung Secondary School).

Problem 248. Let \( ABCD \) be a convex quadrilateral such that line \( CD \) is tangent to the circle with side \( AB \) as diameter. Prove that line \( AB \) is tangent to the circle with side \( CD \) as diameter if and only if lines \( BC \) and \( AD \) are parallel.

**Solution.** Jeff CHEN (Virginia, USA) and Koyrtis G. CHRYSSOSTOMOS (Larissa, Greece, teacher).
Problem 249. Let \( h_2 = \frac{AB}{2} \). Let \( F \) be the midpoint of \( CD \) and let \( h_2 \) be the distance from \( F \) to \( AB \). Observe that the areas of \( \triangle DEF \) and \( \triangle DEF = CD \cdot AB \). Now

\[
\frac{a^4}{b^4} + \cdots + \frac{a^4}{b^4} \leq \frac{17}{10} (a^2 + \cdots + a^2).
\]

Solution. Jeff CHEN (Virginia, USA).

For \( x, y \) in \([1,2]\), we have

\[
\frac{1}{2} \leq \frac{x}{y} \leq 2
\]

\[
\Rightarrow \frac{y}{2} \leq x \leq 2y
\]

\[
\Rightarrow x + y \leq 5xy/2.
\]

Let \( x = a \) and \( y = b \), then \( a^2 + b^2 \leq 5ab/2 \). Summing and manipulating, we get

\[
\sum_{i=1}^{n} a_i b_i \leq -\frac{5}{2} \sum_{i=1}^{n} (a_i + b_i)^2 = -\frac{5}{2} \sum_{i=1}^{n} a_i^2.
\]

Let \( x = (a_i^2/b_i^2)^{1/2} \) and \( y = (a_i/b_i)^{1/2} \). Then \( x/y = a_i/b_i \) in \([1,2]\). So \( a_i^2 + b_i \leq 5a_i^2/2 \). Summing, we get

\[
\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} a_i b_i \leq -\frac{5}{2} \sum_{i=1}^{n} a_i^2.
\]

Adding the two displayed inequalities, we get

\[
\frac{a^4}{b^4} + \cdots + \frac{a^4}{b^4} \leq \frac{17}{10} (a^2 + \cdots + a^2).
\]

Problem 250. Prove that every region with a convex polygon boundary cannot be dissected into finitely many regions with nonconvex quadrilateral boundaries.

Solution. YUNG Fai.

Assume the contrary that there is a dissection of the region into nonconvex quadrilateral \( R_1, R_2, \ldots, R_n \). For a nonconvex quadrilateral \( R_n \), there is a vertex where the angle is \( \theta_i > 180^\circ \), which we refer to as the large vertex of the quadrilateral. The three other vertices, where the angles are less than \( 180^\circ \) will be referred to as small vertices.

Since the boundary of the region is a convex polygon, all the large vertices are in the interior of the region. At a large vertex, one angle is \( \theta_i > 180^\circ \), while the remaining angles are angles of small vertices of some of the quadrilaterals and add up to \( 360^\circ - \theta_i \). Now

\[
\sum_{i=1}^{n} (360^\circ - \theta_i)
\]

accounts for all the angles associated with all the small vertices. This is a contradiction since this will leave no more angles from the quadrilaterals to form the angles of the region.

Olympiad Corner

(continued from page 1)

Part 2, Day 1 (June 8, 2005)

Problem 1. Determine all triples of positive integers \((a, b, c)\), such that \(a + b + c\) is the least common multiple of \(a, b\) and \(c\).

Problem 2. Let \( a, b, c, d \) be positive real numbers. Prove

\[
\frac{a + b + c + d}{abcd} \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}.
\]

Problem 3. In an acute-angled triangle \(ABC\), circle \(k_1\) with diameter \(AC\) and \(k_2\) with diameter \(BC\) are drawn. Let \(E\) be the foot of \(B\) on \(AC\) and \(F\) be the foot of \(A\) on \(BC\). Furthermore, let \(L\) and \(N\) be the points in which the line \(BE\) intersects with \(k_1\) (with \(L\) lying on the segment \(BE\)) and \(K\) and \(M\) be the points in which the line \(AF\) intersects with \(k_2\) (with \(K\) on the segment \(AF\)). Prove that \(KLMN\) is a cyclic quadrilateral.

Part 2, Day 2 (June 9, 2005)

Problem 4. The function \(f\) is defined for all integers \(0, 1, 2, \ldots, 2005\), assuming non-negative integer values in each case. Furthermore, the following conditions are fulfilled for all values of \(x\) for which the function is defined:

\[
\begin{align*}
 f(2x + 1) &= f(2x), \\
 f(3x + 1) &= f(3x) \\
 \text{and } f(5x + 1) &= f(5x).
\end{align*}
\]

How many different values can the function assume at most?

Problem 5. Determine all sextuples \((a, b, c, d, e, f)\) of real numbers, such that the following system of equations is fulfilled:

\[
\begin{align*}
 4a &= (b+c+d+e), \\
 4b &= (c+d+e+f), \\
 4c &= (d+e+f+a), \\
 4d &= (e+f+a+b), \\
 4e &= (f+a+b+c), \\
 4f &= (a+b+c+d).
\end{align*}
\]

Problem 6. Let \(Q\) be a point in the interior of a cube. Prove that an infinite number of lines passing through \(Q\) exists, such that \(Q\) is the mid-point of the line-segment joining the two points \(P\) and \(R\) in which the line and the cube intersect.

Angle Bisectors Bisect Arcs

(continued from page 2)

Let \(H\) be the orthocenter of \(\triangle ABC\). Let line \(AH\) intersect \(BC\) at \(D\) and the circumcircle of \(\triangle ABC\) again at \(A_2\). Note

\[
\angle A_2BC = \angle A_2AC \\
= \angle DAC \\
= 90^\circ - \angle ACD \\
= \angle HBC.
\]

Similarly, we have \(\angle A_2CB = \angle HCB\). Then \(\triangle BAC \cong \triangle HBC\). Since \(A_1\) is the midpoint of arc \(BA_2C\), it is at least as far from chord \(BC\) as \(A_2\). So the area of \(\triangle BAC\) is at least the area of \(\triangle B_2AC\). Then the area of quadrilateral \(BA_1CH\) is at least twice the area of \(\triangle ABHC\).

Cutting hexagon \(AC_1B_1A_2B_2C_1\) into three quadrilaterals with common vertex \(H\) and comparing with cutting \(\triangle ABC\) into three triangles with common vertex \(H\) in terms of areas, we get the conclusion of (ii).

Remarks. In the solution of (ii), we saw the orthocenter \(H\) of \(\triangle ABC\) has the property that \(\triangle A_1B_1C \cong \triangle ABC\) (hence, also \(HD = A_2D\)). These are useful facts for problems related to the orthocenters involving the circumcircles.