Olympiad Corner

The 2008 APMO was held in March. Here are the problems.

Problem 1. Let $ABC$ be a triangle with $\angle A < 60^\circ$. Let $X$ and $Y$ be the points on the sides $AB$ and $AC$, respectively, such that $CA+AX = CB+BX$ and $BA+AY = BC+CY$. Let $P$ be the point in the plane such that the lines $PX$ and $PY$ are perpendicular to $AB$ and $AC$, respectively. Prove that $\angle BPC < 120^\circ$.

Problem 2. Students in a class form groups each of which contains exactly three members such that any two distinct groups have at most one member in common. Prove that, when the class size is 46, there is a set of 10 students in which no group is properly contained.

Problem 3. Let $\Gamma$ be the circumcircle of a triangle $ABC$. A circle passing through points $A$ and $C$ meets the sides $BC$ and $BA$ at $D$ and $E$, respectively. The lines $AD$ and $CE$ meet $\Gamma$ again at $G$ and $H$, respectively. The tangent lines of $\Gamma$ at $A$ and $C$ meet the line $DE$ at $L$ and $M$, respectively. Prove that the lines $LH$ and $MG$ meet at $\Gamma$.

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Point Set Combinatorics

Kin Y. Li

Problems involving sets of points in the plane or in space often appear in math competitions. We will look at some typical examples. The solutions of these problems provide us the basic ideas to attack similar problems.

The following are some interesting examples.

Example 1. (2001 USA Math Olympiad) Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

Solution. Let $A, B$ be arbitrary distinct points and consider a regular hexagon $ABCDEF$ in the plane. Let lines $CD$ and $EF$ intersect at $G$. Let $L$ be the line through $G$ perpendicular to line $DE$.

Observe that $\triangle CEG$ and $\triangle DFG$ are symmetric with respect to $L$ and hence they have the same incenter. So $c+e+g = d+f+g$. Also, $\triangle ACE$ and $\triangle BDF$ are symmetric with respect to $L$ and have the same incenter. So $a+c+e = b+d+f$. Subtracting these two equations, we see $a=b$.

Comments. This outstanding elegant solution was due to Michael Hamburg, who was given a handsome cash prize as a Clay Math Institute award by the USAMO Committee.

Example 2. (1987 IMO Shortlisted Problem) In space, is there an infinite set $M$ of points such that the intersection of $M$ with every plane is nonempty and finite?

Solution. Yes, there is such a set $M$. For example, let $M = \{(t^3, t^3, t) : t \in \mathbb{R}\}$.

Then, for every plane with equation $Ax + By + Cz + D = 0$, the intersection points are found by solving $A^2 + B^2 + C^2 + D = 0$, which has at least one solution (since $A$ or $B$ or $C$ is nonzero) and at most five solutions (since the degree is at most five).

Example 3. (1963 Beijing Mathematics Competition) There are $2n + 3$ $(n \geq 1)$ given points on a plane such that no three of them are collinear and no four of them are concyclic.

Is it always possible to draw a circle through three of them so that half of the other $2n$ points are inside and half are outside the circle?

Solution. Yes, it is always possible.

Take the convex hull of these points, i.e. the smallest convex set containing them. The boundary is a polygon with vertices from the given points. Let $AB$ be a side of the polygon. Since no three are collinear, no other given points are on $AB$. By convexity, the other points $C_1, C_2, ..., C_{2n+1}$ are on the same side of line $AB$. Since no four are collinear, angles $AC_iB$ are all distinct, say $\angle AC_1B < \angle AC_2B < ... < \angle AC_{2n+1}B$.

Then $C_1, C_2, ..., C_n$ are inside the circle through $A, B$ and $C_{n+1}$ and $C_{n+2}, C_{n+3}, ..., C_{2n+1}$ are outside.
Example 4. (1941 Moscow Math. Olympiad) On a plane are given \( n \) points such that every three of them is inside some circle of radius 1. Prove that all these points are inside some circle of radius 1.

Solution. For every three of the \( n \) given points, consider the triangle they formed. If the triangle is an acute triangle, then draw their circumcircle, otherwise take the longest side and draw the circle having that side as the diameter. By the given condition, all these circles have radius less than 1.

Let \( S \) be one of these circles with minimum radius, say \( S \) arose from considering points \( A, B, C \).

Assume one of the given points \( D \) is not inside \( S \).

If \( \triangle ABC \) is acute, then \( D \) is on the same side as one of \( A, B, C \) with respect to the line through the other two points, say \( D \) and \( A \) are on the same side of line \( BC \). Then the circle drawn for \( B, C, D \) would be their circumcircle and would have a radius greater than the radius of \( S \), a contradiction.

If \( \triangle ABC \) is not acute and \( S \) is the circle with diameter \( AB \), then the circle drawn for \( A, B, D \) would have \( AB \) as a chord and not as a diameter, which implies that circle has a radius greater than the radius of \( S \), a contradiction.

Therefore, all \( n \) points are inside or on \( S \). Since the radius of \( S \) was less than 1, we can take the circle of radius 1 at the same center as \( S \) to contain all \( n \) points.

In the next example, we will consider a problem in space and the solution will involve a basic fact from solid geometry. Namely, proofs in Book XI, Problems 20 and 21 of his Elements.

For the left inequality, we may assume that \( \angle BAC \) is the largest of the three angles about vertex \( A \). Let \( E \) be on side \( BC \) so that \( \angle BAD = \angle BAE \). Let \( X, Y, Z \) be on rays \( AB, AC, AD \) respectively, and \( AX = AY = AZ \). Then \( \triangle AXZ \cong \triangle AXY \) and we have \( XZ = XY \). Let line \( XY \) intersect line \( AC \) at \( W \). Since \( XZ + ZW > XW \), cancelling \( XZ = XY \), we have \( ZW > YW \). Comparing triangles \( WAZ \) and \( WAY \), we have \( WA = WA, AZ = AY \), so \( ZW > YW \) implies \( \angle ZAW > \angle YAW \). Then

\[
\angle BAC = \angle XAY + \angle YAW < \angle XAZ + \angle ZAW = \angle BAD + \angle DAC.
\]

For the right inequality, by the left inequality, we have

\[
\angle BDC \leq \angle DBA + \angle ABC, \\
\angle BCD \leq \angle BCA + \angle ACD, \\
\angle CDB \leq \angle CDG + \angle ADB.
\]

Adding them, we get 180° is less than or equal to the sum of the six angles on the right. Now the sum of these six angles and the three angles about \( A \) is 3×180°. So the sum of the three angles about \( A \) is less than or equal to 360°.

Example 5. (1969 All Soviet Math. Olympiad) There are \( n \) given points in space with no three collinear. Prove that there is a way to order the points as \( A_1, A_2, \ldots, A_n \) such that whenever \( 1 \leq i < j < k \leq n \), we have \( \angle A_iA_jA_k > 120° \).

Solution. Take two furthest points among these \( n \) points and call them \( A_1 \) and \( A_n \).

For every two points \( X, Y \) among the other \( n-2 \) points, since \( A_1A_n \) is the longest side in both \( \Delta A_1AX \) and \( \Delta A_jA_k \), we have \( \angle XA_1A_n \leq 60° \) and \( \angle YA_jA_k \leq 60° \). About vertex \( A_1 \) of the tetrahedron \( A_1A_jA_k \), we have

\[
\angle XA_1Y \leq \angle XA_1A_n + \angle YA_jA_k < 60° + 60° = 120°.
\]

Similarly, \( \angle XA_1Y < 120° \).

Also, \( A_1X \neq A_1Y \) (since otherwise, the two equal angles in \( \Delta XA_1Y \) cannot be greater than 90° and so only \( \angle XA_1Y \) can be greater than 120°, which will contradict the inequality above). Now order the points by its distance to \( A_1 \) so that \( A_1A_2 \leq A_2A_3 \leq \cdots \leq A_{n-1}A_n \).

For \( 1 < j < k \leq n \), taking \( X = A_j \) and \( Y = A_k \) in the inequality above, we get

\[
\angle A_jA_kA_1 \leq 120°.
\]

Since \( A_1A_j > A_1A_k \), so in \( \Delta A_1A_2A_j \), \( \angle A_1A_2A_j > 120° \).

For \( 1 < i < j < k \leq n \), we have \( \angle A_iA_jA_k > 120° \) and \( \angle A_iA_kA_j > 120° \) by the last paragraph. Then, about vertex \( A_i \) of the tetrahedron \( A_iA_jA_k \), we have \( \angle A_iA_jA_k \leq \angle A_iA_jA_k + \angle A_iA_kA_1 < 60° + 60° = 120° \).

Hence, in \( \angle A_iA_jA_k \), we have \( \angle A_iA_jA_k > 120° \).

Example 6. (1994 All Russian Math. Olympiad) There are \( k \) points, \( 2 \leq k \leq 50 \), inside a convex 100-sided polygon. Prove that we can choose at most \( 2k \) vertices from this 100-sided polygon so that the \( k \) points are inside the polygon with the chosen points as vertices.

Solution. Let \( M = A_1A_2 \cdots A_n \) be the boundary of the convex hull of the \( k \) points. Hence, \( n \leq k \). Let \( O \) be a point inside \( M \). From \( i = 1 \) to \( n \), let ray \( OA_i \) intersect the 100-sided polygon at \( B_i \). Let \( M' \) be the boundary of the convex hull of \( B_1, B_2, \ldots, B_n \).

For every point \( P \) on or inside \( M \), the line \( OP \) intersects \( M \) at two sides, say \( A_{i-1}A_i \) and \( A_{i+1}A_i \). By the definition of the points \( B_i \)'s, we see the line \( OP \) intersects \( B_iB_{i+1} \) and \( B_{i-1}B_i \). Let \( S, T \) be the points such that \( S, T \) are in \( M' \). Then \( O \) and \( P \) are in \( M' \). Thus \( M' \) contains \( M \).

Let \( M' = C_1C_2 \cdots C_m \). Then \( m \leq n \leq k \). Observe that all \( C_i \)'s are on the 100-sided polygon. Now each \( C_i \) is a vertex or between two consecutive vertices of the 100-sided polygon. Let \( G \) be the set of all these vertices. Then \( G \) has at most \( 2k \) points and the polygon with vertices from \( G \) contains the \( k \) points.

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**Problem Corner**

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong. The deadline for submitting solutions is **May 20, 2008**.

**Problem 296.** Let \( n > 1 \) be an integer. From a \( n \times n \) square, one \( 1 \times 1 \) corner square is removed. Determine (with proof) the least positive integer \( k \) such that the remaining areas can be partitioned into \( k \) triangles with equal areas.

*(Source: 1992 Shanghai Math Contest)*

**Problem 297.** Prove that for every pair of positive integers \( p \) and \( q \), there exist an integer coefficient polynomial \( f(x) \) and an open interval with length \( 1/q \) on the real axis such that for every \( x \) in the interval, \( f(x) - p/q < 1/q^2 \).

*(Source: 1983 Finnish Math Olympiad)*

**Problem 298.** The diagonals of a convex quadrilateral \( ABCD \) intersect at \( O \). Let \( M_1 \) and \( M_2 \) be the centroids of \( \triangle AOB \) and \( \triangle COD \) respectively. Let \( H_1 \) and \( H_2 \) be the orthocenters of \( \triangle BOC \) and \( \triangle AOD \) respectively. Prove that \( M_1 \parallel H_1 \parallel H_2 \).

*(Source: 1996 Shanghai Math Contest)*

**Problem 299.** Determine (with proof) the least positive integer \( n \) such that in every way of partitioning \( S = \{1, 2, \ldots, n\} \) into two subsets, one of the subsets will contain two distinct numbers \( a \) and \( b \) such that \( ab \) is divisible by \( a + b \).

*(Source: 1996 Shanghai Math Contest)*

**Problem 300.** Prove that in base 10, every odd positive integer has a multiple all of whose digits are odd.

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**Solutions**

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**Problem 291.** Prove that if a convex polygon lies in the interior of another convex polygon, then the perimeter of the inner polygon is less than the perimeter of the outer polygon.

*Solution. Jeff CHEN (Virginia, USA), HO Kin Fai (HKUST, Math Year 3) and Fai YUNG.*

We will define a sequence of convex polygons \( P_0, P_1, \ldots, P_{n-1} \). Let the outer convex polygon be \( P_0 \) and the inner convex polygon be \( A_1A_2 \ldots A_n \). For \( i = 1 \) to \( n-1 \), let the line \( A_{i-1}A_{i+1} \) intersect \( P_{i-1} \) at \( B_i \), \( B_{i+1} \). The line \( A_{i-1}A_{i+1} \) divides \( P_{i-1} \) into two parts with one part enclosing \( A_1A_2 \ldots A_n \). Let \( P_i \) be the polygon formed by putting the segment \( B_iB_{i+1} \) together with the part of \( P_{i-1} \) enclosing \( A_1A_2 \ldots A_n \). Note \( P_{n-1} \) is \( A_1 \ldots A_n \). Finally, the perimeter of \( P_i \) is less than the perimeter of \( P_{i-1} \) because the length of \( B_iB_{i+1} \), being the shortest distance between \( B_i \) and \( B_{i+1} \), is less than the length of the part of \( P_{i-1} \) removed to form \( P_i \).

**Commended solvers:** Salem MALIKIĆ (Sarajevo College, 4\(^{th}\) Grade, Sarajevo, Bosnia and Herzegovina), Raul A. SIMON (Santiago, Chile) and Simon YAU Chi-Keung (City University of Hong Kong).

**Problem 292.** Let \( k_1 < k_2 < k_3 < \ldots \) be positive integers with no two of them are consecutive. For every \( m = 1, 2, 3, \ldots \), let \( S_m = k_1 + k_2 + \cdots + k_m \). Prove that for every positive integer \( n \), the interval \( [S_n, S_{n+1}] \) contains at least one perfect square number.

*(Source: 1996 Shanghai Math Contest)*

**Solution. Jeff CHEN (Virginia, USA), G.R.A. 20 Problem Solving Group (Roma, Italy), HO Kin Fai (HKUST, Math Year 3), Salem MALIKIĆ (Sarajevo College, 4\(^{th}\) Grade, Sarajevo, Bosnia and Herzegovina) and Raul A. Simon (Santiago, Chile).*

There is a nonnegative integer \( a \) such that \( a^2 < S_n \leq (a+1)^2 \). We have

\[
S_n = k_1 + k_2 + \cdots + k_n < k_1 + (k_2 - 2) + \cdots + (k_n - 2n + 2) = n(k_n - n + 1).
\]

By the AM-GM inequality,

\[
a < \sqrt{S_n} < \frac{n + (k_n - n + 1)}{2} = \frac{k_n + 1}{2}.
\]

Then

\[
(a + 1)^2 = a^2 + 2a + 1 < S_n + (k_n + 1) + 1 \leq S_n + k_{n+1} = S_{n+1}.
\]

**Commended solvers:** Simon YAU

**Problem 293.** Let \( CH \) be the altitude of triangle \( ABC \) with \( \angle ACB = 90^\circ \). The bisector of \( \angle ABC \) intersects \( CH \), \( CB \) at \( P \), \( M \) respectively. The bisector of \( \angle ABC \) intersects \( CH \), \( CA \) at \( Q \), \( N \) respectively. Prove that the line passing through the midpoints of \( PM \) and \( QN \) is parallel to line \( AB \).

*(Source: 52\(^{nd}\) Belorussian Math. Olympiad)*

**Solution. Jeff CHEN (Virginia, USA) and CHEUNG Wang Chi (Magdalene College, University of Cambridge, England) and Salem MALIKIĆ (Sarajevo College, 4\(^{th}\) Grade, Sarajevo, Bosnia and Herzegovina).*

Let \( E, F \) be the midpoints of \( QN \), \( PM \) respectively. Let \( X, Y \) be the intersection of \( CE, CF \) with \( AB \) respectively. Now

\[
\angle CPM = 90^\circ - \angle CAM = 90^\circ - \angle BAM = \angle APH = \angle CPB.
\]

So \( CM = CP \). Then \( CF \perp AF \). Since \( AF \) bisects \( \angle CAY \), by ASA, \( \triangle CAF \cong \triangle YAF \). So \( CF = FY \). Similarly, \( CE = EX \). By the midpoint theorem, we have \( EF \parallel \) to line \( XY \), which is the same as line \( AB \).

**Commended solvers:** Konstantine ZELATOR (University of Toledo, Toledo, Ohio, USA).

**Problem 294.** For three nonnegative real numbers \( x, y, z \) satisfying the condition \( xy + yz + zx = 3 \), prove that

\[
x^2 + y^2 + z^2 + 3xyz \geq 6.
\]

**Solution. Jeff CHEN (Virginia, USA), CHEUNG Wang Chi (Magdalene College, University of Cambridge, England), Ovidiu FURDUI (Cluj, Romania), MA Ka Hei (Wah Yan College, Kowloon) and Salem MALIKIĆ (Sarajevo College, 4\(^{th}\) Grade, Sarajevo, Bosnia and Herzegovina).*

Let \( p = x + y + z \), \( q = xy + yz + zx \) and \( r = xyz \). Now
Problem 295. There are 2\(n\) distinct points in space, where \(n \geq 2\). No four of them are on the same plane. If \(n^2 + 1\) pairs of them are connected by line segments, then prove that there are at least \(n\) distinct triangles formed.

(Source: 1989 Chinese IMO team training problem)

Solution. Jeff CHEN (Virginia, USA) and CHEUNG Wang Chi (Magdalen College, University of Cambridge, England).

We prove by induction on \(n\). For \(n = 2\), say the points are \(A, B, C, D\). For five segments connecting them, only one pair of them is not connected, say they are \(A\) and \(B\). Then triangles \(ACD\) and \(BCD\) are formed.

Suppose the case \(n = k\) is true. Consider the case \(n = k + 1\). We first claim there is at least one triangle. Suppose \(AB\) is one such connected segment. Let \(\alpha, \beta, \gamma\) be the number of segments connecting \(A, B\) to the other \(2n - 2 = 2k\) points respectively.

If \(\alpha + \beta > 2k + 1\), then \(A, B\) are both connected to one of the other \(2k\) points, hence a triangle is formed.

If \(\alpha + \beta \leq 2k\), then the other \(2k\) points have at least \((k + 1)^2 + 1 = (2k + 1)^2 + 1\) segments connecting them. By the case \(n = k\), there is a triangle in these \(2k\) points.

So the claim is established. Now take such a triangle, say \(ABC\). Let \(a, \beta, \gamma\) be the number of segments connecting \(A, B, C\) to the other \(2k - 1\) points respectively.

If \(\alpha + \beta + \gamma \geq 3k - 1\), then let \(D_1, D_2, \ldots, D_m\) be all the points among the other \(2k - 1\) points connecting to at least one of \(A\) or \(B\) or \(C\). The number of segments to \(D_i\) from \(A\) or \(B\) or \(C\) is \(n_i = 1, 2\) or 3. Checking each of these three cases, we see there are at least \(n_i = 1\) triangles having \(D_i\) as a vertex and the two other vertices from \(A, B, C\). So there are \(\sum n_i \geq 2k + 1\) such triangles, each having one \(D_i\) vertex, plus triangle \(ABC\), resulting in at least \(k + 1\) triangles.

If \(\alpha + \beta + \gamma \leq 3k - 2\), then the sum of \(\alpha + \beta, \gamma+\alpha, \beta+\gamma\) is at most \(6k - 4\). Hence the least of them cannot be \(2k - 1\) or more, say \(\alpha + \beta \leq 2k - 2\). Then removing \(A\) and \(B\) and all segments connected to at least one of them, we have at least \((k + 1)^2 + 1 - (2k + 1) = k^2 + 1\) segments left for the remaining \(2k\) points. By the case \(n = k\), we have \(k\) triangles. These plus triangle \(ABC\) result in at least \(k + 1\) triangles. The induction is complete.

Commended solvers: Raúl A. SIMON (Santiago, Chile) and Simon YAU Chi-Keung (City University of Hong Kong).

Olympiad Corner (continued from page 1)

Problem 4. Consider the function \(f: \mathbb{N}_0 \rightarrow \mathbb{N}_0\), where \(\mathbb{N}_0\) is the set of all non-negative integers, defined by the following conditions:

(i) \(f(0) = 0\), (ii) \(f(2n) = 2f(n)\) and (iii) \(f(2n + 1) = n + 2f(n)\) for all \(n \geq 0\).

(a) Determine the three sets \(L := \{ n \mid f(n) < f(n + 1) \}, E := \{ n \mid f(n) = f(n + 1) \},\) and \(G := \{ n \mid f(n) > f(n + 1) \}\).

(b) For each \(k \geq 0\), find a formula for \(a_k := \max \{ f(n) \mid 0 \leq n \leq 2^k \}\) in terms of \(k\).

Problem 5. Let \(a, b, c\) be integers satisfying \(0 < a < c - 1\) and \(1 < b < c\). For each \(k\), \(0 \leq k \leq a\), let \(r_k = 0 \leq r_k < c\) be the remainder of \(kb\) when divided by \(c\). Prove that the two sets \(\{ r_0, r_1, r_2, \ldots, r_a \}\) and \(\{ 0, 1, 2, \ldots, a \}\) are different.

Point Set Combinatorics (continued from page 2)

Example 7. (1987 Chinese IMO Team Selection Test) There are \(2n\) distinct points in space, where \(n \geq 2\). No four of them are on the same plane. If \(n^2 + 1\) pairs of them are connected by line segments, prove that there are two triangles sharing a common side.

Solution. We prove by induction on \(n\). For \(n = 2\), say the points are \(A, B, C, D\). For five segments connecting them, only one pair of them is not connected, say they are \(A\) and \(B\). Then triangles \(ACD\) and \(BCD\) are formed and the side \(CD\) is common to them.

Suppose the case \(n = k\) is true. Consider the case \(n = k + 1\). Suppose \(AB\) is one such connected segment. Let \(a, \beta, \gamma\) be the number of segments connecting \(A\) to the other \(2n - 2 = 2k\) points respectively.

Case 1. If \(\alpha + \beta \geq 2k + 2\), then there are points \(C, D\) among the other \(2k\) points such that \(AC, BC, AD, BD\) are connected. Then triangles \(ABC\) and \(ABD\) are formed and the side \(AB\) is common to them.

Case 2. If \(\alpha + \beta \leq 2k\), then removing \(A, B\) and all segments connecting to at least one of them, there would still be at least \((k + 1)^2 + 1 - (2k + 1) = k^2 + 1\) segments left for the remaining \(2k\) points. By the case \(n = k\), there would exist two triangles sharing a common side among them.

Case 3. Assume cases 1 and 2 do not occur for all the connected segments. Then take any connected segment \(AB\) and we have \(\alpha + \beta = 2k + 1\). There would then be a point \(C\) among the other \(2k\) points such that triangle \(ABC\) is formed.

Let \(g\) be the number of segments connecting \(C\) to the other \(2k - 1\) points respectively. Since cases 1 and 2 do not occur, we have\(\beta + \gamma = 2k + 1\) and \(\alpha + \gamma = 2k + 1\), too. However, this would lead to \((\alpha + \beta) + (\beta + \gamma) + (\gamma + \alpha) = 6k + 3\), which is contradictory as the left side is even and the right side is odd.

One cannot help to notice the similarity between the last example and problem 295 in the problem corner. Naturally this raise the question: when \(n\) is large, again if \(n^2 + 1\) pairs of the points are connected by line segments, would we be able to get more pairs of triangles sharing common sides? Any information or contribution for this question from the readers will be appreciated.