Mathematical Excalibur

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Olympiad Corner

Below are the 2007 Asia Pacific Math Olympiad problems.

Problem 1. Let S be a set of 9 distinct integers all of whose prime factors are at most 3. Prove that S contains 3 distinct integers such that their product is a perfect cube.

Problem 2. Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter, and H be the orthocenter of the triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.

Problem 3. Consider n disks $C_1, C_2, \cdots, C_n$ in a plane such that for each $1 \le i < n$, the center C_i is on the circumference of C_{i+1}, and the center of C_n is on the circumference of C_1. Define the score of such an arrangement of n disks to be the number of pairs (i, j) for which C_i properly contains C_j. Determine the maximum possible score.

Problem 4. Let x, y and z be positive real numbers such that $\sqrt{x}+\sqrt{y}+\sqrt{z} =$ 1. Prove that

$$\frac{x^2+yz}{\sqrt{2x^2(y+z)}}+\frac{y^2+zx}{\sqrt{2y^2(z+x)}}+\frac{z^2+xy}{\sqrt{2z^2(x+y)}} \ge 1.$$

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 31, 2007***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI
Department of Mathematics
The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

From *How to Solve It* to Problem Solving in Geometry

K. K. Kwok
Munsang College (HK Island)

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Geometry is the science of
correct reasoning on
incorrect figures.

If you can't solve a problem,
then there is an easier
problem you can solve, find
it.
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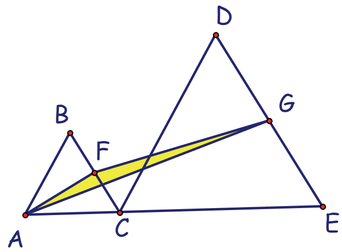
George Pölya

幾何是：在靜止中看出動態，從變幻中覓得永恆

數學愛好者，強

Example 1. In the figure below, C is a point on AE. ΔABC and ΔCDE are equilateral triangles. F and G are the midpoints of BC and DE respectively. If the area of ΔABC is 24 cm^2, the area of ΔCDE is 60 cm^2, find the area of ΔAFG.



Idea and solution outline:
This question is easy enough and can be solved by many different approaches. One of them is to recognize that the extensions of AF and CG are parallel. (Why? At what angles do they intersect line AE?) Thus $[AFC] = [AFG]$.

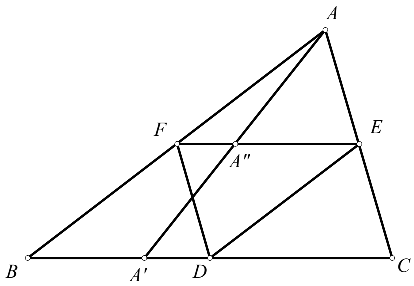
Example 2. In ΔABC, $AB = AC$. A point P on the plane satisfies $\angle ABP = \angle ACP$. Show that P is either on BC or on the perpendicular bisector of BC.

Solution:
Apply the sine law to ΔABP and ΔACP, we have

$$\sin\angle APB = \frac{AB\sin\angle ABP}{AP} = \frac{AC\sin\angle ACP}{AP} = \sin\angle APC.$$

Thus, either $\angle APB = \angle APC$ or $\angle APB + \angle APC = 180^\circ$. The first case implies $\Delta ABP \cong \Delta ACP$, so $BP = CP$ and P lies on the perpendicular bisector of BC. The second case implies P lies on BC.

Example 3.[Tournament of Towns1993] Vertices A, B and C of a triangle are connected to points A', B' and C' lying in their respective opposite sides of the triangle (not at vertices). Can the midpoints of the segments AA', BB' and CC' lie in a straight line?

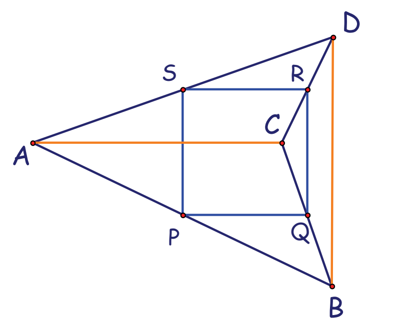


Solution outline:
Let D, E and F be midpoints of BC, AC, and AB respectively. Given any point A' on BC, let AA' intersect EF at A''. Then it is easy to see that A'' is indeed the midpoint of AA'.

Therefore, the midpoints of the segments AA', BB' and CC' lie respectively on EF, DF and DE, and cannot be collinear.

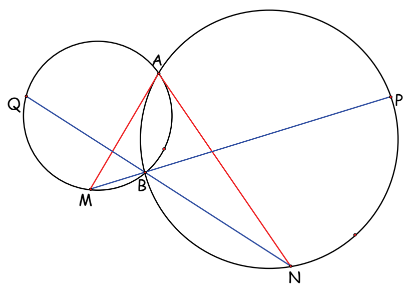
Example 4. [Tournament of Towns 1993] Three angles of a non-convex, non-self-intersecting quadrilateral are equal to 45 degrees (i.e. the last equals 225 degrees). Prove that the midpoints of its sides are vertices of a square.

Idea:
Do you know a similar, but easier problem? For example, the famous *Varignon Theorem*: By joining the midpoints of the sides of an arbitrary quadrilateral, a parallelogram is formed.



Solution outline:
Extend BC to cut AD at O. Then ΔOAB and ΔOCD are both isosceles right-angled triangle. It follows that a 90° rotation about O will map A into B and C into D, so that $AC = BD$ and they are perpendicular to each other.

Example 5. [Tournament of Towns 1994] Two circles intersect at the points A and B. Tangent lines drawn to both of the circles at the point A intersect the circles at the points M and N. The lines BM and BN intersect the circles once more at the points P and Q respectively. Prove that the segments MP and NQ are equal.



Idea:
MP and NQ are sides of the triangles ΔAQN and ΔAMP respectively, so it is natural for us to prove that the two triangles are congruent. It is easy to observe that the two triangles are similar, so what remains to prove is either $AQ = AM$ or $AP = AN$. Note that we can transmit the information between the two circles by using the theorem on alternate segment at A.

Solution outline:

(1) Observe that $\Delta AQN \sim \Delta AMP$.

(2) $AP = AN$ follows from computing

$\angle APN = \angle APB + \angle BPN$

$= \angle ANB + \angle BAN$ [$\angle$s in same segment]

$= \angle ANB + \angle AQN$ [$\angle$ in alt. segment]

$= 180^\circ - \angle QAN$

$= 180^\circ - \angle MAP$ [by step (1)]

$= \angle AMB + \angle APB$

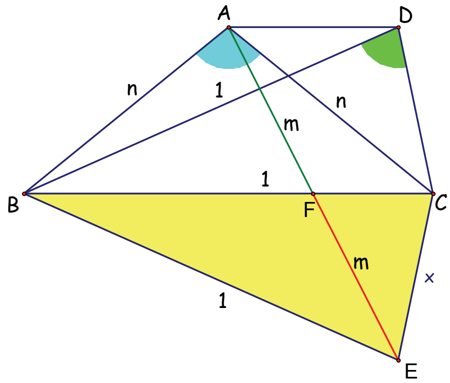
$= \angle AMB + \angle MAB$ [$\angle$ in alt. segment]

$= \angle ABP$ [ext. $\angle$ of Δ]

$= \angle ANP$ [$\angle$s in the same segment].

Example 6. $ABCD$ is a trapezium with $AD \,//\, BC$. It is known that $BC = BD = 1$, $AB = AC$, $CD < 1$ and $\angle BAC + \angle BDC = 180^\circ$, find CD.

Idea:
The condition $\angle BAC + \angle BDC = 180^\circ$ leads us to consider a cyclic quadrilateral. If we reflect ΔBDC across BC, a cyclic quadrilateral is formed.



Solution outline:
(1) Let E be the reflection of D across BC.

(2) $\angle BAC + \angle BDC = 180^\circ$
$\Rightarrow \angle BAC + \angle BEC = 180^\circ$
$\Rightarrow ABEC$ is cyclic,

$AD \,//\, BC \Rightarrow AF = FE,$

$AB = AC \Rightarrow \angle BEF = \angle FEC$

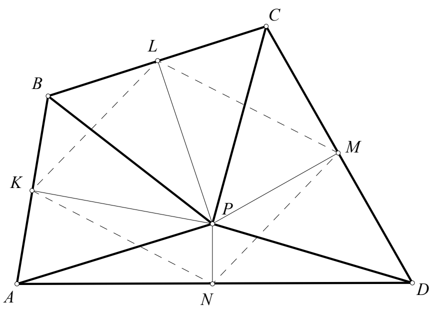
$$\Rightarrow \frac{FC}{BF} = \frac{EC}{BE} = EC\,.$$

(3) Let $AF = FE = m$, $AB = AC = n$ and $DC = EC = x$. It follows from Ptolemy's theorem that $AE\times BC = AC\times BE + AB\times EC$, i.e. $2m = n\,(1 + x)$. Now

$$\frac{2}{1+x} = \frac{n}{m} = \frac{AC}{AF} = \frac{BE}{BF} = \frac{BC}{BF} = \frac{BF+FC}{BF}$$
$$= 1 + \frac{FC}{BF} = 1 + \frac{EC}{BE} = 1 + x\,,$$

i.e. $(1 + x)^2 = 2$. Therefore, $x = \sqrt{2} - 1$.

Example 7. [Tournament of Towns 1995] Let P be a point inside a convex quadrilateral $ABCD$. Let the angle bisector of $\angle APB$, $\angle BPC$, $\angle CPD$ and $\angle DPA$ meet AB, BC, CD and DA at K, L, M and N respectively. Find a point P such that $KLMN$ is a parallelogram.



Idea:
The *angle bisector theorem* enables us to replace the ratios that K, L, N and M divided the sides of the quadrilateral by the ratios of the distance from P to A, B, C and D. For instance, we have

$$\frac{AK}{KB} = \frac{AP}{BP} \text{ and } \frac{AN}{ND} = \frac{AP}{DP}$$

If $BP = DP$, we have $\dfrac{AK}{KB} = \dfrac{AN}{ND}$ and hence $KN//BD$. Similarly, we have $LM//BD$ and so $KN//LM$.

Therefore, we shall look for a point P such that $BP = DP$ and $AP = CP$.

Solution outline:
(1) Let P be the point of intersection of the perpendicular bisectors of the diagonals AC and BD. Then $AP = CP$ and $BP = DP$.

(2) By the angle bisector theorem, we have

$$\frac{AK}{KB} = \frac{AP}{BP} = \frac{AP}{DP} = \frac{AN}{ND}$$

and so $KN \,//\, BD$. Similarly, $LM//BD$, $KL//AC$ and $MN//AC$.

Hence $KN//LM$ and $KL//NM$, which means that $KLMN$ is a parallelogram.

Remark: Indeed, point P in the solution above is the only point that satisfies the condition given in the problem.

Example 8. [IMO 2001] Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$ac + bd = (b + d + a - c)(b + d - a + c)$.

Prove that $ab + cd$ is not prime.

Remark: This is a difficult problem in number theory. However, we would like to present a solution aided by geometrical insights!

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***May 31, 2007.***

Problem 271. There are 6 coins that look the same. Five of them have the same weight, each of these is called a good coin. The remaining one has a different weight from the 5 good coins and it is called a bad coin. Devise a scheme to weigh groups of the coins using a scale (not a balance) three times only to determine the bad coin and its weight.

Problem 272. ΔABC is equilateral. Find the locus of all point Q inside the triangle such that

$$\angle QAB+\angle QBC+\angle QCA=90^\circ.$$

Problem 273. Let R and r be the circumradius and the inradius of triangle ABC. Prove that

$$\frac{\cos A}{\sin^2 A}+\frac{\cos B}{\sin^2 B}+\frac{\cos C}{\sin^2 C}\ge\frac{R}{r}.$$

(*Source: 2000 Beijing Math Contest*)

Problem 274. Let $n < 11$ be a positive integer. Let p_1, p_2, p_3, p be prime numbers such that $p_1+p_3^n$ is prime. If $p_1+p_2=3p$, $p_2+p_3=p_1^n(p_1+p_3)$ and $p_2>9$, then determine $p_1p_2p_3^n$.

(*Source: 1997 Hubei Math Contest*)

Problem 275. There is a group of children coming from 11 countries (at least one child from each of the 11 countries). Their ages are from 7 to 13. Prove that there are 5 children in the group, for each of them, the number of children in the group with the same age is greater than the number of children in the group from the same country.

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Solutions

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Problem 266. Let

$$N=1+10+10^2+\cdots+10^{1997}.$$

Determine the 1000th digit after the decimal point of the square root of N in base 10. (*Source: 1998 Putnam Exam*)

Solution. **Jeff CHEN** (Virginia, USA), **Irfan GLOGIC** (Sarajevo College, 4th grade, Sarajevo, Bosnia and Herzegovina), **Salem MALIKIĆ** (Sarajevo College, 3rd grade, Sarajevo, Bosnia and Herzegovina), **Anna Ying PUN** (HKU, Math, Year 1) and **Fai YUNG**.

The answer is the same as the unit digit of $10^{1000}\sqrt{N}$. We have

$$10^{1000}\sqrt{N}=10^{1000}\sqrt{\frac{10^{1998}-1}{9}}=\frac{\sqrt{10^{3998}-10^{2000}}}{3}.$$

Since

$$(10^{1999}-7)^2<10^{3998}-10^{2000}<(10^{1999}-4)^2,$$

so it follows that $10^{1000}\sqrt{N}$ is between $(10^{1999}-7)/3=33\cdots31$ and $(10^{1999}-4)/3=33\cdots32$. Therefore, the answer is 1.

Commended solvers: **Simon YAU** and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 6).

Problem 267. For any integer a, set

$$n_a=101a-100\cdot 2^a.$$

Show that for $0\le a, b, c, d\le 99$, if

$$n_a+n_b\equiv n_c+n_d \pmod{10100},$$

then $\{a,b\}=\{c,d\}$. (*Source: 1994 Putnam Exam*)

Solution. **Jeff CHEN** (Virginia, USA), **Irfan GLOGIC** (Sarajevo College, 4th grade, Sarajevo, Bosnia and Herzegovina), **Salem MALIKIĆ** (Sarajevo College, 3rd grade, Sarajevo, Bosnia and Herzegovina), **Anna Ying PUN** (HKU, Math, Year 1) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 6).

If $n_a+n_b\equiv n_c+n_d \pmod{10100}$, then $a+b\equiv n_a+n_b\equiv n_c+n_d\equiv c+d \pmod{100}$ and $2^a+2^b\equiv n_a+n_b\equiv n_c+n_d\equiv 2^c+2^d \pmod{101}$.

By Fermat's little theorem, $2^{100}\equiv 1 \pmod{101}$ and so $2^{a+b}\equiv 2^{c+d} \pmod{101}$. Next

$$\begin{aligned}(2^a-2^c)(2^a-2^d)&=2^a(2^a-2^c-2^d)+2^{c+d}\\&\equiv 2^a(-2^b)+2^{a+b}\\&=0 \pmod{101}.\end{aligned}$$

So $2^a\equiv 2^c \pmod{101}$ or $2^a\equiv 2^d \pmod{101}$.

Now we claim that if $0\le s\le t\le 99$ and $2^s\equiv 2^t \pmod{101}$, then $s=t$. To see this, let k be the least positive integer such that $2^k\equiv 1 \pmod{101}$. Dividing 100 by k, we get $100=kq+r$ with $0\le r<k$. Since $2^r=2^{100-kq}\equiv 1 \pmod{101}$ too, so $r=0$, then k is a divisor of 100.

Clearly, $1<2^1, 2^2, 2^4, 2^5<101$ and $2^{10}=1024\equiv 14 \pmod{101}$, $2^{20}\equiv 14^2\equiv -6 \pmod{101}$, $2^{25}\equiv(-6)32\equiv 10 \pmod{101}$, $2^{50}\equiv 10^2\equiv -1 \pmod{101}$. Hence $k=100$. Finally $2^{t-s}\equiv 1 \pmod{101}$ and $0\le t-s<100$ imply $t-s=0$, proving the claim.

By the claim, we get $a=c$ or $a=d$. From $a+b\equiv c+d \pmod{100}$ and $0\le a, b, c, d\le 99$, we get $a=c$ implies $b=d$ and similarly $a=d$ implies $b=c$. The conclusion follows.

Problem 268. In triangle ABC, $\angle ABC=\angle ACB=40^\circ$. Points P and Q are inside the triangle such that $\angle PAB=\angle QAC=20^\circ$ and $\angle PCB=\angle QCA=10^\circ$. Must B, P, Q be collinear? Give a proof. (*Source: 1994 Shanghai Math Competition*)

Solution. **Jeff CHEN** (Virginia, USA), **Courtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher), **Irfan GLOGIC** (Sarajevo College, 4th grade, Sarajevo, Bosnia and Herzegovina), **Kelvin LEE** (Winchester College, England) **Salem MALIKIĆ** (Sarajevo College, 3rd grade, Sarajevo, Bosnia and Herzegovina) and **NG Ngai Fung** (STFA Leung Kau Kui College).

Since lines AP, BP, CP concur, by the trigonometric form of Ceva's theorem,

$$\frac{\sin\angle CBP\sin\angle BAP\sin\angle PCA}{\sin\angle PBA\sin\angle PAC\sin\angle PCB}=1,$$

which implies

$$\frac{\sin\angle CBP}{\sin\angle PBA}=\frac{\sin 80^\circ\sin 10^\circ}{\sin 20^\circ\sin 30^\circ}=\frac{\cos 10^\circ\sin 10^\circ}{\sin 20^\circ/2}=1.$$

So $\angle CBP=\angle PBA=20^\circ$. Replacing P by Q above, we similarly have

$$\frac{\sin\angle CBQ}{\sin\angle QBA}=\frac{\sin 20^\circ\sin 30^\circ}{\sin 80^\circ\sin 10^\circ}=1.$$

So $\angle QBA=\angle CBQ=20^\circ$. Then B, P, Q are on the bisector of $\angle ABC$.

Commended solvers: **CHIU Kwok Sing** (Belilios Public School), **FOK Pak Hei** (Pui Ching Middle School), **Anna Ying PUN** (HKU, Math, Year 1) and **Simon YAU**.

Problem 269. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0=0$, $a_{n+1}=f(a_n)$ for all $n\ge 0$. Prove that if there exists a positive integer m for which $a_m=0$, then either $a_1=0$ or

$a_2 = 0$. (*Source: 2000 Putnam Exam*)

Solution. **Irfan GLOGIC** (Sarajevo College, 4th grade, Sarajevo, Bosnia and Herzegovina), **Salem MALIKIĆ** (Sarajevo College, 3rd grade, Sarajevo, Bosnia and Herzegovina) and **Anna Ying PUN** (HKU, Math, Year 1).

Observe that for any integers m and n, $m-n$ divides $f(m)-f(n)$ since for all nonnegative integer k, m^k-n^k has $m-n$ as a factor. For nonnegative integer n, let $b_n=a_{n+1}-a_n$, then by the last sentence, b_n divides b_{n+1} for all n.

Since $a_0 = a_m = 0$, $a_1 = a_{m+1}$ and so $b_0=b_m$. If $b_0 = 0$, then $a_1 = a_{m+1} = b_m+a_m = 0$.

If $b_0 \neq 0$, then using b_n divides b_{n+1} for all n and $b_0=b_m$, we get $b_n = \pm\, b_0$ for $n=1,2,\cdots,m$. Since $b_0+b_1+\cdots+b_m = a_m-a_0 = 0$, half of the integers $b_0, \cdots, b_m$ are positive and half are negative. Then there is $k < m$ such that $b_{k-1} = -b_k$, which implies $a_{k-1}=a_{k+1}$. Then $a_m=a_{m+2}$ and so $0=a_m=a_{m+2}=f(f(a_m))=f(f(a_0))=a_2$.

Problem 270. The distance between any two of the points *A, B, C, D* on a plane is at most 1. Find the minimum of the radius of a circle that can cover these four points. (*Source 1998 Tianjin Math Competition*)

Solution. **Jeff CHEN** (Virginia, USA).

<u>Case 1</u>: (one of the point, say D, is inside or on a side of ΔABC) If ΔABC is acute, then one of the angle, say $\angle BAC \geq 60^\circ$. By the extended sine law, the circumcircle of ΔABC covers the four points with diameter

$$2R = \frac{BC}{\sin\angle BAC} \leq \frac{2}{\sqrt{3}}.$$

(Note equality occurs in case ΔABC is equilateral.) If ΔABC is right or obtuse, then the circle using the longest side as diameter covers the four points with $R \leq 1/2$.

<u>Case 2</u>: ($ABCD$ is a convex quadrilateral) If there is a pair of opposite angles, say angles A and C, are at least 90°, then the circle with BD as diameter will cover the four points with $R \leq 1/2$. Otherwise, there is a pair of neighboring angles, say angles A and B, both of which are less than 90°.

If $\angle ADB \geq \angle ACB \geq 90^\circ$, then the circle with AB as diameter covers the four points and radius $R \leq 1/2$.

If $\angle ADB \geq \angle ACB$ and $\angle ACB < 90^\circ$, then D is in or on the circumcircle of ΔABC with radius $R \leq 1/\sqrt{3}$ as in case 1.

So summarizing all cases, we see the minimum radius that works for all possible arrangements of *A,B,C and D* is $R = 1/\sqrt{3}$.

Commended solvers: **NG Ngai Fung** (STFA Leung Kau Kui College) and **Anna Ying PUN** (HKU, Math, Year 1).

Olympiad Corner

(continued from page 1)

Problem 5. A regular (5×5)-array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all possible positions of this light.

From *How to Solve It* to Problem Solving in Geometry

(continued from page 2)

Idea:
Observe that

$$ac + bd = (b + d + a - c)(b + d - a + c)$$
$$\Leftrightarrow a^2 + c^2 - ac = b^2 + d^2 + bd$$

The last equality suggests one to think about using the cosine law as follow:

$$\begin{aligned} & a^2 + c^2 - 2ac\cos 60^\circ \\ &= a^2 + c^2 - ac \\ &= b^2 + d^2 + bd \\ &= b^2 + d^2 - 2bd\cos 120^\circ. \end{aligned}$$

Solution:

(1) **Lemma**: Let x, y, and z be positive integers with $z < x$ and $z < y$. If xy/z is an integer, then xy/z is composite.
[Can you prove this lemma? Is there any trivial case you can see immediately? How about proving the lemma by mathematical induction in z?]

The case $z = 1$ is trivial. In case $z > 1$, inductively suppose the lemma is true for all positive integers z' less than z. Then z has a prime divisor p, say $z = pz'$. Since xy/z is an integer, either p divides x or p divides y, say p divides x. Then $x = px'$. So $xy/z=x'y/z'$ with $z' < x'$ and $z' < z < y$. By the induction hypothesis, $xy/z=x'y/z'$ is composite.

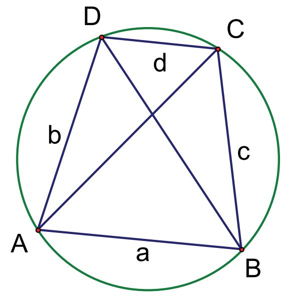
(2) The equality

$$ac + bd = (b + d + a - c)(b + d - a + c)$$

is equivalent to

$$a^2 + c^2 - ac = b^2 + d^2 + bd$$

In view of this, we can construct cyclic quadrilateral $ABCD$ with $AB = a$, $BC = c$, $CD = d$, $DA = b$, $\angle ABC = 60^\circ$ and $\angle ADC = 120^\circ$.



(3) Considering the ratios of areas and using Ptolemy's theorem, we have

$$\frac{AC}{BD} = \frac{ab+cd}{ac+bd} \text{ and } AC \times BD = ad + bc.$$

(4) Therefore,

$$\frac{ab+cd}{ac+bd} = \frac{AC}{BD} = \frac{AC^2}{AC\times BD} = \frac{a^2+c^2-ac}{ad+bc},$$

which implies

$$ab + cd = \frac{(ac+bd)(a^2+c^2-ac)}{ad+bc} \qquad (*).$$

(5) To get the conclusion from the lemma, it remains to show

$$ad + bc < ac + bd$$

and $$ad + bc < a^2 + c^2 - ac.$$

Now

$$\begin{aligned} & (ac + bd) - (ad + bc) \\ &= (a - b)(c - d) > 0 \\ \Rightarrow\ & ad + bc < ac + bd. \end{aligned}$$

Also,

$$\begin{aligned} & (ab + cd) - (ac + bd) \\ &= (a - d)(b - c) > 0 \\ \Rightarrow\ & ab + cd > ac + bd \\ \Rightarrow\ & ad + bc < a^2 + c^2 - ac \quad \text{(by (*))}. \end{aligned}$$

Now the result follows from the lemma.