Olympiad Corner

Below were the problems of the 10th China Hong Kong Math Olympiad, which was held on November 24, 2007. It was a three hour exam.

**Problem 1.** Let $D$ be a point on the side $BC$ of triangle $ABC$ such that $AB+BD = AC+CD$. The line segment $AD$ cuts the incircle of triangle $ABC$ at $X$ and $Y$ with $X$ closer to $A$. Let $E$ be the point of contact of the incircle of triangle $ABC$ on the side $BC$. Show that

(i) $EY$ is perpendicular to $AD$.
(ii) $XD$ is parallel to $A'E$, where $I$ is the incentre of the triangle $ABC$ and $A'$ is the midpoint of $BC$.

**Problem 2.** Is there a polynomial $f$ of degree 2007 with integer coefficients, such that $f(n), f(f(n)), f(f(f(n)))$, ... are pairwise relatively prime for every integer $n$? Justify your claims.

**Problem 3.** In a school there are 2007 male and 2007 female students. Each student joins not more than 100 clubs in the school. It is known that any two students of opposite genders have joined at least one common club.

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**Inequalities with Product Condition**

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There are many inequality problems that have $n$ positive variables $a_1, a_2, \ldots, a_n$ (generally $n = 3$) such that their product is 1. There are several ways to solve this kind of problems. One common method is to change these variables by letting

$$a_1 = \left(\frac{x_1}{x_n}\right)^{\alpha}, \quad a_2 = \left(\frac{x_2}{x_n}\right)^{\alpha}, \ldots, a_n = \left(\frac{x_n}{x_n}\right)^{\alpha},$$

where $x_1, x_2, \ldots, x_n$ are positive real numbers and generally $\alpha = 1$. Here are some examples on the usage of these substitutions.

**Example 1.** If $a, b, c$ are positive real numbers such that $abc = 1$, then prove that

$$\frac{a}{ab+1} + \frac{b}{bc+1} + \frac{c}{ca+1} \geq \frac{3}{2}.$$

**Solution.** Since $abc = 1$, we can find positive $x, y, z$ such that $a = x/y$, $b = y/z$, $c = z/x$ (for example, $x=1=abc$, $y=bc$ and $z=c$). Then

$$\frac{a}{ab+1} = \frac{x}{(x+y)(y+z)+z/x},$$

where the inequality follows from Nesbitt’s inequality applied to $x, y$ and $z$. (Editor—Nesbitt’s inequality asserts that if $r, s, t > 0$, then

$$\frac{r}{r+s+t} + \frac{s}{t+r+s} + \frac{t}{r+s+t} \geq \frac{3}{2}.$$

It follows by writing the left side as

$$\frac{x}{x+y+z} + \frac{y}{x+y+z} + \frac{z}{x+y+z} \geq \frac{3}{2},$$

where the inequality sign is due to the Cauchy-Schwarz inequality.) Equality occurs if and only if the three variables are equal.

**Example 2.** (2004 Russian Math Olympiad) Prove that if $a > 0$ and $abc = 1$, then

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \geq \frac{3}{2}.$$

**Example 3.** If $a, b, c > 0$ and $abc = 1$, then prove that

$$\frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{a+b+c} \geq \frac{1}{1/a} + \frac{1}{1/b} + \frac{1}{1/c}.$$
Solution. Let us perform the following substitutions
\[ a = \frac{x}{y}, \quad b = \frac{z}{x}, \quad c = \frac{1}{z}, \quad d = \frac{y}{t} \]
with \( x, y, z, t > 0 \) (for example, \( x = 1 = abcd, y = bcd, z = b \) and \( t = bc \)). Then after simple transformations, our inequality becomes
\[ \frac{y}{x + z} + \frac{x}{z + t} + \frac{z}{y + t} + \frac{t}{x + y} \geq 2. \]

Let \( I \) be the left side of this inequality and
\[ J = (x + y + z + t)^2. \]

By the Cauchy-Schwarz inequality, we easily get \( IJ \geq (x + y + z + t)^2 \). Then
\[ I \geq \frac{(x + y + z + t)^2}{y + x + z + x + z + x + t + y + z + z + t + t + x + y}. \]

So it is enough to prove that
\[ \frac{(x + y + z + t)^2}{y + x + z + x + z + x + t + y + z + z + t + t + x + y} \geq 2, \]
which is equivalent to
\[ x^2 + y^2 + z^2 + t^2 \geq 2(yz + xt). \]

This one is equivalent to
\[ (x - t)^2 + (y - z)^2 \geq 0, \]
which is obviously true.

For equality case to occur, we must have \( x = t \) and \( y = z \), which directly imply \( a = c \) and \( b = d \) so \( ab = 1 \) and therefore \( b = d = 1/a = 1/c \) is the equality case.

Example 5. (Crux 3147) Let \( n \geq 3 \) and let \( x_1, x_2, \ldots, x_n \) be positive real numbers such that \( x_1 x_2 \cdots x_n = 1 \). For \( n = 3 \) and \( n = 4 \) prove that
\[ \frac{1}{x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + x_3^2 + x_2 x_3 + x_4} \geq \frac{n}{2}. \]

Solution. We consider the substitutions
\[ x_i = \sqrt[n]{a_1 a_2 \cdots a_n}. \]

The inequality becomes
\[ \frac{a_1}{a_2 + \sqrt[n]{a_1 a_3}} + \frac{a_2}{a_3 + \sqrt[n]{a_1 a_3}} + \cdots + \frac{a_n}{a_1 + \sqrt[n]{a_1 a_n}} \geq \frac{n}{2}. \]

Since
\[ \sqrt[n]{a_1 a_2} \leq \frac{a_1 + a_2}{2}, \ldots, \sqrt[n]{a_{n-1} a_n} \leq \frac{a_{n-1} + a_n}{2}, \]
by the AM-GM inequality, it suffices to show that
\[ \frac{a_1}{a_2 + \sqrt[n]{a_1 a_3}} + \frac{a_2}{a_3 + \sqrt[n]{a_1 a_3}} + \cdots + \frac{a_n}{a_1 + \sqrt[n]{a_1 a_n}} \geq \frac{n}{4}. \]

Let \( J \) be the left side of this inequality and
\[ J = a_1 a_2 a_3 a_4 \cdots a_n. \]

By the Cauchy-Schwarz inequality, we have
\[ (a_1 a_2 + a_3 + \cdots + a_n)^2 \geq (n \cdot a_1 a_2 a_3 a_4 \cdots a_n). \]

This one is equivalent to
\[ 4(a_1 + a_2 + \cdots + a_n) \geq n(a_1 a_2 + a_3 + a_4 + \cdots + a_n). \]

For \( n = 4 \), by expansion, we can see the inequality is actually an identity. For \( n = 3 \), the inequality is equivalent to
\[ a_1^2 + a_2^2 + a_3^2 \geq a_1 a_2 + a_1 a_3 + a_2 a_3, \]
which is true because
\[ 2(a_1^2 + a_2^2 + a_3^2) - 2(a_1 a_2 + a_1 a_3 + a_2 a_3) = (a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_1 - a_3)^2 \geq 0. \]

This completes the proof. Equality holds if and only if \( a_i = 1 \) for all \( i \).

NOTE: This problem appeared in the May 2006 issue of the Crux Mathematicorum. It was proposed by Vasile Cirtoaje and Gabriel Dospinescu. No complete solution was received (except the above solution of the proposers).

Example 6. (Crux 2023) Let \( a, b, c, d, e \) be positive real numbers such that \( abcde = 1 \). Prove that
\[ \frac{a + abc}{1 + ab + abc} + \frac{b + bcd}{1 + bc + bcd} + \frac{c + cde}{1 + cd + cde} + \frac{d + dea}{1 + de + dea} + \frac{e + eab}{1 + ea + eab} \geq \frac{10}{3}. \]

Solution. Again we consider the standard substitutions
\[ a = \frac{x}{y}, \quad b = \frac{y}{z}, \quad c = \frac{z}{t}, \quad d = \frac{t}{u}, \quad e = \frac{u}{x}, \]
where \( x, y, z, t, u > 0 \).

Now we have
\[ \frac{a + abc}{1 + ab + abc} \geq \frac{1}{y + 1/4}. \]

Writing the other relations and letting
\[ a_1 = \frac{1}{x}, \quad a_2 = \frac{1}{y}, \quad a_3 = \frac{1}{z}, \quad a_4 = \frac{1}{t}, \quad a_5 = \frac{1}{u} \]
we have to show that if \( a_1, a_2, a_3, a_4, a_5 > 0 \), then
\[ \sum_{cyclic} a_2 + a_4 \geq \frac{10}{5}. \]

(Editor—The notation
\[ \sum_{cycle} f(a_1, a_2, \ldots, a_n) \]
for \( n \) variables \( a_1, a_2, \ldots, a_n \) is a shorthand notation for
\[ \sum_{i=1}^{n} f(a_{i_1}, a_{i_2}, \ldots, a_{i_n}), \]
where \( a_{i_1} \cdots a_{i_n} \) is the cycle of \( 1, 2, \ldots, n \).)

Let \( I \) be the left side of inequality (*),
\[ I = \sum_{cycle} (a_1 + a_2)(a_1 + a_3 + a_4) \]
and \( S = a_1 + a_2 + a_3 + a_4 + a_5 \). Using the Cauchy-Schwarz inequality, we easily get
\[ I \geq \left( \sum_{cycle} (a_1 + a_2) \right)^2 = (2S)^2 = 4S^2. \]

So to prove \( I \geq 10/3 \), it is enough to show
\[ \frac{4S^2}{J} \geq \frac{10}{3}. \]

Now comparing \( S^2 \) and \( J \), we can observe that \( 2S^2 - J \) equals
\[ T = (a_2 + a_3)^2 + (a_1 + a_4)^2 + (a_1 + a_3)^2 + (a_2 + a_4)^2. \]

Using this relation, (**) can be rewritten as
\[ 12S^2 \geq 10J = (2S^2 - T) \]
\[ = 20S^2 - 10T. \]

This simplifies to \( 5T \geq 4S^2 \). Finally, writing \( S = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 \), we can get \( 5T \geq 4S^2 \) from the Cauchy-Schwarz inequality easily.

Again equality occurs if and only if all the \( a_i \)'s are equal, which corresponds to the case \( a = b = c = d = e = 1 \).
Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong. The deadline for submitting solutions is January 15, 2008.

Problem 286. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that there exists a real number $y$ such that the sum of $x_i - y$, $x_2 - y$, $\ldots$, $x_n - y$ is at most $(n-1)/2$. (Here $|x| = x - [x]$, where $[x]$ is the greatest integer less than or equal to $x$.)

Can $y$ always be chosen to be one of the $x_i$’s?

Problem 287. Determine (with proof) all nonempty subsets $A, B, C$ of the set of all positive integers $\mathbb{Z}^+$ satisfying

1. $A \cap B = B \cap C = C \cap A = \emptyset$;
2. $A \cup C = \mathbb{Z}^+$;
3. for every $a \in A, b \in B$ and $c \in C$, we have $c+a, a+b, b+c \in B$ and $a+b \in C$.

Problem 288. Let $H$ be the orthocenter of triangle $ABC$. Let $P$ be a point in the plane of the triangle such that $P$ is different from $A, B, C$.

Let $L, M, N$ be the feet of the perpendiculars from $H$ to lines $PA, PB, PC$ respectively. Let $X, Y, Z$ be the intersection points of lines $LH, MH, NH$ with lines $BC, CA, AB$ respectively.

Prove that $X, Y, Z$ are on a line perpendicular to line $PH$.

Problem 289. Let $a$ and $b$ be positive numbers such that $a+b < 1$. Prove that

$$\frac{(a-1)^2 + b(2a-b)}{(b-1)^2 + a(2b-a)} \geq \min \left( \frac{a}{b}, \frac{b}{a} \right).$$

Problem 290. Prove that for every integer $a$ greater than 2, there exist infinitely many positive integers $n$ such that $a^n - 1$ is divisible by $n$.

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Solutions

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Due to an editorial mistake in the last issue, solution to problems 279 by Li ZHOU (Polk Community College, Winter Haven, Florida USA) was overlooked and his name was not listed among the solvers. We express our apology to him.

Problem 281. Let $N$ be the set of all positive integers. Prove that there exists a function $f: \mathbb{N} \to \mathbb{N}$ such that $f(f(x)) = x^2$ for all $x$. (Source: 1998 Romanian Math Olympiad)

Solution 1. Let $x$ be a positive integer. We define $f(x)$ as follows:

- If $x$ is odd, then $f(x) = x^2$.
- If $x$ is even, then $f(x) = \sqrt{x}$.

Observe that if $n$ is under case 1, then $f(n)$ will be under case 2. Similarly, if $n$ is under case 2, then $f(n)$ will be under case 1.

Problem 282. Let $a, b, c, A, B, C$ be real numbers, $a \neq 0$ and $A \neq 0$. For every real number $x$,

$$|ax^2+bx+c| \leq |A^2+Bx+C|.

Prove that $|b^2-4ac| \leq 4|A^2-Bx+C|$. (Source: 2003 Putnam Exam)

Solution. Let $A, B, C$ be the roots of the equation $x^2+Ax+Bx+C=0$. Then the given inequality implies for every real $x$,

$$A^2+Bx+C \geq \max \{|a|, |b^2-4ac|\}.$$
4(AC−ac). Hence, $4ac−b^2 ≤ 4AC−B^2$. (†)

Using (**) and (***) we have

$$b^2 = (B−b)^2 + (B+b)^2 \leq 2((A−a)(C−c)+(A+a)(C+c))$$
$$= 4(AC+ac).$$

Then $−4ac−b^2 ≤ 4AC−B^2$. Along with (†) we have

$$b^2−4ac = ±(4ac−b^2) ≤ 4AC−B^2 = |B^2−4AC|.$$  

Problem 283. P is a point inside $\triangle ABC$. Lines $AC$ and $BP$ intersect at $Q$. Lines $AB$ and $CP$ intersect at $R$. It is known that $AR=RQ=CP$ and $CQ=PQ$. Find $\triangle BRC$ with proof. (Source: 2003 Japanese Math Olympiad)

Solution. Stephen KIM (Toronto, Canada).

Let $S$ be the point on segment $CR$ such that $RS=CP=AR$. Since $CQ=PQ$, we have

$$\angle ACS = \angle QPC = \angle BPR.$$  

Also, since $RS=CP$, we have

$$SC=CR−RS=CR−CP=RP.$$  

Considering line $CR$ cutting $\triangle ABQ$, by Menelaus’ theorem, we have

$$\frac{RB}{PQ}\cdot\frac{AC}{AR}\cdot\frac{BP}{CQ}=1.$$  

Since $AR=RB$ and $CQ=PQ$, we get $AC=BP$. Hence, $\triangle ACS \cong \triangle BPR$. Then $AS=BR=AR=CP=RS$ and so $\triangle ARS$ is equilateral. Therefore, $\angle BRC=120^\circ$.

Commended solvers: FOK Pak Hei (Pui Ching Middle School, Form 6), Kelvin LEE (Trinity College, Cambridge, England), Salem MALIKIĆ (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina), NG Ngai Fung (STFA Leung Kau Kui College, Form 5) and YEUNG Wai Kit (STFA Leung Kau Kui College, Form 7).

Problem 284. Let $p$ be a prime number. Integers $x$, $y$, $z$ satisfy $0 < x < y < z < p$. If $x^2$, $y^2$, $z^2$ have the same remainder upon dividing by $p$, then prove that $x^2+y^2+z^2$ is divisible by $x+y+z$. (Source: 2003 Polish Math Olympiad)

Solution. George Scott ALDA, José Luis DÍAZ-BARRERO (Universitat Politècnica de Catalunya, Barcelona, Spain), EZZAKI Mahmoud (Omar Ibn Abdelaziz, Morocco), Stephen KIM (Toronto, Canada), Kelvin LEE (Trinity College, Cambridge, England) and Salem MALIKIĆ (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).

Since $x^3 ≡ y^3 ≡ z^3 \pmod{p}$, so

$$p \mid x^3+y^3=(x−y)(x^2+xy+y^2).$$

Since $0 < x < y < z < p$ and $p$ is prime, we have $p \nmid x−y$ and hence

$$p \mid x^2+xy+y^2.$$  

Similarly,

$$p \mid x^2+yz+z^2.$$  

By (1) and (2), $p$ divides $(x^2+xy+y^2+(y^2+yz+z^2)=(x−z)(x+y+z)$.

Since $0 < z < x < p$, we have $p \mid x+y+z$.

Also, $0 < x < y < z < p$ implies $x+y+z = p$ or $2p$ and $p > 3$. Now

$$x+y+z \equiv x^2+y^2+z^2 \pmod{2}.$$  

Thus, it remains to show $p \mid x^2+y^2+z^2$.

Now $x^2+y^2+z^2 = x(x+y+z)+y^2−xz$. From (1), we get

$$p \mid x^2−xz.$$  

Similarly,

$$p \mid x^2−zy.$$  

Adding the right sides of (1) to (6), we get

$$p \mid 3(x^2+y^2+z^2).$$

Since $p > 3$ is prime, we get $p \mid x^2+y^2+z^2$ as desired.

Commended solvers: YEUNG Wai Kit (STFA Leung Kau Kui College, Form 7).

Problem 285. Determine the largest positive integer $N$ such that for every way of putting all numbers 1 to 400 into a $20 \times 20$ table (1 number per cell), one can always find a row or a column having two numbers with difference not less than $N$. (Source: 2003 Russian Math Olympiad)

Solution. Jeff CHEN (Virginia, USA) and Stephen KIM (Toronto, Canada).

The answer is 209. We first show $N \leq 209$. Divide the table into a left and a right half, each of dimension $20 \times 10$. Put 1 to 200 row wise in increasing order into the left half. Similarly, put 201 to 400 row wise in increasing order into the right half. Then the difference of two numbers in the same row is at most $210−1=209$ and the difference of two numbers in the same column is at most $191−1=190$. So $N \leq 209$.

Next we will show $N \geq 209$. Let $M_1 = \{1,2,\ldots,91\}$ and $M_2 = \{300, 301, \ldots, 400\}$.

Color a row or a column red if and only if it contains a number in $M_1$. Similarly, color a row or a column blue if and only if it contains a number in $M_2$. We claim that

1. the number of red rows plus the number of red columns is at least 20 and
2. the number of blue rows plus the number of blue columns is at least 21.

Hence, there is a row or a column that is colored red and blue. So two of the numbers in that row or column have a difference of at least $300−91=209$.

For claim (1), let there be $i$ red rows and $j$ red columns. Since the numbers in $M_1$ can only be located at the intersections of these red rows and columns, we have $ij \geq 91$. By the AM-GM inequality,

$$i+j \geq 2\sqrt{ij} \geq 2\sqrt{91} > 19.$$  

Similarly, claim (2) follows from the facts that there are 101 numbers in $M_2$ and $2\sqrt{101} > 20$.

Olympiad Corner

(continued from page 1)

Problem 3. (Cont.) Show that there is a club with at least 11 male and 11 female members.

Problem 4. Determine if there exist positive integer pairs $(m,n)$, such that

(i) the greatest common divisor of $m$ and $n$ is 1, and $m \leq 2007$, and
(ii) for any $k=1,2,\ldots,2007$,

$$\left[\frac{nk}{m}\right] = \left[\sqrt{k}\right].$$

(Here $[x]$ stands for the greatest integer less than or equal to $x$.)