Mathematical Excalibur

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Olympiad Corner

The 2008 APMO was held in March. Here are the problems.

Problem 1. Let ABC be a triangle with $\angle A < 60°$. Let X and Y be the points on the sides AB and AC, respectively, such that $CA+AX = CB+BX$ and $BA+AY = BC+CY$. Let P be the point in the plane such that the lines PX and PY are perpendicular to AB and AC, respectively. Prove that $\angle BPC < 120°$.

Problem 2. Students in a class form groups each of which contains exactly three members such that any two distinct groups have at most one member in common. Prove that, when the class size is 46, there is a set of 10 students in which no group is properly contained.

Problem 3. Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides BC and BA at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines of Γ at A and C meet the line DE at L and M, respectively. Prove that the lines LH and MG meet at Γ.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 20, 2008***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Point Set Combinatorics

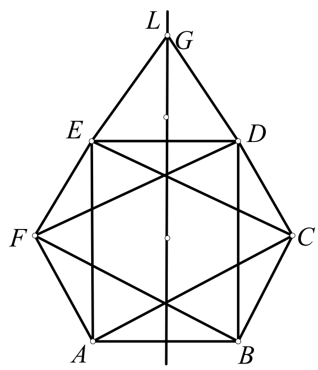
Kin Y. Li

Problems involving sets of points in the plane or in space often appear in math competitions. We will look at some typical examples. The solutions of these problems provide us the basic ideas to attack similar problems.

The following are some interesting examples.

Example 1. (*2001 USA Math Olympiad*) Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

Solution. Let A, B be arbitrary distinct points and consider a regular hexagon $ABCDEF$ in the plane. Let lines CD and EF intersect at G. Let L be the line through G perpendicular to line DE.



Observe that ΔCEG and ΔDFG are symmetric with respect to L and hence they have the same incenter. So $c+e+g = d+f+g$. Also, ΔACE and ΔBDF are symmetric with respect to L and have the same incenter. So $a+c+e=b+d+f$. Subtracting these two equations, we see $a=b$.

Comments: This outstanding elegant solution was due to Michael Hamburg, who was given a handsome cash prize as a Clay Math Institute award by the USAMO Committee.

Example 2. (*1987 IMO Shortlisted Problem*) In space, is there an infinite set M of points such that the intersection of M with every plane is nonempty and finite?

Solution. Yes, there is such a set M. For example, let

$$M = \{(t^5, t^3, t) : t \in \mathbb{R}\}.$$

Then, for every plane with equation $Ax + By + Cz + D = 0$, the intersection points are found by solving

$$At^5 + Bt^3 + Ct + D = 0,$$

which has at least one solution (since A or B or C is nonzero) and at most five solutions (since the degree is at most five).

Example 3. (*1963 Beijing Mathematics Competition*) There are $2n + 3$ ($n \geq 1$) given points on a plane such that no three of them are collinear and no four of them are concyclic.

Is it always possible to draw a circle through three of them so that half of the other $2n$ points are inside and half are outside the circle?

Solution. Yes, it is always possible.

Take the *convex hull* of these points, i.e. the smallest convex set containing them. The boundary is a polygon with vertices from the given points.

Let AB be a side of the polygon. Since no three are collinear, no other given points are on AB. By convexity, the other points $C_1, C_2, \cdots, C_{2n+1}$ are on the same side of line AB. Since no four are collinear, angles AC_iB are all distinct, say

$$\angle AC_1B < \angle AC_2B < \cdots < \angle AC_{2n+1}B.$$

Then $C_1, C_2, \cdots, C_n$ are inside the circle through A, B and C_{n+1} and $C_{n+2}, C_{n+3}, \cdots, C_{2n+1}$ are outside.

Example 4. (*1941 Moscow Math. Olympiad*) On a plane are given n points such that every three of them is inside some circle of radius 1. Prove that all these points are inside some circle of radius 1.

Solution. For every three of the n given points, consider the triangle they formed. If the triangle is an acute triangle, then draw their circumcircle, otherwise take the longest side and draw the circle having that side as the diameter. By the given condition, all these circles have radius less than 1.

Let S be one of these circles with minimum radius, say S arose from considering points A, B, C.

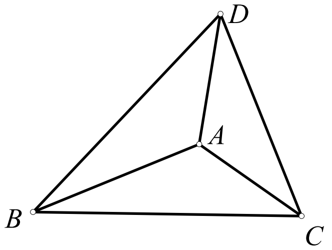
Assume one of the given points D is not inside S.

If ΔABC is acute, then D is on the same side as one of A,B,C with respect to the line through the other two points, say D and A are on the same side of line BC. Then the circle drawn for B,C,D would be their circumcircle and would have a radius greater than the radius of S, a contradiction.

If ΔABC is not acute and S is the circle with diameter AB, then the circle drawn for A, B, D would have AB as a chord and not as a diameter, which implies that circle has a radius greater than the radius of S, a contradiction.

Therefore, all n points are inside or on S. Since the radius of S was less than 1, we can take the circle of radius 1 at the same center as S to contain all n points.

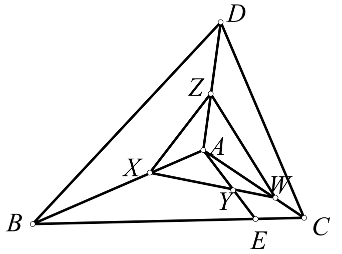
In the next example, we will consider a problem in space and the solution will involve a basic fact from solid geometry. Namely,



about vertex A of a tetrahedron $ABCD$, we have

$$\angle BAC \le \angle BAD + \angle DAC \le 360^\circ.$$

Nowadays, very little solid geometry is taught in school. So let's recall Euclid's proofs in Book XI, Problems 20 and 21 of his *Elements*.



For the left inequality, we may assume that $\angle BAC$ is the largest of the three angles about vertex A. Let E be on side BC so that $\angle BAD = \angle BAE$. Let X, Y, Z be on rays AB, AC, AD respectively, and $AX=AY=AZ$. Then $\Delta AXZ \cong \Delta AXY$ and we have $XZ=XY$. Let line XY intersect line AC at W. Since $XZ + ZW > XW$, cancelling $XZ = XY$ from both sides, we have $ZW > YW$. Comparing triangles WAZ and WAY, we have $WA=WA$, $AZ=AY$, so $ZW > YW$ implies $\angle ZAW > \angle YAW$. Then

$$\begin{aligned}\angle BAC &= \angle XAY + \angle YAW \\ &< \angle XAZ + \angle ZAW \\ &= \angle BAD + \angle DAC.\end{aligned}$$

For the right inequality, by the left inequality, we have

$$\angle DBC \le \angle DBA + \angle ABC,$$
$$\angle BCD \le \angle BCA + \angle ACD,$$
$$\angle CDB \le \angle CDA + \angle ADB.$$

Adding them, we get 180˚ is less than or equal to the sum of the six angles on the right. Now the sum of these six angles and the three angles about A is 3×180˚. So the sum of the three angles about A is less than or equal to 360˚.

Example 5. (*1969 All Soviet Math. Olympiad*) There are n given points in space with no three collinear. For every three of them, they form a triangle having an angle greater than 120°. Prove that there is a way to order the points as A_1, A_2, …, A_n such that whenever $1 \le i < j < k \le n$, we have

$$\angle A_iA_jA_k > 120°.$$

Solution. Take two furthest points among these n points and call them A_1 and A_n.

For every two points X, Y among the other $n-2$ points, since A_1A_n is the longest side in both ΔA_1XA_n and ΔA_1YA_n, we have $\angle XA_1A_n < 60°$ and $\angle YA_1A_n < 60°$. About vertex A_1 of the tetrahedron A_1A_nXY, we have

$$\begin{aligned}\angle XA_1Y &\le \angle XA_1A_n + \angle YA_1A_n \\ &< 60°+60° = 120°.\end{aligned}$$

Similarly, $\angle XA_nY < 120°$.

Also, $A_1X \ne A_1Y$ (since otherwise, the two equal angles in ΔXA_1Y cannot be greater than 90˚ and so only $\angle XA_1Y$ can be greater than 120°, which will contradict the inequality above). Now order the points by its distance to A_1 so that $A_1A_2 < A_1A_3 < \cdots < A_1A_n$.

For $1 < j < k \le n$, taking $X=A_j$ and $Y=A_k$ in the inequality above, we get $\angle A_jA_1A_k < 120°$. Since $A_1A_k > A_1A_j$, so in $\Delta A_1A_jA_k$, $\angle A_1A_jA_k > 120°$.

For $1 < i < j < k \le n$, we have $\angle A_1A_iA_j > 120°$ and $\angle A_1A_iA_k > 120°$ by the last paragraph. Then, about vertex A_i of the tetrahedron $A_iA_jA_kA_1$, we have $\angle A_jA_iA_k < 120°$. Next since $A_1A_k > A_1A_j > A_1A_i$, about vertex A_k of the tetrahedron $A_kA_jA_iA_1$, we have

$$\begin{aligned}\angle A_iA_kA_j &\le \angle A_iA_kA_1 + \angle A_jA_kA_1 \\ &< 60°+60° = 120°.\end{aligned}$$

Hence, in $\Delta A_iA_jA_k$, we have $\angle A_iA_jA_k > 120°$.

Example 6. (*1994 All Russian Math. Olympiad*) There are k points, $2 \le k \le 50$, inside a convex 100-sided polygon. Prove that we can choose at most $2k$ vertices from this 100-sided polygon so that the k points are inside the polygon with the chosen points as vertices.

Solution. Let $M = A_1A_2 \cdots A_n$ be the boundary of the convex hull of the k points. Hence, $n \le k$. Let O be a point inside M. From i=1 to n, let ray OA_i intersect the 100-sided polygon at B_i. Let M' be the boundary of the convex hull of $B_1, B_2, \cdots, B_n$.

For every point P on or inside M, the line OP intersects M at two sides, say A_iA_{i+1} and A_jA_{j+1}. By the definition of the points B_i's, we see the line OP intersects B_iB_{i+1} and B_jB_{j+1}, say at points S and T respectively. Since B_i, B_{i+1}, B_j and B_{j+1} are in M', so S, T are in M'. Then O and P are in M'. Thus M' contains M.

Let $M' = C_1C_2 \cdots C_m$. Then $m \le n \le k$. Observe that all C_i's are on the 100-sided polygon. Now each C_i is a vertex or between two consecutive vertices of the 100-sided polygon. Let G be the set of all these vertices. Then G has at most $2k$ points and the polygon with vertices from G contains the k points.

(Continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***May 20, 2008.***

Problem 296. Let $n > 1$ be an integer. From a $n\times n$ square, one 1×1 corner square is removed. Determine (with proof) the least positive integer k such that the remaining areas can be partitioned into k triangles with equal areas.
(*Source 1992 Shanghai Math Contest*)

Problem 297. Prove that for every pair of positive integers p and q, there exist an integer-coefficient polynomial $f(x)$ and an open interval with length $1/q$ on the real axis such that for every x in the interval, $|f(x) - p/q| < 1/q^2$.
(*Source:1983 Finnish Math Olympiad*)

Problem 298. The diagonals of a convex quadrilateral $ABCD$ intersect at O. Let M_1 and M_2 be the centroids of ΔAOB and ΔCOD respectively. Let H_1 and H_2 be the orthocenters of ΔBOC and ΔDOA respectively. Prove that $M_1M_2 \perp H_1H_2$.

Problem 299. Determine (with proof) the least positive integer n such that in every way of partitioning $S = \{1,2,\ldots,n\}$ into two subsets, one of the subsets will contain two distinct numbers a and b such that ab is divisible by $a+b$.

Problem 300. Prove that in base 10, every odd positive integer has a multiple all of whose digits are odd.

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Solutions

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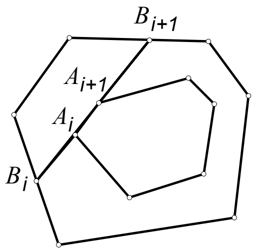
Problem 291. Prove that if a convex polygon lies in the interior of another convex polygon, then the perimeter of the inner polygon is less than the perimeter of the outer polygon.

Solution. **Jeff CHEN** (Virginia, USA), **HO Kin Fai** (HKUST, Math Year 3) and **Fai YUNG**.



We will define a sequence of convex polygons $P_0, P_1, \ldots, P_{n-1}$. Let the outer convex polygon be P_0 and the inner convex polygon be $A_1A_2\ldots A_n$. For $i = 1$ to $n-1$, let the line A_iA_{i+1} intersect P_{i-1} at B_i, B_{i+1}. The line A_iA_{i+1} divides P_{i-1} into two parts with one part enclosing $A_1A_2\ldots A_n$. Let P_i be the polygon formed by putting the segment B_iB_{i+1} together with the part of P_{i-1} enclosing $A_1A_2\ldots A_n$. Note P_{n-1} is $A_1A_2\ldots A_n$. Finally, the perimeter of P_i is less than the perimeter of P_{i-1} because the length of B_iB_{i+1}, being the shortest distance between B_i and B_{i+1}, is less than the length of the part of P_{i-1} removed to form P_i.

Commended solvers: **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina), **Raúl A. SIMON** (Santiago, Chile) and **Simon YAU Chi-Keung** (City University of Hong Kong).

Problem 292. Let $k_1 < k_2 < k_3 < \cdots$ be positive integers with no two of them are consecutive. For every $m = 1, 2, 3, \ldots$, let $S_m = k_1+k_2+\cdots+k_m$. Prove that for every positive integer n, the interval $[S_n, S_{n+1})$ contains at least one perfect square number.
(*Source: 1996 Shanghai Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **HO Kin Fai** (HKUST, Math Year 3), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina) and **Raúl A. SIMON** (Santiago, Chile).

There is a nonnegative integer a such that $a^2 < S_n \le (a+1)^2$. We have

$$\begin{aligned} S_n &= k_n + k_{n-1} + \cdots + k_1 \\ &< k_n + (k_n - 2) + \cdots + (k_n - 2n+2) \\ &= n(k_n - n+1). \end{aligned}$$

By the *AM-GM* inequality,

$$a < \sqrt{S_n} < \frac{n + (k_n - n + 1)}{2} = \frac{k_n + 1}{2}.$$

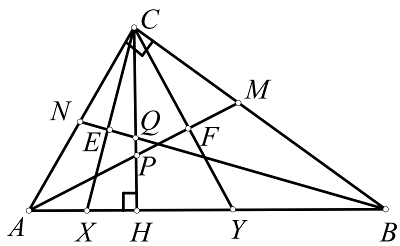
Then

$$\begin{aligned} (a+1)^2 = a^2+2a+1 &< S_n + (k_n+1) + 1 \\ &\le S_n + k_{n+1} = S_{n+1}. \end{aligned}$$

Commended solvers: **Simon YAU Chi-Keung** (City University of Hong Kong).

Problem 293. Let CH be the altitude of triangle ABC with $\angle ACB = 90°$. The bisector of $\angle BAC$ intersects CH, CB at P, M respectively. The bisector of $\angle ABC$ intersects CH, CA at Q, N respectively. Prove that the line passing through the midpoints of PM and QN is parallel to line AB.
(*Source: 52nd Belorussian Math. Olympiad*)

Solution. **Jeff CHEN** (Virginia, USA) and **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England) and **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).



Let E, F be the midpoints of QN, PM respectively. Let X, Y be the intersection of CE, CF with AB respectively. Now

$$\begin{aligned} \angle CMP &= 90° - \angle CAM \\ &= 90° - \angle BAM \\ &= \angle APH = \angle CPM. \end{aligned}$$

So $CM=CP$. Then $CF \perp AF$. Since AF bisects $\angle CAY$, by ASA, $\Delta CAF \cong \Delta YAF$. So $CF=FY$. Similarly, $CE=EX$. By the midpoint theorem, we have EF parallel to line XY, which is the same as line AB.

Commended solvers: **Konstantine ZELATOR** (University of Toledo, Toledo, Ohio, USA).

Problem 294. For three nonnegative real numbers x, y, z satisfying the condition $xy+yz+zx = 3$, prove that

$$x^2 + y^2 + z^2 + 3xyz \ge 6.$$

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Ovidiu FURDUI** (Cimpia - Turzii, Cluj, Romania), **MA Ka Hei** (Wah Yan College, Kowloon) and **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).

Let $p = x+y+z$, $q = xy+yz+zx$ and $r = xyz$. Now

$$p^2 - 9 = x^2 + y^2 + z^2 - xy - yz - zx$$
$$= \frac{(x-y)^2 + (y-z)^2 + (z-x)^2}{2} \geq 0,$$

So $p \geq 3$. By Schur's inequality (see Math Excalibur, vol. 10, no. 5, p. 2, column 2), $12p = 4pq \leq p^3 + 9r$. Since

$$p^2 = x^2+y^2+z^2+2(xy+yz+zx)$$
$$= x^2+y^2+z^2+6,$$

we get

$$3xyz = 3r \geq 9r/p$$
$$\geq 12 - p^2$$
$$= 6 - (x^2 + y^2 + z^2).$$

Problem 295. There are $2n$ distinct points in space, where $n \geq 2$. No four of them are on the same plane. If $n^2 + 1$ pairs of them are connected by line segments, then prove that there are at least n distinct triangles formed.
(*Source: 1989 Chinese IMO team training problem*)

Solution. **Jeff CHEN** (Virginia, USA) and **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England).

We prove by induction on n. For n=2, say the points are A,B,C,D. For five segments connecting them, only one pair of them is not connected, say they are A and B. Then triangles ACD and BCD are formed.

Suppose the case n=k is true. Consider the case n=k+1. We first claim there is at least one triangle. Suppose AB is one such connected segment. Let α, β be the number of segments connecting A, B to the other $2n$−2=$2k$ points respectively.

If $\alpha+\beta > 2k+1$, then A, B are both connected to one of the other $2k$ points, hence a triangle is formed.

If $\alpha+\beta \leq 2k$, then the other $2k$ points have at least $(k+1)^2 + 1 - (2k+1) = k^2 + 1$ segments connecting them. By the case n=k, there is a triangle in these $2k$ points.

So the claim is established. Now take one such triangle, say ABC. Let α, β, γ be the number of segments connecting A, B, C to the other $2k$−1 points respectively.

If $\alpha+\beta+\gamma \geq 3k-1$, then let $D_1, D_2, \ldots, D_m$ ($m \leq 2k-1$) be all the points among the other $2k$−1 points connecting to at least one of A or B or C. The number of segments to D_i from A or B or C is n_i = 1 or 2 or 3. Checking each of these three cases, we see there are at least n_i-1 triangles having D_i as a vertex and the two other vertices from A, B, C. So there are

$$\sum_{i=1}^{m}(n_i - 1) \geq 3k - 1 - m \geq k$$

triangles, each having one D_i vertex, plus triangle ABC, resulting in at least k+1 triangles.

If $\alpha+\beta+\gamma \leq 3k-2$, then the sum of $\alpha+\beta$, $\gamma+\alpha$, $\beta+\gamma$ is at most $6k-4$. Hence the least of them cannot be $2k-1$ or more, say $\alpha+\beta \leq 2k-2$. Then removing A and B and all segments connected to at least one of them, we have at least $(k+1)^2+1-(2k+1)=k^2+1$ segments left for the remaining $2k$ points. By the case n=k, we have k triangles. These plus triangle ABC result in at least k+1 triangles. The induction is complete.

Commended solvers: **Raúl A. SIMON** (Santiago, Chile) and **Simon YAU Chi-Keung** (City University of Hong Kong).

Olympiad Corner

(continued from page 1)

Problem 4. Consider the function $f: \mathbb{N}_0 \to \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all non-negative integers, defined by the following conditions:

(i) $f(0) = 0$, (ii) $f(2n) = 2f(n)$ and (iii) $f(2n+1) = n+2f(n)$ for all $n \geq 0$.

(a) Determine the three sets L:={ $n \mid f(n) < f(n+1)$}, E:={ $n \mid f(n) = f(n+1)$}, and G:={$n \mid f(n) > f(n+1)$}.

(b) For each $k \geq 0$, find a formula for a_k:= max{ $f(n) \mid 0 \leq n \leq 2^k$} in terms of k.

Problem 5. Let a, b, c be integers satisfying $0 < a < c-1$ and $1 < b < c$. For each k, $0 \leq k \leq a$, let r_k, $0 \leq r_k < c$, be the remainder of kb when dived by c. Prove that the two sets $\{r_0, r_1, r_2, \ldots, r_a\}$ and $\{0, 1, 2, \ldots, a\}$ are different.

Point Set Combinatorics

(continued from page 2)

Example 7. (*1987 Chinese IMO Team Selection Test*) There are $2n$ distinct points in space, where $n \geq 2$. No four of them are on the same plane. If $n^2 + 1$ pairs of them are connected by line segments, then prove that there are two triangles sharing a common side.

Solution. We prove by induction on n. For n=2, say the points are A,B,C,D. For five segments connecting them, only one pair of them is not connected, say they are A and B. Then triangles ACD and BCD are formed and the side CD is common to them.

Suppose the case n=k is true. Consider the case n=k+1. Suppose AB is one such connected segment. Let α, β be the number of segments connecting A, B to the other $2n - 2 = 2k$ points respectively.

Case 1. If $\alpha+\beta \geq 2k+2$, then there are points C, D among the other $2k$ points such that AC, BC, AD, BD are connected. Then triangles ABC and ABD are formed and the side AB is common to them.

Case 2. If $\alpha+\beta \leq 2k$, then removing A, B and all segments connecting to at least one of them, there would still be at least $(k+1)^2 + 1 - (2k+1) = k^2+1$ segments left for the remaining $2k$ points. By the case $n = k$, there would exist two triangles sharing a common side among them.

Case 3. Assume cases 1 and 2 do not occur for all the connected segments. Then take any connected segment AB and we have $\alpha+\beta = 2k+1$. There would then be a point C among the other $2k$ points such that triangle ABC is formed.

Let γ be the number of segments connecting C to the other $2k-1$ points respectively. Since cases 1 and 2 do not occur, we have

$$\beta + \gamma = 2k+1 \text{ and } \gamma + \alpha = 2k+1,$$

too. However, this would lead to

$$(\alpha + \beta) + (\beta + \gamma) + (\gamma + \alpha) = 6k + 3,$$

which is contradictory as the left side is even and the right side is odd.

One cannot help to notice the similarity between the last example and problem 295 in the problem corner. Naturally this raise the question: when n is large, again if $n^2 + 1$ pairs of the points are connected by line segments, would we be able to get more pairs of triangles sharing common sides? Any information or contribution for this question from the readers will be appreciated.