Mathematical Excalibur

Volume 11, Number 3 June 2006 – October 2006

Olympiad Corner

The following were the problems of the IMO 2006.

Day 1 (July 12, 2006)

Problem 1. Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

Problem 2. Let P be a regular 2006-gon. A diagonal of P is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*.

Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

Problem 3. Determine the least real number M such that the inequality

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| \leq M(a^2+b^2+c^2)^2$$

holds for all real numbers a, b and c.

(continued on page 4)

Editors: 張百康 (CHEUNG Pak-Hong), Munsang College, HK
高子眉 (KO Tsz-Mei)
梁達榮 (LEUNG Tat-Wing)
李健賢 (LI Kin-Yin), Dept. of Math., HKUST
吳鏡波 (NG Keng-Po Roger), ITC, HKPU

Artist: 楊秀英 (YEUNG Sau-Ying Camille), MFA, CU

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 25, 2006***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI
Department of Mathematics
The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

Summation by Parts

Kin Y. Li

In calculus, we have a formula called <u>*integration by parts*</u>

$$\int_s^t f(x)g(x)dx = F(t)g(t)-F(s)g(s) - \int_s^t F(x)g'(x)dx,$$

where $F(x)$ is an anti-derivative of $f(x)$. There is a discrete version of this formula for series. It is called <u>*summation by parts*</u>, which asserts

$$\sum_{k=1}^n a_kb_k = A_nb_n - \sum_{k=1}^{n-1} A_k(b_{k+1}-b_k),$$

where $A_k = a_1+a_2+\cdots+a_k$. This formula follows easily by observing that $a_1 = A_1$ and for $k > 1$, $a_k = A_k - A_{k-1}$ so that

$$\begin{aligned}\sum_{k=1}^n a_kb_k &= A_1b_1+(A_2-A_1)b_2+\cdots+(A_n-A_{n-1})b_n\\ &= A_nb_n - A_1(b_2-b_1)-\cdots-A_{n-1}(b_n-b_{n-1})\\ &= A_nb_n - \sum_{k=1}^{n-1} A_k(b_{k+1}-b_k).\end{aligned}$$

From this identity, we can easily obtain some famous inequalities.

<u>Abel's Inequality.</u> Let $m \leq \sum_{i=1}^k a_i \leq M$ for $k = 1,2,\ldots,n$ and $b_1 \geq b_2 \geq \cdots \geq b_n > 0$. Then

$$b_1m \leq \sum_{k=1}^n a_kb_k \leq b_1M.$$

<u>Proof.</u> Let $A_k = a_1+a_2+\cdots+a_k$. Applying summation by parts, we have

$$\sum_{k=1}^n a_kb_k = A_nb_n + \sum_{k=1}^{n-1} A_k(b_k-b_{k+1}).$$

The right side is at least

$$mb_n + \sum_{k=1}^{n-1} m(b_k-b_{k+1}) = mb_1$$

and at most

$$Mb_n + \sum_{k=1}^{n-1} M(b_k-b_{k+1}) = Mb_1.$$

<u>K. L. Chung's Inequality.</u> Suppose $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $k = 1,2,\ldots,n$. Then

$$\sum_{i=1}^n a_i^2 \leq \sum_{i=1}^n b_i^2.$$

<u>Proof.</u> Applying summation by parts and Cauchy-Schwarz' inequality, we have

$$\begin{aligned}\sum_{i=1}^n a_i^2 &= \left(\sum_{i=1}^n a_i\right)a_n + \sum_{k=1}^{n-1}\left(\sum_{i=1}^k a_i\right)(a_k-a_{k+1})\\ &\leq \left(\sum_{i=1}^n b_i\right)a_n + \sum_{k=1}^{n-1}\left(\sum_{i=1}^k b_i\right)(a_k-a_{k+1})\\ &= \sum_{i=1}^n a_ib_i\\ &\leq \left(\sum_{i=1}^n a_i^2\right)^{1/2}\left(\sum_{i=1}^n b_i^2\right)^{1/2}.\end{aligned}$$

Squaring and simplifying, we get

$$\sum_{i=1}^n a_i^2 \leq \sum_{i=1}^n b_i^2.$$

Below we will do some more examples to illustrate the usefulness of the summation by parts formula.

<u>Example 1.</u> (*1978 IMO*) Let n be a positive integer and $a_1, a_2, \cdots, a_n$ be a sequence of distinct positive integers. Prove that

$$\sum_{k=1}^n \frac{a_k}{k^2} \geq \sum_{k=1}^n \frac{1}{k}.$$

<u>Solution.</u> Since the a_i's are distinct positive integers, $A_k = a_1 + a_2 + \cdots + a_k$ is at least $1 + 2 + \cdots + k = k(k + 1)/2$.

Applying summation by parts, we have

$$\begin{aligned}\sum_{k=1}^n \frac{a_k}{k^2} &= \frac{A_n}{n^2} + \sum_{k=1}^{n-1} A_k\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right)\\ &\geq \frac{n(n+1)/2}{n^2} + \sum_{k=1}^{n-1}\frac{k(k+1)}{2}\frac{(2k+1)}{k^2(k+1)^2}\\ &= \frac{1}{2}\left(1+\frac{1}{n}+\sum_{k=1}^{n-1}\frac{2k+1}{k(k+1)}\right)\\ &= \frac{1}{2}\left(1+\frac{1}{n}+\sum_{k=1}^{n-1}\left(\frac{1}{k}+\frac{1}{k+1}\right)\right)\\ &= \frac{1}{2}\left(\sum_{k=1}^n\frac{1}{k}+\sum_{k=0}^{n-1}\frac{1}{k+1}\right)\\ &= \sum_{k=1}^n \frac{1}{k}.\end{aligned}$$

Example 2. (*1982 USAMO*) If x is a positive real number and n is a positive integer, then prove that

$$[nx] \ge \frac{[x]}{1} + \frac{[2x]}{2} + \frac{[3x]}{3} + \cdots + \frac{[nx]}{n},$$

where $[t]$ denotes the greatest integer less than or equal to t.

Solution. Let $a_k = [kx]/k$. Then

$$A_k = \sum_{i=1}^{k} \frac{[ix]}{i}.$$

In terms of A_k, we are to prove $[nx] \ge A_n$. The case $n = 1$ is easy. Suppose the cases 1 to $n-1$ are true. Applying summation by parts, we have

$$\sum_{k=1}^{n}[kx] = \sum_{k=1}^{n} a_k k = A_n n - \sum_{k=1}^{n-1} A_k.$$

Using this and the inductive hypothesis,

$$\begin{aligned} A_n n &= \sum_{k=1}^{n}[kx] + \sum_{k=1}^{n-1} A_k \\ &\le \sum_{k=1}^{n}[kx] + \sum_{k=1}^{n-1}[kx] \\ &= [nx] + \sum_{k=1}^{n-1}\big([kx] + [(n-k)x]\big) \\ &\le [nx] + \sum_{k=1}^{n-1}[kx + (n-k)x] \\ &= n[nx], \end{aligned}$$

which yields case n.

Example 3. Consider a polygonal line $P_0P_1P_2\ldots P_n$ such that $\angle P_0P_1P_2 = \angle P_1P_2P_3 = \cdots = \angle P_{n-2}P_{n-1}P_n$, all measure in counterclockwise direction. If $P_0P_1 > P_1P_2 > \cdots > P_{n-1}P_n$, show that P_0 and P_n cannot coincide.

Solution. Let a_k be the length of $P_{k-1}P_k$. Consider the complex plane. Each P_k corresponds to a complex number. We may set $P_0 = 0$ and $P_1 = a_1$. Let $\theta = \angle P_0P_1P_2$ and $z = -\cos\theta + i\sin\theta$, then $P_n = a_1 + a_2 z + \cdots + a_n z^{n-1}$. Applying summation by parts, we get

$$P_n = (a_1 - a_2) + (a_2 - a_3)(1+z) + \cdots + a_n(1 + z + \cdots + z^{n-1}).$$

If $\theta = 0$, then $z = 1$ and $P_n > 0$. If $\theta \ne 0$, then assume $P_n = 0$. We get $P_n(1-z) = 0$, which implies

$$(a_1 - a_2)(1-z) + (a_2 - a_3)(1-z^2) + \cdots + a_n(1-z^n) = 0.$$

Then

$$(a_1 - a_2) + (a_2 - a_3) + \cdots + a_n = (a_1 - a_2)z + (a_2 - a_3)z^2 + \cdots + a_n z^n.$$

However, since $|z| = 1$ and $z \ne 1$, by the triangle inequality,

$$\begin{aligned} &|(a_1 - a_2)z + (a_2 - a_3)z^2 + \cdots + a_n z^n| \\ &< |(a_1 - a_2)z| + |(a_2 - a_3)z^2| + \cdots + |a_n z^n| \\ &= (a_1 - a_2) + (a_2 - a_3) + \cdots + a_n, \end{aligned}$$

which is a contradiction to the last displayed equation. So $P_n \ne 0 = P_0$.

Example 4. Show that the series $\sum_{k=1}^{\infty} \frac{\sin k}{k}$ converges.

Solution. Let $a_k = \sin k$ and $b_k = 1/k$. Using the identity

$$\sin m \sin\frac{1}{2} = \frac{\cos(m - \frac12) - \cos(m + \frac12)}{2},$$

we get

$$A_k = \sin 1 + \cdots + \sin k = \frac{\cos\frac12 - \cos(k + \frac12)}{2\sin\frac12}.$$

Then $|A_k| \le 1/(\sin \frac12)$ and hence

$$\lim_{n\to\infty} A_n b_n = 0.$$

Applying summation by parts, we get

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\sin k}{k} &= \lim_{n\to\infty} \sum_{k=1}^{n} a_k b_k \\ &= \lim_{n\to\infty}\Big(A_n b_n - \sum_{k=1}^{n-1} A_k (b_{k+1} - b_k)\Big) \\ &= \sum_{k=1}^{\infty} A_k \left(\frac{1}{k} - \frac{1}{k+1}\right). \end{aligned}$$

Since

$$\sum_{k=1}^{\infty}\left|A_k\left(\frac1k - \frac{1}{k+1}\right)\right| \le \frac{1}{\sin\frac12}\sum_{k=1}^{\infty}\left(\frac1k - \frac{1}{k+1}\right) = \frac{1}{\sin\frac12},$$

so $\sum_{k=1}^{\infty} \frac{\sin k}{k}$ converges.

Example 5. Let $a_1 \ge a_2 \ge \cdots \ge a_n$ with $a_1 \ne a_n$, $\sum_{i=1}^{n} x_i = 0$ and $\sum_{i=1}^{n} |x_i| = 1$. Find the least number m such that

$$\left|\sum_{i=1}^{n} a_i x_i\right| \le m(a_1 - a_n)$$

always holds.

Solution. Let $S_i = x_1 + x_2 + \cdots + x_i$. Let

$$p = \sum_{x_i > 0} x_i, \quad q = -\sum_{x_i < 0} x_i.$$

Then $p - q = 0$ and $p + q = 1$. So $p = q = \frac12$. Thus, $-\frac12 \le S_k \le \frac12$ for $k = 1,2,\cdots,n$.

Applying summation by parts, we get

$$\begin{aligned} \left|\sum_{i=1}^{n} a_i x_i\right| &= \left|S_n a_n - \sum_{k=1}^{n-1} S_k (a_{k+1} - a_k)\right| \\ &\le \sum_{k=1}^{n-1} |S_k| (a_k - a_{k+1}) \\ &\le \sum_{k=1}^{n-1} \frac12 (a_k - a_{k+1}) \\ &= \frac12 (a_1 - a_n). \end{aligned}$$

When $x_1 = 1/2$, $x_n = -1/2$ and all other $x_i = 0$, we have equality. So the least such m is 1/2.

Example 6. Prove that for all real numbers $a_1, a_2, \ldots, a_n$, there is an integer m among $1,2,\ldots,n$ such that if

$$0 \le \theta_n \le \theta_{n-1} \le \cdots \le \theta_1 \le \frac{\pi}{2},$$

then $\left|\sum_{i=1}^{n} a_i \sin\theta_i\right| \le \left|\sum_{i=1}^{m} a_i\right|$.

Solution. Let $A_i = a_1 + a_2 + \cdots + a_i$ and $b_i = \sin\theta_i$, then $1 \ge b_1 \ge b_2 \ge \cdots \ge b_n \ge 0$. Next let $|A_m|$ be the maximum among $|A_1|, |A_2|, \ldots, |A_n|$. With $a_{n+1} = b_{n+1} = 0$, we apply summation by parts to get

$$\begin{aligned} \left|\sum_{i=1}^{n} a_i \sin\theta_i\right| &= \left|\sum_{i=1}^{n+1} a_i b_i\right| \\ &= \left|\sum_{i=1}^{n} A_i (b_{i+1} - b_i)\right| \\ &\le \sum_{i=1}^{n} |A_m| (b_i - b_{i+1}) \\ &= |A_m| b_1 \\ &\le |A_m|. \end{aligned}$$

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***November 25, 2006.***

Problem 256. Show that there is a rational number q such that

$$\sin 1^\circ \sin 2^\circ \cdots \sin 89^\circ \sin 90^\circ = q\sqrt{10}.$$

Problem 257. Let $n > 1$ be an integer. Prove that there is a unique positive integer $A < n^2$ such that $[n^2/A] + 1$ is divisible by n, where $[x]$ denotes the greatest integer less than or equal to x.
(*Source: 1993 Jiangsu Math Contest*)

Problem 258. (*Due to Mihaiela Vizental and Alfred Eckstein, Arad, Romaina*) Show that if A, B, C are in the interval $(0, \pi/2)$, then

$$f(A,B,C) + f(B,C,A) + f(C,A,B) \geq 3,$$

where $f(x,y,z) = \dfrac{4\sin x + 3\sin y + 2\sin z}{2\sin x + 3\sin y + 4\sin z}$.

Problem 259. Let AD, BE, CF be the altitudes of acute triangle ABC. Through D, draw a line parallel to line EF intersecting line AB at R and line AC at Q. Let P be the intersection of lines EF and CB. Prove that the circumcircle of $\triangle PQR$ passes through the midpoint M of side BC.
(*Source: 1994 Hubei Math Contest*)

Problem 260. In a class of 30 students, number the students 1, 2, …, 30 from best to worst ability (no two with the same ability). Every student has the same number of friends in the class, where friendships are mutual. Call a student <u>*good*</u> if his ability is better than more than half of his friends. Determine the maximum possible number of good students in this class.
(*Source: 1998 Hubei Math Contest*)

****

Solutions

****

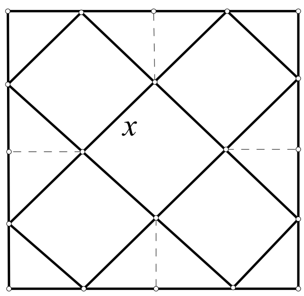
Problem 251. Determine with proof the largest number x such that a cubical gift of side x can be wrapped completely by folding a unit square of wrapping paper (without cutting).

Solution. **CHAN Tsz Lung** (Math, HKU) and **Jeff CHEN** (Virginia, USA).

Let A and B be two points inside or on the unit square such that the line segment AB has length d. After folding, the distance between A and B <u>along the surface of the cube</u> will be at most d because the line segment AB on the unit square after folding will provide one path between the two points along the surface of the cube, which may or may not be the shortest possible.

In the case A is the center of the unit square and B is the point opposite to A on the surface of the cube with respect to the center of the cube, then the distance along the surface of the cube between them is at least $2x$. Hence, $2x \leq d \leq \sqrt{2}/2$. Therefore, $x \leq \sqrt{2}/4$.

The maximum $x = \sqrt{2}/4$ is attainable can be seen by considering the following picture of the unit square.



Commended solvers: **Alex O Kin-Chit** (STFA Cheng Yu Tung Secondary School) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7).

Problem 252. Find all polynomials $f(x)$ with integer coefficients such that for every positive integer n, $2^n - 1$ is divisible by $f(n)$.

Solution. **Jeff CHEN** (Virginia, USA) and **G.R.A. 20 Math Problem Group** (Roma, Italy).

We will prove that the only such polynomials $f(x)$ are the constant polynomials 1 and −1.

Assume $f(x)$ is such a polynomial and $|f(n)| \neq 1$ for some $n > 1$. Let p be a prime which divides $f(n)$, then p also divides $f(n+kp)$ for every integer k. Therefore, p divides $2^{n+kp} - 1$ for all integers $k \geq 0$.

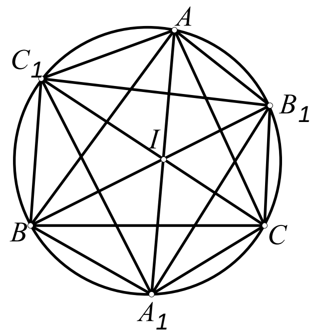
When $k = 0$, p divides $2^n - 1$, which implies $2^n \equiv 1 \pmod{p}$. By Fermat's little theorem, $2^p \equiv 2 \pmod{p}$. Finally, when $k = 1$, we get

$$1 \equiv 2^{n+p} = 2^n\, 2^p \equiv 1\cdot 2 = 2 \pmod{p}$$

implying p divides $2 - 1 = 1$, which is a contradiction.

Problem 253. Suppose the bisector of $\angle BAC$ intersect the arc opposite the angle on the circumcircle of ΔABC at A_1. Let B_1 and C_1 be defined similarly. Prove that the area of $\Delta A_1B_1C_1$ is at least the area of ΔABC.

Solution. **CHAN Tsz Lung** (Math, HKU), **Jeff CHEN** (Virginia, USA) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7).



By a well-known property of the incenter I (see page 1 of <u>*Mathematical Excalibur*</u>, vol. 11, no. 2), we have $AC_1 = C_1I$ and $AB_1 = B_1I$. Hence, $\Delta AC_1B_1 \cong \Delta IC_1B_1$. Similarly, $\Delta BA_1C_1 \cong \Delta IA_1C_1$ and $\Delta CB_1A_1 \cong \Delta IB_1A_1$. Letting $[\cdots]$ denote area, we have

$$[AB_1CA_1BC_1] = 2[A_1B_1C_1].$$

If ΔABC is not acute, say $\angle BAC$ is not acute, then

$$[ABC] \leq \frac{1}{2}[ABA_1C]$$

$$\leq \frac{1}{2}[AB_1CA_1BC_1] = [A_1B_1C_1].$$

Otherwise, ΔABC is acute and we can apply the fact that

$$[ABC] \leq \frac{1}{2}[AB_1CA_1BC_1] = [A_1B_1C_1]$$

(see example 6 on page 2 of <u>*Mathematical Excalibur*</u>, vol. 11, no. 2).

Commended solvers: **Samuel Liló Abdalla** (Brazil) and **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher).

Problem 254. Prove that if a, b, $c > 0$, then

$$\sqrt{abc}\,(\sqrt{a}+\sqrt{b}+\sqrt{c})+(a+b+c)^2 \ge 4\sqrt{3abc\,(a+b+c)}.$$

Solution 1. **José Luis Díaz-Barrero** (Universitat Politècnica de Catalunya, Barcelona, Spain) and **G.R.A. 20 Math Problem Group** (Roma, Italy).

Dividing both sides by $\sqrt{abc(a+b+c)}$, the inequality is equivalent to

$$\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{a+b+c}}+\frac{(\sqrt{a+b+c})^3}{\sqrt{abc}} \ge 4\sqrt{3}.$$

By the AM-GM inequality,

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \ge 3(\sqrt{abc})^{1/3}.$$

Therefore, it suffices to show

$$\frac{3(\sqrt{abc})^{1/3}}{\sqrt{a+b+c}}+\frac{(\sqrt{a+b+c})^3}{\sqrt{abc}}=\frac{3}{t}+t^3 \ge 4\sqrt{3},$$

where again by the AM-GM inequality,

$$t=\frac{\sqrt{a+b+c}}{(\sqrt{abc})^{1/3}}=\sqrt{\frac{a+b+c}{(abc)^{1/3}}} \ge \sqrt{3}.$$

By the AM-GM inequality a third time,

$$\frac{3}{t}+t^3=\frac{3}{t}+\frac{t^3}{3}+\frac{t^3}{3}+\frac{t^3}{3} \ge \frac{4t^2}{\sqrt{3}} \ge 4\sqrt{3}.$$

Solution 2. **Alex O Kin-Chit** (STFA Cheng Yu Tung Secondary School).

By the AM-GM inequality, we have

$$a+b+c \ge 3(abc)^{1/3} \qquad (1)$$

and

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \ge 3(abc)^{1/6}. \qquad (2)$$

Applying (2), (1), the AM-GM inequality and (1) in that order below, we have

$$\begin{aligned}
&\sqrt{abc}\,(\sqrt{a}+\sqrt{b}+\sqrt{c})+(a+b+c)^2\\
&\ge 3(abc)^{2/3}+3(abc)^{1/3}(a+b+c)\\
&\ge 4\left(3(abc)^{2/3}(abc)(a+b+c)^3\right)^{1/4}\\
&\ge 4\left(3(abc)^{2/3}(abc)3(abc)^{1/3}(a+b+c)^2\right)^{1/4}\\
&= 4\sqrt{3abc(a+b+c)}.
\end{aligned}$$

Commended solvers: **Samuel Liló Abdalla** (Brazil), **CHAN Tsz Lung** (Math, HKU), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7).

Problem 255. Twelve drama groups are to do a series of performances (with some groups possibly making repeated performances) in seven days. Each group is to see every other group's performance at least once in one of its day-offs.

Find with proof the minimum total number of performances by these groups.

Solution. **CHAN Tsz Lung** (Math, HKU).

Here are three important observations:

(1) Each group perform at least once.

(2) If more than one groups perform on the same day, then each of these groups will have to perform on another day so the other groups can see its performance in their day-offs.

(3) If a group performs exactly once, on the day it performs, it is the only group performing.

We will show the minimum number of performances is 22. The following performance schedule shows the case 22 is possible.

Day 1: Group 1
Day 2: Group 2
Day 3: Groups 3, 4, 5, 6
Day 4: Groups 7, 8, 9, 3
Day 5: Groups 10, 11, 4, 7
Day 6: Groups 12, 5, 8, 10
Day 7: Groups 6, 9, 11, 12.

Assume it is possible to do at most 21 performances. Let k groups perform exactly once, then $k + 2(12 - k) \le 21$ will imply $k \ge 3$.

<u>Case 1: Exactly 3 groups perform exactly once</u>, say group 1 on day 1, group 2 on day 2 and group 3 on day 3.

(a) If at least 4 groups perform on one of the remaining 4 days, say groups 4, 5, 6, 7 on day 4, then by (2), each of them has to perform on one of the remaining 3 days. By the pigeonhole principle, two of groups 4, 5, 6, 7 will perform on the same day again later, say groups 4 and 5 perform on day 5. Then they will have to perform separately on the last 2 days for the other to see. Then groups 1, 2, 3 once each, groups 4, 5 thrice each and groups 6, 7, …, 12 twice each at least , resulting in at least

$$3 + 2 \times 3 + 7 \times 2 = 23$$

performances, contradiction.

(b) If at most 3 groups perform on each of the remaining 4 days, then there are at most $3 \times 4 = 12$ slots for performances. However, each of groups 4 to 12 has to perform at least twice, yielding at least $9 \times 2 = 18$ (> 12) performances, contradiction.

<u>Case 2: More than 3 groups perform exactly once, say k groups with $k > 3$.</u> By argument similar to case 1(a), we see at most 3 groups can perform on each of the remaining $7 - k$ days (meaning at most $3(7 - k)$ performance slots). Again, the remaining $12 - k$ groups have to perform at least twice, yielding $2(12 - k) \le 3(7 - k)$, which implies $k \le -3$, contradiction.

Commended solvers: **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7) and **Raúl A. SIMON** (Santiago, Chile).

Comments: This was a problem in the 1994 Chinese IMO team training tests. In the Chinese literature, there is a solution using the famous Sperner's theorem which asserts that for a set with n elements, the number of subsets so that no two with one contains the other is at most $\binom{n}{[n/2]}$. We hope to present this solution in a future article.

Olympiad Corner

(continued from page 1)

<u>**Day 2 (July 13, 2006)**</u>

Problem 4. Determine all pairs (x,y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Problem 5. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\cdots P(P(x))\cdots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

Problem 6. Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.