Olympiad Corner

The 9th China Hong Kong Math Olympiad was held on Dec. 2, 2006. The following were the problems.

Problem 1. Let $M$ be a subset of \{1, 2, ..., 2006\} with the following property: For any three elements $x$, $y$ and $z$ ($x < y < z$) of $M$, $x + y$ does not divide $z$. Determine the largest possible size of $M$. Justify your claim.

Problem 2. For a positive integer $k$, let $f(k)$ be the square of the sum of the digits of $k$. (For example $f(123) = (1+2+3)^2 = 36$.) Let $f_1(k) = k f(k)$. Determine the value of $f_{2006}(2006)$. Justify your claim.

Problem 3. A convex quadrilateral $ABCD$ with $AC \neq BD$ is inscribed in a circle with center $O$. Let $E$ be the intersection of diagonals $AC$ and $BD$. If $P$ is a point inside $ABCD$ such that $\angle PAB + \angle PCB = \angle PBC + \angle PDC = 90^\circ$, prove that $O$, $P$ and $E$ are collinear.

Problem 4. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive numbers. If there exists a positive number $M$ such that for every $n = 1, 2, 3, \ldots$, $a_1^2 + a_2^2 + \ldots + a_n^2 < M a_n^2$, then prove that there exists a positive number $M'$ such that for every $n = 1, 2, 3, \ldots$,

$$a_1 + a_2 + \ldots + a_n < M' a_n.$$

Following are some useful facts:

1. If $P$ is outside $C$, then recall $P'$ is found by drawing tangents from $P$ to $C$, say tangent at $X$ and $Y$. Then $P' = OP \cap XY$, where $\cap$ denotes intersection. By symmetry, $OP \perp XY$. So the polar $p$ of $P$ is the line $XY$.

2. Conversely, for distinct points $X, Y$ on $C$, the pole of the line $XY$ is the intersection of the tangents at $X$ and $Y$. Also, it is the point $P$ on the perpendicular bisector of $XY$ such that $O, X, P, Y$ are concyclic since $\angle OXP = 90^\circ = \angle OYP$.

3. Let $x, y, z$ be the polars of distinct points $X, Y, Z$ respectively. Then $Z = x \cap y \iff z = XY$.

\textbf{Proof.} By La Hire’s theorem, $Z$ on $x \cap y \iff X$ on $z$ and $Y$ on $z \iff z = XY$.

4. Let $W, X, Y, Z$ be on $C$. The polar $p$ of $P = XY \cap WZ$ is the line through $Q = WX \cap ZY$ and $R = ZX \cap WY$.

\textbf{Proof.} Let $T, S$ be the poles of $s = XY$, $t = WZ$ respectively. Then $P = s \cap t$. By fact (3), $S = x \cap y$, $T = w \cap z$ and $P = ST$. For hexagon $WXXYYZ$, we have $Q = WX \cap ZY$, $S = XX \cap YY$, $R = ZX \cap WY$, where $XX$ denotes the tangent line at $X$. By Pascal’s theorem (see Mathematical Excalibur, vol. 10, no. 3, p.1), $Q, S, R$ are collinear. Similarly, considering the hexagon $XWWYZZ$, we see $Q, T, R$ are collinear. Therefore, $p = ST = QR$.

Next we will present some examples using the pole-polar transformation.

\textbf{Example 1.} Let $UV$ be a diameter of a semicircle. $PQ$ are two points on the semicircle with $UP < UQ$. The tangents to the semicircle at $P$ and $Q$ meet at $R$. If $S = UP \cap VQ$, then prove that $RS \perp UV$.

\textbf{Solution (due to CHENG Kei Tsi).} Let $K = PQ \cap UV$. With respect to the circle, by fact (4), the polar of $K$ passes through $UP \cap VQ = S$. Since the tangents to the semicircle at $P$ and $Q$ meet at $R$, by fact (1), the polar of $R$ is $PQ$. Since $K$ is on line $PQ$, which is the polar of $R$, by La Hire’s theorem, $R$ is on the polar of $K$. So the polar of $K$ is the line $RS$. As $K$ is on the diameter $UV$ extended, by the definition of polar, we get $RS \perp UV$.
Example 2. Quadrilateral $ABCD$ has an inscribed circle $\Gamma$ with sides $AB$, $BC$, $CD$, $DA$ tangent to $\Gamma$ at $G$, $H$, $K$, $L$ respectively. Let $AB \cap CD = E$, $AD \cap BC = F$ and $GK \cap HL = P$. If $O$ is the center of $\Gamma$, then prove that $OP \perp EF$.

Solution. Consider the pole-polar transformation with respect to the inscribed circle. By fact (1), the polars of $E$, $F$ are lines $GK$, $HL$ respectively. Since $GK \cap HL = P$, by fact (3), the polar of $P$ is line $EF$. By the definition of polar, we get $OP \perp EF$.

Example 3. (1997 Chinese Math Olympiad) Let $ABCD$ be a cyclic quadrilateral. Let $AB \cap CD = P$ and $AD \cap BC = Q$. Let the tangents from $Q$ meet the circumcircle of $ABCD$ at $E$ and $F$. Prove that $P$, $E$, $F$ are collinear.

Solution. Consider the pole-polar transformation with respect to the circumcircle of $ABCD$. Since $P = AB \cap CD$, by fact (4), the polar of $P$ passes through $AD \cap BC = Q$. By La Hire’s theorem, $P$ is on the polar of $Q$, which by fact (1), is the line $EF$.

Example 4. (1998 Austrian-Polish Math Olympiad) Distinct points $A$, $B$, $C$, $D$, $E$, $F$ lie on a circle in that order. The tangents to the circle at the points $A$ and $D$, the lines $BF$ and $CE$ are concurrent. Prove that the lines $AD$, $BC$, $EF$ are either parallel or concurrent.

Solution. Let $O$ be the center of the circle and $X = AA \cap DD \cap BB \cap CE$.

If $BC \parallel EF$, then by symmetry, lines $BC$ and $EF$ are perpendicular to line $OX$. Since $AD \perp OX$, we get $BC \parallel EF \parallel AD$.

If lines $BC$, $EF$ intersect, then by fact (4), the polar of $X = CE \cap BF$ passes through $BC \parallel EF$. Since the tangents at $A$ and $D$ intersect at $X$, by fact (1), the polar of $X$ is line $AD$. Therefore, $AD$, $BC$, $EF$ are concurrent in this case.

Example 5. (2006 China Western Math Olympiad) As in the figure below, $AB$ is a diameter of a circle with center $O$. $C$ is a point on $AB$ extended. A line through $C$ cuts the circle with center $O$ at $D$, $E$. $OF$ is a diameter of the circumcircle of $\triangle BOD$ with center $O_1$. Line $CF$ intersect the circumcircle again at $G$. Prove that $O, A, E, G$ are concyclic.

Solution (due to WONG Chiu Wai). Let $AE \parallel BD = P$. By fact (4), the polar of $P$ with respect to the circle having center $O$ is the line through $BA \cap DE = C$ and $AD \cap EB = H$. Then $OP \perp CH$. Let $Q = OP \cap CH$.

We claim $Q = G$. Once this shown, we will have $P = BD \cap OG$. Then $PE.PM = PD.PB = PG.PO$, which implies $O, A, E, G$ are concyclic.

To show $Q = G$, note that $\angle PQH$, $\angle PDH$ and $\angle PEH$ are $90^\circ$, which implies $P, E, Q, H, D$ are concyclic. Then $\angle PQD = \angle PED = \angle DBO$, which implies $Q, D, B, O$ are concyclic. Therefore, $Q = G$ since they are both the point of intersection (other than $O$) of the circumcircle of $\triangle BOD$ and the circle with diameter $OC$.

Example 6. (2006 China Hong Kong Math Olympiad) A convex quadrilateral $ABCD$ with $AC \neq BD$ is inscribed in a circle with center $O$. Let $E$ be the intersection of diagonals $AC$ and $BD$. If $P$ is a point inside $ABCD$ such that $\angle PAB + \angle PCB = \angle PBC + \angle PDC = 90^\circ$, prove that $O, P$ and $E$ are collinear.

Solution (due to WONG Chiu Wai). Let $\Gamma$, $\Gamma_1$, $\Gamma_2$ be the circumcircles of quadrilateral $ABCD$, $\Delta PAC$, $\Delta PBD$ with centers $O$, $O_1$, $O_2$ respectively. We first show that the polar of $O_1$ with respect to $\Gamma$ is line $AC$. Since $OO_1$ is the perpendicular bisector of $AC$, by fact (1), all we need to show is that $\angle AOC + \angle AO_1C = 180^\circ$.

For this, note

\[ \angle APC = 360^\circ - (\angle PAB + \angle PCB + \angle ABC) = 270^\circ - \angle ABC = 90^\circ + \angle ACD \]

and so

\[ \angle AO_1C = 2(180^\circ - \angle APC) = 2(90^\circ - \angle ACD) = 180^\circ - 2\angle ACD = 180^\circ - \angle AOC. \]

Similarly, the polar of $O_2$ with respect to $\Gamma$ is line $BD$. By fact (3), since $E = AC \cap BD$, the polar of $E$ with respect to $\Gamma$ is line $O_1O_2$. So $OE \perp O_1O_2$.

Next we will consider radical axis and radical center, see Mathematical Excalibur, vol. 4, no. 3, p. 2.) Among $\Gamma$, $\Gamma_1$, $\Gamma_2$, two of the pairwise radical axes are lines $AC$ and $BD$. This implies $E$ is the radical center. Since $\Gamma_1$, $\Gamma_2$ intersect at $P$, so $PE$ is the radical axis of $\Gamma_1$, $\Gamma_2$, which implies $PE \perp O_1O_2$. Combining with $OE \perp O_1O_2$ proved above, we see $O$, $P$ and $E$ are collinear.

(continued on page 4)
Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong. The deadline for submitting solutions is January 25, 2007.

Problem 261. Prove that among any 13 consecutive positive integers, one of them has sum of its digits (in base 10) divisible by 7.

Problem 262. Let O be the center of the circumscribed triangle of $\Delta ABC$ and let $AD$ be a diameter. Let the tangent at $D$ to the circumscribed interline $BC$ at $P$. Let line $PO$ intersect lines $AC, AB$ at $M, N$ respectively. Prove that $OM = ON$.

Problem 263. For positive integers $m, n$, consider a $(2m+1) \times (2n+1)$ table, where in each cell, there is exactly one ant. For a prime number $p$, consider a $(2^m+1) \times (2^n+1)$ table. Prove that there exists a cell with no ant.

Problem 264. For a prime number $p > 3$ and arbitrary integers $a, b$, prove that $ab^p - ba^p$ is divisible by $6p$.

Problem 265. Determine (with proof) the maximum of

$$\sum_{j=1}^{n} (x_j^3 - x_j^2)$$

where $x_1, x_2, \ldots, x_n$ are nonnegative real numbers whose sum is 1.

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Solutions

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Problem 256. Show that there is a rational number $q$ such that

$$\sin 1 \cdot \sin 2 \cdot \ldots \cdot \sin 89^\circ = q \sqrt{10}.$$

Solution 1. Jeff CHEN (Virginia, USA), Koyrtis G. CHRYSSOSTOMOS (Larissa, Greece, teacher), G.R.A. 20 Math Problem Group (Roma, Italy) and D. Kipp JOHNSON (Valley Catholic School, Teacher, Beaverton, Oregon, USA).

Let $\omega = e^{2\pi i/180}$. Then

$$P(z) = \sum_{k=0}^{179} z^k \prod_{j=1}^{179} (z - \omega^j).$$

Using $\sin x = \frac{e^{ix} - e^{-ix}}{2i} = \frac{e^{2ix} - 1}{2ie^{ix}}$, we have

$$\prod_{k=1}^{90} \sin k^4 = \prod_{k=1}^{90} \omega^{k^4} = \frac{\omega^{90}}{2^{179}90^{180}}.$$

Also, $\prod_{k=1}^{90} \sin k^4 = \prod_{k=1}^{90} \omega^{k^4} = \frac{\omega^{90}}{2^{179}90^{180}}.$

Then

$$\prod_{k=1}^{90} \sin k^4 = \frac{90}{2^{179}}.$$

Therefore, $\prod_{k=1}^{90} \sin k^4 = \frac{3}{2^{179}} \sqrt{10}.$

Problem 257. Let $n > 1$ be an integer. Prove that there is a unique positive integer $A < n^2$ such that $[n^2/A] + 1$ is divisible by $n$, where $[x]$ denotes the greatest integer less than or equal to $x$. (Source: 1993 Jiangsu Math Contest)

Solution. Jeff CHEN (Virginia, USA), G.R.A. 20 Math Problem Group (Roma, Italy) and Fai YUNG.

We claim the unique number is $A = n^2 + 1$.

If $n = 2$, then $1 \leq A < n^2 = 4$ and only $A = 3$ works. If $n > 2$, then $[n^2/A] + 1$ is divisible by $n$ implies $n^2 + 1 \geq \left[ \frac{n^2}{A} \right] + 1 \geq n$. This leads to

$$A \leq n^2 - n - 1 = n + 1 - \frac{1}{n - 1}.$$

The case $A = n$ does not work because $[n^2/n] + 1 = n + 1$ is not divisible by $n$ when $n > 1$.

For $0 < A < n$, assume $[n^2/A] + 1 = kn$ for some positive integer $k$. This leads to

$$kn = n^2 - [n^2/A] - 1 = n^2 - kn,$$

which implies $n < kA \leq (n^2/A)n < n + 1$. This is a contradiction as $kA$ is an integer and cannot be strictly between $n$ and $n + 1$.

Problem 258. (Due to Mihaiela Vizental and Alfred Eckstein, Arad, Romania) Show that if $A, B, C$ are in the interval $(0, \pi/2)$, then

$$f(A, B, C) = f(B, C, A) = f(C, A, B) \geq 3,$$

where

$$f(x, y, z) = \frac{4 \sin x + 3 \sin y + 2 \sin z}{2 \sin x + 3 \sin y + 4 \sin z}.$$

Solution. Samuel Lilo Abdalla (Brazil), Koyrtis G. CHRYSSOSTOMOS (Larissa, Greece, teacher) and Fai YUNG.

Note

$$f(x, y, z) + 1 = \frac{6 \sin x + 6 \sin y + 6 \sin z}{2 \sin x + 3 \sin y + 4 \sin z}.$$

For $a, b, c > 0$, by the $AM-HM$ inequality, we have

$$(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9.$$

Multiplying by $\frac{3}{2}$ on both sides, we get

$$(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 6. (*)$$

Let $r = \sin A, s = \sin B, t = \sin C, a = 1/(2r + 3s + 4t), b = 1/(2s + 3t + 4r)$ and
c = 1/(2t + 3r + 4s). Then
\[ \frac{2}{3} \left( \frac{a}{b} + \frac{1}{c} \right) = 6r + 6s + 6t. \]

Using (*), we get
\[ f(A, B, C) + f(B, C, A) + f(C, A, B) + 3 \]
\[ \frac{6r + 6s + 6t}{2r + 3s + 4r} = \frac{2s + 3t + 4r}{2t + 3r + 4s} \geq 6. \]

The result follows.

**Problem 259.** Let $AD$, $BE$, $CF$ be the altitudes of acute triangle $ABC$. Through $D$, draw a line parallel to line $EF$ intersecting line $AB$ at $R$ and line $AC$ at $Q$. Let $P$ be the intersection of lines $EF$ and $CB$. Prove that the circumcircle of $\triangle PQR$ passes through the midpoint $M$ of side $BC$.

(Source: 1994 Hubei Math Contest)

**Solution. Jeff CHEN** (Virginia, USA).

![Diagram](image)

Observe that
1. $\angle BFC = 90^\circ = \angle BEC$ implies $B, E, F, C$ concyclic;
2. $\angle AEB = 90^\circ = \angle ADB$ implies $A, B, D, E$ concyclic.

By (1), we have $\angle ACB = \angle AFE$. From $EF \parallel QR$, we get $\angle AFE = \angle ARQ$. So $\angle ACB = \angle ARQ$. Then $B, Q, R, C$ are concyclic. By the intersecting chord theorem,
\[ RD \cdot QD = BD \cdot CD \quad (*) \]

Since $\angle BEC = 90^\circ$ and $M$ is the midpoint of $BC$, we get $MB = ME$ and $\angle EBM = \angle EBM$. Now
\[ \angle EBM = \angle EPM + \angle EBP \]
\[ \angle EBM = \angle DEM + \angle BED. \]

By (1) and (2), $\angle EBP = \angle BCF = 90^\circ = \angle ABC = \angle BAD = \angle BAD$. So $\angle EPM = \angle DEM$. Then right triangles $EPM$ and $DEM$ are similar. We have $ME/MP = MD/ME$ and so
\[ MB^2 = ME^2 = MD \cdot MP = MD(MD + PD) = MD^2 + PD. \]

Then $MD \cdot PD = MB^2 - MD^2 = (MB - MD)(MB + MD) = BD(MC + MD) = BD \cdot CD$.

Using (*), we get $RD \cdot QD = MD \cdot PD$. By the converse of the intersecting chord theorem, $P, Q, R, M$ are concyclic.

**Commented solvers: Koyritis G. CHRYSOSTOMOS** (Larissa, Greece, teacher).

**Problem 260.** In a class of 30 students, number the students $1, 2, \ldots, 30$ from best to worst ability (no two with the same ability). Every student has the same number of friends in the class, where friendships are mutual. Call a student good if his ability is better than more than half of his friends. Determine the maximum possible number of good students in this class.

(Source: 1998 Hubei Math Contest)

**Solution. Jeff CHEN** (Virginia, USA) and Fai YUNG.

Suppose each student has $m$ friends and $n$ is the maximum number of good students. There are $15m$ pairs of friendship.

For $m$ odd, $m = 2k - 1$ for some positive integer $k$. For $j = 1, 2, \ldots, k$, student $j$ has at least $(2k-j) > k > m/2$ worse friends, hence student $j$ is good. For the other $n-k$ good students, every one of them has at least $k$ worse friends. Then
\[ \sum_{j=1}^{k} (2k-j) + (n-k)k \leq 15(2k - 1). \]

Solving for $n$, we get
\[ n \leq 30.5 - \frac{15}{k} \leq 30.5 - \sqrt{30} < 26. \]

For $m$ even, $m = 2k$ for some positive integer $k$. For $j = 1, 2, \ldots, k$, student $j$ has at least $(2k-1-j) > k = m/2$ worse friends, hence student $j$ is good. For the other $n-k$ good students, every one of them has at least $k+1$ worse friends. Then
\[ \sum_{j=1}^{k-1} (2k+1-j) + (n-k)(k+1) \leq 15-2k. \]

Solving for $n$, we get
\[ n \leq 31.5 - \frac{31}{k+1} \leq 31.5 - \sqrt{62} < 24. \]

Therefore, $n \leq 25$. For an example of $n = 25$, in the odd case, we need to take $k = 5$ (so $m = 9$). Consider the $6 \times 5$ matrix $M$ with $M_{ij} = 5(i-1) + j$. For $M_{ij}$, let his friends be $M_{ij}$, $M_{i,j}$ and $M_{i,j}$ for all $k \neq j$. For $M_{ij}$ with $1 < i < 6$, let his friends be $M_{ij}$, $M_{i,j}$ and $M_{i,j}$ for all $k \neq j$. For $M_{ij}$, let his friends be $M_{ij}$ and $M_{i,j}$ for all $k < 6$ and $k \neq j$. It is easy to check $1$ to $25$ are good.

**Pole and Polar**

(continued from page 2)

**Example 7.** (1998 IMO) Let $I$ be the incenter of triangle $ABC$. Let the incircle of $ABC$ touch the sides $BC$, $CA$ and $AB$ at $K$, $L$ and $M$ respectively. The line through $B$ parallel to $MK$ meets the lines $LM$ and $LK$ at $R$ and $S$ respectively. Prove that angle $RIS$ is acute.

**Solution.** Consider the pole-polar transformation with respect to the incircle. Due to tangency, the polars of $B$, $K$, $L$, $M$ are lines $MK$, $BC$, $CA$, $AB$ respectively. Observe that $B$ is sent to $B' = IB \cap MK$ under the inversion with respect to the incircle. Since $B'$ is on line $MK$, which is the polar of $B$, by La Hire’s theorem, $B$ is on the polar of $B'$. Since $MK || RS$, so the polar of $B'$ is line $RS$. Since $R, B, S$ are collinear, their polars concur at $B'$.

Next, since the polars of $K$, $L$ intersect at $C$ and since $K$, $L$, $S$ are collinear, their polars concur at $C$. Then the polar of $S$ is $B'C$. By the definition of polar, we get $IS \perp B'C$. By a similar reasoning, we also get $IR \perp B'A$. Then \[ \angle RIS = 180^\circ - \angle AB'C. \]

To finish, we will show $B'$ is inside the circle with diameter $AC$, which implies $\angle AB'C > 90^\circ$ and hence $\angle RIS < 90^\circ$. Let $T$ be the midpoint of $AC$.

Then $2B'T = B'C + B'A$
\[ = (B'K + KK) + (B'M + MM) \]
\[ = KK + MM. \]

Since $KK$ and $MM$ are nonparallel, $B'T < KK + MM$.

Therefore, $B'$ is inside the circle with diameter $AC$. 