Mathematical Excalibur

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Olympiad Corner

Below were the problems of the 10th China Hong Kong Math Olympiad, which was held on November 24, 2007. It was a three hour exam.

Problem 1. et D be a point on the side BC of triangle ABC such that $AB+BD = AC+CD$. The line segment AD cut the incircle of triangle ABC at X and Y with X closer to A. Let E be the point of contact of the incircle of triangle ABC on the side BC. Show that

(i) EY is perpendicular to AD,
(ii) XD is $2IA'$, where I is the incentre of the triangle ABC and A' is the midpoint of BC.

Problem 2. Is there a polynomial f of degree 2007 with integer coefficients, such that $f(n)$, $f(f(n))$, $f(f(f(n)))$, … are pairwise relatively prime for every integer n? Justify your claims.

Problem 3. In a school there are 2007 male and 2007 female students. Each student joins not more than 100 clubs in the school. It is known that any two students of opposite genders have joined at least one common club.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 15, 2008***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI
Department of Mathematics
The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

Inequalities with Product Condition

Salem Malikić

(4th Grade Student, Sarajevo College, Bosnia and Herzegovina)

There are many inequality problems that have n positive variables $a_1, a_2, \ldots, a_n$ (generally $n = 3$) such that their product is 1. There are several ways to solve this kind of problems. One common method is to change these variables by letting

$$a_1=\left(\frac{x_2}{x_1}\right)^\alpha, a_2=\left(\frac{x_3}{x_2}\right)^\alpha, \cdots, a_n=\left(\frac{x_1}{x_n}\right)^\alpha,$$

where $x_1, x_2, \ldots, x_n$ are positive real numbers and generally $\alpha=1$. Here are some examples on the usage of these substitutions.

Example 1. If a, b, c are positive real numbers such that $abc = 1$, then prove that

$$\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1}\ge\frac{3}{2}.$$

Solution. Since $abc = 1$, we can find positive x, y, z such that $a = x/y$, $b = y/z$, $c = z/x$ (for example, $x=1=abc$, $y=bc$ and $z=c$). Then

$$\begin{aligned}&\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1}\\&=\frac{x/y}{(x/z)+1}+\frac{y/z}{(y/x)+1}+\frac{z/x}{(z/y)+1}\\&=\frac{zx}{xy+yz}+\frac{xy}{zx+yz}+\frac{yz}{xy+zx}\ge\frac{3}{2},\end{aligned}$$

where the inequality follows from *Nesbitt's inequality* applied to zx, xy and yz. (*Editor*—Nesbitt's inequality asserts that if $r,s,t > 0$, then

$$\frac{r}{s+t}+\frac{s}{t+r}+\frac{t}{r+s}\ge\frac{3}{2}.$$

It follows by writing the left side as

$$\begin{aligned}&\frac{r+s+t}{s+t}+\frac{r+s+t}{t+r}+\frac{r+s+t}{r+s}-3\\&=\left(\frac{s+t}{2}+\frac{t+r}{2}+\frac{r+s}{2}\right)\left(\frac{1}{s+t}+\frac{1}{t+r}+\frac{1}{r+s}\right)-3\\&\ge\left(\frac{1}{\sqrt2}+\frac{1}{\sqrt2}+\frac{1}{\sqrt2}\right)^2-3=\frac{3}{2},\end{aligned}$$

where the inequality sign is due to the Cauchy-Schwarz inequality.) Equality occurs if and only if the three variables are equal.

Example 2. (*2004 Russian Math Olympiad*) Prove that if $n > 3$ and $x_1, x_2, \ldots, x_n > 0$ have product 1, then

$$\frac{1}{1+x_1+x_1x_2}+\frac{1}{1+x_2+x_2x_3}+\cdots+\frac{1}{1+x_n+x_nx_1}>1.$$

Solution. Again we use the substitutions $x_1 = a_2/a_1$, $x_2 = a_3/a_2$, …, $x_n = a_1/a_n$ (say $a_1=1$ and for $i > 1$, $a_i=x_1x_2\cdots x_{i-1}$). Then the inequality is equivalent to

$$\frac{a_1}{a_1+a_2+a_3}+\frac{a_2}{a_2+a_3+a_4}+\cdots+\frac{a_n}{a_n+a_1+a_2}>\sum_{i=1}^{n}\frac{a_i}{a_1+a_2+\cdots+a_n}=1,$$

where the inequality sign is because $n > 3$ and $a_i > 0$ for all i so that $a_i+a_{i+1}+a_{i+2} < a_1+a_2+\cdots+a_n$.

Example 3. If $a, b, c > 0$ and $abc = 1$, then prove that

$$3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Solution. Since $abc = 1$, we can find positive x, y, z such that $a = x/y$, $b = z/x$, $c = y/z$ (for example, $x = 1 = abc$, $y = bc$ and $z=b$). After doing the substitution, the inequality can be rewritten as

$$3+\frac{x^2}{yz}+\frac{z^2}{xy}+\frac{y^2}{zx}\ge\frac{x}{y}+\frac{z}{x}+\frac{y}{z}+\frac{y}{x}+\frac{x}{z}+\frac{z}{y}.$$

Multiplying by xyz on both sides, we get

$$\begin{aligned}&x^3+y^3+z^3+3xyz\\&\ge x^2y+xy^2+y^2z+yz^2+z^2x+zx^2,\end{aligned}$$

which is just *Schur's inequality* (see *vol.* 10, *no.* 5, *p.* 2 of *Math Excalibur*). Since x,y,z are positive, equality holds if and only if $x = y = z$, that is $a = b = c$.

Example 4. (*Mathlinks Contest*) Prove that if $a,b,c,d > 0$ and $abcd = 1$, then

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+d)}+\frac{1}{d(1+a)}\ge 2.$$

Solution. Let us perform the following substitutions

$$a=\frac{x}{y},b=\frac{z}{x},c=\frac{t}{z},d=\frac{y}{t}$$

with $x,y,z,t>0$ (for example, $x=1=abcd$, $y=bcd$, $z=b$ and $t=bc$). Then after simple transformations, our inequality becomes

$$\frac{y}{x+z}+\frac{x}{z+t}+\frac{z}{y+t}+\frac{t}{x+y}\geq 2.$$

Let I be the left side of this inequality and

$$J=y(x+z)+x(z+t)+z(y+t)+t(x+y).$$

By the Cauchy-Schwarz inequality, we easily get $IJ\geq(x+y+z+t)^2$. Then

$$\frac{y}{x+z}+\frac{x}{z+t}+\frac{z}{y+t}+\frac{t}{x+y}=I$$
$$\geq\frac{(x+y+z+t)^2}{yx+yz+xz+xt+zy+zt+tx+ty}.$$

So it is enough to prove that

$$\frac{(x+y+z+t)^2}{yx+yz+xz+xt+zy+zt+tx+ty}\geq 2,$$

which is equivalent to

$$x^2+y^2+z^2+t^2\geq 2(yz+xt).$$

This one is equivalent to

$$(x-t)^2+(y-z)^2\geq 0,$$

which is obviously true.

For equality case to occur, we must have $x=t$ and $y=z$, which directly imply $a=c$ and $b=d$ so $ab=1$ and therefore $b=d=1/a=1/c$ is the equality case.

Example 5. (*Crux* 3147) Let $n\geq 3$ and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1x_2\cdots x_n=1$. For $n=3$ and $n=4$ prove that

$$\frac{1}{x_1^2+x_1x_2}+\frac{1}{x_2^2+x_2x_3}+\cdots+\frac{1}{x_n^2+x_nx_1}\geq\frac{n}{2}.$$

Solution. We consider the substitutions

$$x_1=\sqrt{\frac{a_2}{a_1}},x_2=\sqrt{\frac{a_3}{a_2}},\ldots,x_n=\sqrt{\frac{a_1}{a_n}}.$$

The inequality becomes

$$\frac{a_1}{a_2+\sqrt{a_1a_3}}+\frac{a_2}{a_3+\sqrt{a_2a_4}}+\cdots+\frac{a_n}{a_1+\sqrt{a_na_2}}\geq\frac{n}{2}.$$

Since

$$\sqrt{a_1a_3}\leq\frac{a_1+a_3}{2},\ldots,\sqrt{a_na_2}\leq\frac{a_n+a_2}{2}$$

by the *AM-GM* inequality, it suffices to show that

$$\frac{a_1}{a_1+2a_2+a_3}+\frac{a_2}{a_2+2a_3+a_4}+\cdots+\frac{a_n}{a_n+2a_1+a_2}\geq\frac{n}{4}.$$

Let I be the left side of this inequality and

$$J=a_1(a_1+2a_2+a_3)+\cdots+a_n(a_n+2a_1+a_2).$$

By the Cauchy-Schwarz inequality, we have $IJ\geq(a_1+a_2+\cdots+a_n)^2$. Thus, to prove $I\geq n/4$, it suffices to show that $(a_1+a_2+\cdots+a_n)^2/J\geq n/4$, which is equivalent to

$$4(a_1+a_2+\cdots a_n)^2$$
$$\geq n(a_1(a_1+2a_2+a_3)+\cdots+a_n(a_n+2a_1+a_2)).$$

For $n=4$, by expansion, we can see the inequality is actually an identity. For $n=3$, the inequality is equivalent to

$$a_1^2+a_2^2+a_3^2\geq a_1a_2+a_2a_3+a_3a_1,$$

which is true because

$$2(a_1^2+a_2^2+a_3^2)-2(a_1a_2+a_2a_3+a_3a_1)$$
$$=(a_1-a_2)^2+(a_2-a_3)^2+(a_3-a_1)^2$$
$$\geq 0.$$

This completes the proof. Equality holds if and only if $x_i=1$ for all i.

NOTE: This problem appeared in the May 2006 issue of the Crux Mathematicorum. It was proposed by Vasile Cîrtoaje and Gabriel Dospinescu. No complete solution was received (except the above solution of the proposers).

Example 6. (*Crux* 2023) Let a,b,c,d,e be positive real numbers such that $abcde=1$. Prove that

$$\frac{a+abc}{1+ab+abcd}+\frac{b+bcd}{1+bc+bcde}+\frac{c+cde}{1+cd+cdea}$$
$$+\frac{d+dea}{1+de+deab}+\frac{e+eab}{1+ea+eabc}\geq\frac{10}{3}.$$

Solution. Again we consider the standard substitutions

$$a=\frac{x}{y},b=\frac{y}{z},c=\frac{z}{t},d=\frac{t}{u},e=\frac{u}{x},$$

where $x,y,z,t,u>0$.

Now we have

$$\frac{a+abc}{1+ab+abcd}=\frac{1/y+1/t}{1/x+1/z+1/u}.$$

Writing the other relations and letting

$$a_1=\frac{1}{x},a_2=\frac{1}{y},a_3=\frac{1}{z},a_4=\frac{1}{t},a_5=\frac{1}{u},$$

we have to show that if $a_1, a_2, a_3, a_4, a_5>0$, then

$$\sum_{cyclic}\frac{a_2+a_4}{a_1+a_3+a_5}\geq\frac{10}{3}. \qquad (*)$$

(*Editor*—The notation

$$\sum_{cyclic}f(a_1,a_2,\ldots,a_n)$$

for n variables $a_1, a_2, \ldots, a_n$ is a shorthand notation for

$$\sum_{i=1}^{n}f(a_i,a_{i+1},\ldots,a_{i+n}),$$

where $a_{i+j}=a_{i+j-n}$ when $i+j>n$.)

Let I be the left side of inequality (*),

$$J=\sum_{cyclic}(a_2+a_4)(a_1+a_3+a_5)$$

and $S=a_1+a_2+a_3+a_4+a_5$. Using the Cauchy-Schwarz inequality, we easily get

$$IJ\geq\left(\sum_{cyclic}(a_2+a_4)\right)^2=(2S)^2=4S^2.$$

So to prove $I\geq 10/3$, it is enough to show

$$\frac{4S^2}{J}\geq\frac{10}{3}. \qquad (**)$$

Now comparing S^2 and J, we can observe that $2S^2-J$ equals

$$T=(a_2+a_4)^2+(a_1+a_4)^2+(a_3+a_5)^2$$
$$+(a_2+a_5)^2+(a_1+a_3)^2.$$

Using this relation, (**) can be rewritten as

$$12S^2\geq 10J=10(2S^2-T)$$
$$=20S^2-10T.$$

This simplifies to $5T\geq 4S^2$. Finally, writing $5=1^2+1^2+1^2+1^2+1^2$, we can get $5T\geq 4S^2$ from the Cauchy-Schwarz inequality easily.

Again equality occurs if and only if all the a_i's are equal, which corresponds to the case $a=b=c=d=e=1$.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***January 15, 2008.***

Problem 286. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that there exists a real number y such that the sum of $\{x_1-y\}, \{x_2-y\}, \ldots, \{x_n-y\}$ is at most $(n-1)/2$. (Here $\{x\} = x - [x]$, where $[x]$ is the greatest integer less than or equal to x.)

Can y always be chosen to be one of the x_i's ?

Problem 287. Determine (with proof) all nonempty subsets A, B, C of the set of all positive integers $\mathbb{Z}^+$ satisfying

(1) $A\cap B=B\cap C=C\cap A=\varnothing$;
(2) $A\cup B\cup C=\mathbb{Z}^+$;
(3) for every $a\in A$, $b\in B$ and $c\in C$, we have $c+a\in A$, $b+c\in B$ and $a+b\in C$.

Problem 288. Let H be the orthocenter of triangle ABC. Let P be a point in the plane of the triangle such that P is different from A, B, C.

Let L, M, N be the feet of the perpendiculars from H to lines PA, PB, PC respectively. Let X, Y, Z be the intersection points of lines LH, MH, NH with lines BC, CA, AB respectively.

Prove that X, Y, Z are on a line perpendicular to line PH.

Problem 289. Let a and b be positive numbers such that $a+b<1$. Prove that

$$\frac{(a-1)^2+b(2a-b)}{(b-1)^2+a(2b-a)} \ge \min\left\{\frac{a}{b},\frac{b}{a}\right\}.$$

Problem 290. Prove that for every integer a greater than 2, there exist infinitely many positive integers n such that a^n-1 is divisible by n.

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Solutions

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Due to an editorial mistake in the last issue, solution to problems 279 by **Li ZHOU** (Polk Community College, Winter Haven, Florida USA) was overlooked and his name was not listed among the solvers. We express our apology to him.

Problem 281. Let N be the set of all positive integers. Prove that there exists a function $f: N\to N$ such that $f(f(n)) = n^2$ for all n in N. (*Source: 1978 Romanian Math Olympiad*)

***Solution* 1. George Scott ALDA, Jeff CHEN** (Virginia, USA), **NGOO Hung Wing** (HKUST, Math Year 1), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 7) and **Fai YUNG**.

Let x_k be the k-th term of the sequence

2,3,5,6,7,8,10,11,12,13,14,15,17,18,19,…

of all positive integers that are not perfect squares in increasing order. By taking square roots (repeatedly) of an integer $n > 1$, we will eventually get to one of the x_k's. So every integer $n > 1$ is of the 2^m-th power of x_k for some nonnegative integer m and positive integer k.

We define $f(1)=1$. For $n > 1$, if n is the 2^m-th power of x_k, then we define $f(n)$ as follow:

<u>case 1</u>: if k is odd, then $f(n)$ is the 2^m-th power of x_{k+1};

<u>case 2</u>: if k is even, then $f(n)$ is the 2^{m+1}-st power of x_{k-1}.

Observe that if n is under case 1, then $f(n)$ will be under case 2. Similarly, if n is under case 2, then $f(n)$ will be under case 1. In computing $f(f(n))$, we have to apply case 2 once so that m increases by 1 and the k value goes up once and down once. Therefore, we have $f(f(n)) = n^2$ for all n in N.

***Solution 2.* GRA20 Problem Solving Group** (Roma, Italy) and **Kelvin LEE** (Trinity College, Cambridge, England).

We first define a function $g: N\to N$ such that $g(g(n)) = 2n$. Let p be an odd prime and let $\mathrm{ord}_p(n)$ be the greatest nonnegative integer α such that $p^\alpha \mid n$. If $\mathrm{ord}_p(n)$ is even, then let $g(n)=2pn$, otherwise let $g(n)=n/p$.

Next we will check $g(g(n)) = 2n$. If $\mathrm{ord}_p(n)$ is even, then $\mathrm{ord}_p(g(n)) = \mathrm{ord}_p(2pn)$ is odd and so $g(g(n)) = g(2pn) = 2pn/p = 2n$.

If $\mathrm{ord}_p(n)$ is odd, then $\mathrm{ord}_p(g(n)) = \mathrm{ord}_p(n/p)$ is even and so $g(g(n)) = g(n/p) = 2p(n/p) = 2n$.

Define $f(1)=1$. For an integer $n > 1$, let

$$n=\prod_{k=1}^{r} p_k^{\alpha_k}\text{ , where all } \alpha_k > 0,$$

be the prime factorization of n, then we define

$$f(n)=\prod_{k=1}^{r} p_k^{g(\alpha_k)}.$$

Finally, we have

$$f\big(f(n)\big)=\prod_{k=1}^{r} p_k^{g(g(\alpha_k))}=\prod_{k=1}^{n} p_k^{2\alpha_k}=n^2.$$

Problem 282. Let a, b, c, A, B, C be real numbers, $a\ne 0$ and $A\ne 0$. For every real number x,

$$|ax^2+bx+c| \le |Ax^2+Bx+C|.$$

Prove that $|b^2-4ac| \le |B^2-4AC|$. (*Source: 2003 Putnam Exam*)

***Solution.* Samuel Liló ABDALLA** (ITA, São Paulo, Brazil), **Jeff CHEN** (Virginia, USA), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 7).

We have

$$|a|=\lim_{x\to\infty}\frac{|ax^2+bx+c|}{x^2}\le\lim_{x\to\infty}\frac{|Ax^2+Bx+C|}{x^2}=|A|.$$

If $B^2-4AC > 0$, then $Ax^2+Bx+C=0$ has two distinct real roots x_0 and x_1. By the given inequality, these will also be roots of $ax^2+bx+c=0$. So $b^2-4ac > 0$. Then

$$|B^2-4AC| = A^2(x_0-x_1)^2 \ge a^2(x_0-x_1)^2 = |b^2-4ac|.$$

If $B^2-4AC\le 0$, then by replacing A by $-A$ or a by $-a$ if necessary, we may assume $A\ge a>0$. Since $A > 0$ and $B^2-4AC\le 0$, so for every real number x, $Ax^2+Bx+C \ge 0$. Then the given inequality implies for every real x,

$$Ax^2+Bx+C \ge \pm(ax^2+bx+c)\,. \qquad (*)$$

Then $(A-a)x^2 + (B-b)x + (C-c) \ge 0$. This implies

$$(B-b)^2 \le 4(A-a)(C-c). \qquad (**)$$

Similarly,

$$(B+b)^2 \le 4(A+a)(C+c). \qquad (***)$$

Then

$$(B^2-b^2)^2 \le 16(A^2-a^2)(C^2-c^2) \le 16(AC-ac)^2,$$

which implies $B^2-b^2 \le 4|AC-ac|$.

Taking $x = 0$ in (*), we get $C \ge |c|$. Since $A\ge a>0$, we get $B^2-b^2 \le$

$4(AC-ac)$. Hence,

$$4ac-b^2 \le 4AC-B^2. \qquad (\dagger)$$

Using (**) and (***), we have

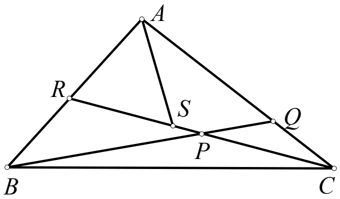
$$B^2+b^2=\frac{(B-b)^2+(B+b)^2}{2}$$
$$\le 2\big((A-a)(C-c)+(A+a)(C+c)\big)$$
$$=4(AC+ac).$$

Then $-(4ac-b^2) \le 4AC-B^2$. Along with (†), we have

$$|b^2-4ac| = \pm(4ac-b^2)$$
$$\le 4AC-B^2=|B^2-4AC|.$$

Problem 283. P is a point inside $\triangle ABC$. Lines AC and BP intersect at Q. Lines AB and CP intersect at R. It is known that $AR=RB=CP$ and $CQ=PQ$. Find $\angle BRC$ with proof. (*Source: 2003 Japanese Math Olympiad*)

Solution. **Stephen KIM** (Toronto, Canada).



Let S be the point on segment CR such that $RS=CP=AR$. Since $CQ=PQ$, we have

$$\angle ACS=\angle QPC=\angle BPR.$$

Also, since $RS=CP$, we have

$$SC=CR-RS=CR-CP=RP.$$

Considering line CR cutting $\triangle ABQ$, by Menelaus' theorem, we have

$$\frac{RB}{AR}\cdot\frac{PQ}{BP}\cdot\frac{AC}{CQ}=1.$$

Since $AR=RB$ and $CQ=PQ$, we get $AC=BP$. Hence, $\triangle ACS \cong \triangle BPR$. Then $AS=BR=AR=CP=RS$ and so $\triangle ARS$ is equilateral. Therefore, $\angle BRC=120°$.

Commended solvers: **FOK Pak Hei** (Pui Ching Middle School, Form 6), **Kelvin LEE** (Trinity College, Cambridge, England), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina), **NG Ngai Fung** (STFA Leung Kau Kui College, Form 5) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 7).

Problem 284. Let p be a prime number. Integers x, y, z satisfy $0<x<y<z<p$. If x^3, y^3, z^3 have the same remainder upon dividing by p, then prove that $x^2+y^2+z^2$ is divisible by $x+y+z$. (*Source: 2003 Polish Math Olympiad*)

Solution. **George Scott ALDA, José Luis DÍAZ-BARRERO** (Universitat Politècnica de Catalunya, Barcelona, Spain), **EZZAKI Mahmoud** (Omar Ibn Abdelaziz, Morocco), **Stephen KIM** (Toronto, Canada), **Kelvin LEE** (Trinity College, Cambridge, England) and **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).

Since $x^3 \equiv y^3 \equiv z^3 \pmod p$, so

$$p \mid x^3-y^3=(x-y)(x^2+xy+y^2).$$

Since $0<x<y<z<p$ and p is prime, we have $p \nmid x-y$ and hence

$$p \mid x^2+xy+y^2. \qquad (1)$$

Similarly,

$$p \mid y^2+yz+z^2 \qquad (2)$$

and

$$p \mid z^2+zx+x^2. \qquad (3)$$

By (1) and (2), p divides

$$(x^2+xy+y^2)-(y^2+yz+z^2)=(x-z)(x+y+z).$$

Since $0<z-x<p$, we have $p \mid x+y+z$.

Also, $0<x<y<z<p$ implies $x+y+z=p$ or $2p$ and $p>3$. Now

$$x+y+z \equiv x^2+y^2+z^2 \pmod 2).$$

Thus, it remains to show $p \mid x^2+y^2+z^2$.

Now $x^2+xy+y^2=x(x+y+z)+y^2-xz$. From (1), we get

$$p \mid y^2-xz. \qquad (4)$$

Similarly,

$$p \mid x^2-zy \qquad (5)$$

and

$$p \mid z^2-yx. \qquad (6)$$

Adding the right sides of (1) to (6), we get

$$p \mid 3(x^2+y^2+z^2).$$

Since $p>3$ is prime, we get $p \mid x^2+y^2+z^2$ as desired.

Commended solvers: **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 7).

Problem 285. Determine the largest positive integer N such that for every way of putting all numbers 1 to 400 into a 20×20 table (1 number per cell), one can always find a row or a column having two numbers with difference not less than N. (*Source: 2003 Russian Math Olympiad*)

Solution. **Jeff CHEN** (Virginia, USA) and **Stephen KIM** (Toronto, Canada).

The answer is 209. We first show $N \le 209$. Divide the table into a left and a right half, each of dimension 20×10. Put 1 to 200 row wise in increasing order into the left half. Similarly, put 201 to 400 row wise in increasing order into the right half. Then the difference of two numbers in the same row is at most 210−1=209 and the difference of two numbers in the same column is at most 191−1=190. So $N \le 209$.

Next we will show $N \ge 209$. Let $M_1=\{1,2,\ldots,91\}$ and $M_2=\{300, 301, \ldots, 400\}$.

Color a row or a column red if and only if it contains a number in M_1. Similarly, color a row or a column blue if and only if it contains a number in M_2. We claim that

(1) the number of red rows plus the number of red columns is at least 20 and
(2) the number of blue rows plus the number of blue columns is at least 21.

Hence, there is a row or a column that is colored red and blue. So two of the numbers in that row or column have a difference of at least 300−91=209.

For claim (1), let there be i red rows and j red columns. Since the numbers in M_1 can only be located at the intersections of these red rows and columns, we have $ij \ge 91$. By the *AM-GM* inequality,

$$i+j \ge 2\sqrt{ij} \ge 2\sqrt{91} > 19.$$

Similarly, claim (2) follows from the facts that there are 101 numbers in M_2 and $2\sqrt{101} > 20$.

Olympiad Corner

(continued from page 1)

Problem 3. (*Cont.*) Show that there is a club with at least 11 male and 11 female members.

Problem 4. Determine if there exist positive integer pairs (m,n), such that

(i) the greatest common divisor of m and n is 1, and $m \le 2007$,

(ii) for any k=1,2, …, 2007,

$$\left[\frac{nk}{m}\right]=\left[\sqrt{2}\,k\right].$$

(Here $[x]$ stands for the greatest integer less than or equal to x.)