Olympiad Corner

The following are the problems of the 2008 IMO held in Madrid in July.

Problem 1. An acute-angled triangle $ABC$ has orthocenter $H$. The circle passing through $H$ with centre the midpoint of $BC$ intersects the line $BC$ at $A_1$ and $A_2$. Similarly, the circle passing through $H$ with centre the midpoint of $CA$ intersects the line $CA$ at $B_1$ and $B_2$, and the circle passing through $H$ with centre the midpoint of $AB$ intersects the line $AB$ at $C_1$ and $C_2$. Show that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle.

Problem 2. (a) Prove that
\[ (x-1)^2 + y^2 + z^2 \geq 1 \]
for all real numbers $x, y, z$, each different from 1, and satisfying $xyz = 1$.

(b) Prove that equality holds above for infinitely many tripes of rational numbers $x, y, z$, each different from 1, and satisfying $xyz = 1$.

Problem 3. Prove that there exist infinitely many positive integers $n$ such that $n^2 + 1$ has a prime divisor which is greater than $2n + \sqrt{2n}$.

(continued on page 4)

Geometric Transformations II

Kin Y. Li

Below the vector from $X$ to $Y$ will be denoted as $XY$. The notation $\angle ABC = \alpha$ means the ray $BA$ after rotated an angle $\alpha$ (anticlockwise if $\alpha > 0$, clockwise if $\alpha < 0$) will coincide with the ray $BC$.

On a plane, a translation by a vector $v$ (denoted as $T(v)$) moves every point $X$ to a point $Y$ such that $XY = v$. On the complex plane $\mathbb{C}$, if the vector $v$ corresponds to the vector from 0 to $v$, then $T(v)$ has the same effect as the function $f: \mathbb{C} \rightarrow \mathbb{C}$ given by $f(w) = w + v$.

A homothety about a center $C$ and ratio $r$ (denoted as $H(C, r)$) moves every point $X$ to a point $Y$ such that $CY = r CX$. If $C$ corresponds to the complex number $c$, then $H(C, r)$ has the same effect as $f(w) = r(w - c) + c = re^w + (1 - re)c$.

A rotation about a center $C$ by angle $\alpha$ (denoted as $R(C, \alpha)$) moves every point $X$ to a point $Y$ such that $CX = CY$ and $\angle XCY = \alpha$. In $\mathbb{C}$, if $C$ corresponds to the complex number $c$, then $R(C, \alpha)$ has the same effect as $f(w) = e^{i\alpha}(w - c) + c = e^{i\alpha}w + (1 - e^{i\alpha})c$.

A reflection across a line $l$ (denoted as $S(l)$) moves every point $X$ to a point $Y$ such that the line $XY$ is the perpendicular bisector of segment $XY$. In $\mathbb{C}$, let $S(l)$ send 0 to $b$. If $b = 0$ and $l$ is the line through 0 and $e^{i0}$, then $S(l)$ has the same effect as $f(w) = e^{i0}w$. If $b \neq 0$, then let $b = |b| e^{i\theta}$, $e^{i\theta} = -e^{i\theta}$ and $L$ be the vertical line through $|b|/2$. In $\mathbb{C}$, $S(L)$ sends $w$ to $|b|/2$. Using that, $S(l)$ is

\[ f(w) = e^{i\theta}(|b| - we^{-i\theta}) = e^{i\theta}w + b. \]

We have the following useful facts:

Fact 1. If $l_1 \parallel l_2$, then
\[ S(l_1) \circ S(l_2) = T(2A_4A_2), \]
where $A_1$ is on $l_1$ and $A_2$ is on $l_2$ such that the length of $A_1A_2$ is the distance $d$ from $l_1$ to $l_2$.

(Reason): Say $l_1, l_2$ are vertical lines through $A_1 = 0, A_2 = d$. Then $S(l_1), S(l_2)$ are $f_1(w) = -\overline{w}$ and $f_2(w) = -\overline{w} + 2d$.

So $S(l_1) \circ S(l_2)$ is
\[ f_2(f_1(w)) = -(\overline{w} + 2d) = -\overline{w} + 2d, \]
which is $T(2A_4A_2)$.

Fact 2. If $l_1 \parallel l_2$, then
\[ S(l_1) \circ S(l_2) = R(O, \alpha), \]
where $l_1$ intersects $l_2$ at O and $\alpha$ is twice the angle from $l_1$ to $l_2$ in the anticlockwise direction.

(Reason): Say O is the origin, $l_1$ is the x-axis. Then $S(l_1)$ and $S(l_2)$ are
\[ f_1(w) = \overline{w} \ 	ext{and} \ f_2(w) = e^{i\alpha}w, \]
so $S(l_1) \circ S(l_2)$ is $f_2(f_1(w)) = e^{i\alpha}w$, which is $R(O, \alpha)$.

Fact 3. If $\alpha + \beta$ is not a multiple of $360^\circ$, then
\[ R(O, \beta) \circ R(O, \alpha) = R(O, \alpha + \beta), \]
where $\angle OAO_2 = \alpha/2, \angle O_2AO = \beta/2$. If $\alpha + \beta$ is a multiple of $360^\circ$, then
\[ R(O, \beta) \circ R(O, \alpha) = T(O, O_2), \]
where $R(O, \beta)$ sends $O_2$ to $O_1$.

(Reason): Say $O_1$ is 0, $O_2$ is $-1$. Then $R(O_1, \alpha), R(O_2, \beta)$ are $f_1(w) = \overline{w}, f_2(w) = e^{i\alpha}w + (e^{i\alpha}-1)$, so $f_2(f_1(w)) = e^{i\alpha}w + (e^{i\alpha}-1)$. If $e^{i\alpha} \neq 1$, this is a rotation about $e^{i(\alpha-1)/2} = e^{i\alpha/2}$. We have
\[ c' = \frac{\sin(\beta/2)}{\sin((\alpha + \beta)/2)} e^{i(\alpha-1)/2}, \]
\[ c' = -\frac{-\sin(\alpha/2)}{\sin((\alpha + \beta)/2)} e^{-i\beta/2}. \]
If $e^{i\alpha} = 1$, this is a translation by $e^{i\alpha-1} = f_2(0)$.)

Fact 4. If $O_1, O_2, O_3$ are noncollinear, $\alpha_1, \alpha_2, \alpha_3 > 0, \alpha_1 + \alpha_2 + \alpha_3 = 360^\circ$ and
\[ R(O_1, \alpha_1) \circ R(O_2, \alpha_2) \circ R(O_3, \alpha_3) = I, \]
where $I$ is the identity transformation, then $\angle O_1O_2O_3 = \alpha_1/2, \angle O_2O_3O_1 = \alpha_2/2$ and $\angle O_3O_1O_2 = \alpha_3/2$.

(This is just the case $\alpha_2 = 360^\circ - (\alpha_1 + \alpha_3)$ of fact 3.)
Fact 5. Let $O_1 \neq O_2$. For $r_1 r_2 \neq 1$,$$H(O_2, r_2) \circ H(O_1, r_1) = H(O, r_1 r_2)$$for some $O$ on line $O_1 O_2$. For $r_1 r_2 = 1$,$$H(O_2, r_2) \circ H(O_1, r_1) = T(1-r_2) O_1 O_2$$

(Reason: Say $O_1$ is 0, $O_2$ is $c$. Then $H(O_2, r_2) H(O_2, r_2)$ are $f_1(w) = r_1 w, f_2(w) = r_2 w = r_1 r_2 w + (1-r_2)c$. For $r_1 r_2 \neq 1$, this is a homothety about $c = (1-r_2)c/(1-r_2 r_2)$ and ratio $r_1 r_2$. For $r_1 r_2 = 1$, this is a translation by $(1-r_2)c$.)

Next we will present some examples.

Example 1. In $\triangle ABC$, let $E$ be onside $AB$ such that $AE:EB = 1:2$ and $D$ be on side $AC$ such that $AD:DC = 2:1$. Let $F$ be the intersection of $BD$ and $CE$. Determine $FD:FB$ and $FE:FC$.

Solution. We have $H(E, -1/2)$ sends $B$ to $A$ and $H(C, 1/3)$ sends $A$ to $D$. Since $(1/3) \times (-1/2) \neq 1$, by fact 5,$$H(C, 1/3) \circ H(E, -1/2) = H(O, -1/6)$$where the center $O$ is on line $CE$. However, the composition on the left side sends $B$ to $D$. So $O$ is also on line $BD$. Hence, $O$ must be $F$. Then we have $FD:FB = OD:OB = 1:6$.

Similarly, we have$$H(B, 2/3) \circ H(D, -2) = H(F, -4/3)$$sends $C$ to $E$, so $FE:FC = 4:3$.

Example 2. Let $E$ be inside square $ABCD$ such that $EAD = EDA = 15^\circ$. Show that $\triangle EBC$ is equilateral.

Solution. Let $O$ be inside the square such that $\triangle ADO$ is equilateral. Then $R(D, 30^\circ)$ sends $C$ to $O$ and $R(A, 30^\circ)$ sends $O$ to $B$. Since $EAD = 15^\circ = \angle DAE$, by fact 3,$$R(A, 30^\circ) \circ R(D, 30^\circ) = R(E, 60^\circ),$$so $R(E, 60^\circ)$ sends $C$ to $B$. Therefore, $\triangle EBC$ is equilateral.

Example 3. Let $ABEF$ and $ACGH$ be squares outside $\triangle ABC$. Let $M$ be the midpoint of $EG$. Show that $MB = MC$ and $MB \bot MC$.

Solution. Since $\angle CAB = \angle RAD = 180^\circ$, it follows that $\angle BMC = \angle DNB = 180^\circ$.

Now $R(M, \angle BMC)$ sends $B$ to $C$, $R(K, 180^\circ)$ sends $C$ to $D$ and $R(N, \angle DNB)$ sends $D$ to $B$. However, by fact 3,$$R(N, \angle DNB) \circ R(K, 180^\circ) \circ R(M, \angle BMC)$$is a translation and since it sends $B$ to $B$, it must be the identity transformation $I$. By fact 4, $\angle MKN = 90^\circ$.

Example 6. Let $H$ be the orthocenter of $\triangle ABC$ and lie inside it. Let $A', B', C'$ be the circumcenters of $\triangle HBC, \triangle HAC, \triangle HAB$ respectively. Show that $AA', BB', CC'$ are concurrent and identify the point of concurrency.

Solution. For $\triangle ABC$, let $O$ be its circumcenter and $G$ be its centroid. Let the reflection across line $BC$ sends $A$ to $A'$. Then $\angle BAC = \angle BAC'$. Now$$\angle BHC = \angle ABH + \angle BAC + \angle ACH$$is $(90^\circ - \angle ABC) + \angle BAC + (90^\circ - \angle BAC')$ $= 180^\circ - \angle BAC'$. So $A'$ is on the circumcircle of $\triangle HBC$.

Now the reflection across line $BC$ sends $O$ to $A'$, the reflection across line $CA$ sends $O$ to $B'$ and the reflection across line $AB$ sends $O$ to $C'$. Let $D, E, F$ be the midpoints of sides $BC, CA, AB$ respectively. Then $H(G, -1/2)$ sends $\triangle ABC$ to $\triangle DEF$ and $H(O, 2)$ sends $\triangle DEF$ to $\triangle A'B'C'$. Since $(-1/2)^2 \neq 1$, by fact 5,$$H(O, 2) \circ H(G, -1/2) = H(X, -1)$$for some point $X$. Since the composition on the left side sends $\triangle ABC$ to $\triangle A'B'C'$, segments $AA', BB', CC'$ concur at $X$ and in fact $X$ is their common midpoint.
Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong. The deadline for sending solutions is October 31, 2008.

Problem 306. Prove that for every integer $n \geq 48$, every cube can be decomposed into $n$ smaller cubes, where every pair of these small cubes does not have any common interior point and has possibly different sidelengths.

Problem 307. Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ be a polynomial with real coefficients such that $a_0 \neq 0$ and for all real $x$, $f(x)f(2x^2) = f(2x^3+x)$. Prove that $f(x)$ has no real root.

Problem 308. Determine (with proof) the greatest positive integer $n > 1$ such that the system of equations

\[
(x+1)^2 + y_1^2 = (x+2)^2 + y_2^2 = \ldots = (x+n)^2 + y_n^2
\]

has an integral solution $(x, y_1, y_2, \ldots, y_n)$.

Problem 309. In acute triangle $ABC$, $AB > AC$. Let $H$ be the foot of the perpendicular from $A$ to $BC$ and $M$ be the midpoint of $AH$. Let $D$ be the point where the incircle of $ABC$ is tangent to side $BC$. Let line $DM$ intersect the incircle again at $N$. Prove that $\angle BND = \angle CND$.

Problem 310. (Due to Pham Van Thuan) Prove that if $p$, $q$ are positive real numbers such that $p + q = 2$, then

\[
3p^3q^4 + p^2q^4 \leq 4.
\]

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Solutions

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Problem 301. Prove that it is possible to decompose two congruent regular hexagons into a total of six pieces such that they can be rearranged to form an equilateral triangle with no pieces overlapping.

Problem 302. Let $Z$ denote the set of all integers. Determine (with proof) all functions $f: Z \to Z$ such that for all $x, y$ in $Z$, we have $f(x+yf(y)) = f(x) - y$. (Source: 2004 Spanish Math Olympiad)

Problem 303. In base 10, let $N$ be a positive integer with all digits nonzero. Prove that there do not exist two permutations of the digits of $N$, forming numbers that are different (integral) powers of two.

Solution. Glenier L. BELLO-BURGUET (I.E.S. Hermanos D’Elhuyar, Spain), NGUYEN Tho Tung (High School for Gifted Education, Ha Noi University of Education) and PUN Ying Anna (HKU Math Year 3).

Assume there exist two permutations of the digits of $N$, forming the numbers $2^k$ and $2^m$ for some positive integers $k$ and $m$ with $k < m$. Then $2^k < 10^2^m$. So $k \leq m+3$.

Since every number is congruent to its sum of digits (mod 9), we get $2^k \equiv 2^m$ (mod 9). Since $2^k$ and $9$ are relatively prime, we get $2^k2^{-m} \equiv 1$ (mod 9). However, $k - m = 1, 2$ or 3, which contradicts $2^k2^{-m} \equiv 1$ (mod 9).

Problem 304. Let $M$ be a set of 100 distinct lattice points (i.e. coordinates are integers) chosen from the $x$-$y$ coordinate plane. Prove that there are at most 2025 rectangles whose vertices are in $M$ and whose sides are parallel to the $x$-axis or the $y$-axis. (Source: 2003 Chinese IMO Team Training Test)

Solution 1. Glenier L. BELLO-BURGUET (I.E.S. Hermanos D’Elhuyar, Spain) and PUN Ying Anna (HKU Math Year 3).

Let $O$ be a point in $M$. We say a rectangle is good if all its sides are parallel to the $x$ or $y$-axis and all its vertices are in $M$, one of which is $O$. We claim there are at most 81 good rectangles. (Once the claim is proved, we see there can only be at most (81 × 100)/4 = 2025 desired rectangles.)
The division by 4 is due to such rectangle has 4 vertices, hence counted 4 times).

For the proof of the claim, we may assume O is the origin of the plane. Suppose the x-axis contains m points in M other than O and the y-axis contains n points in M other than O. For a point P in M not on either axis, it can only be a vertex of at most one good rectangle. There are at most 99 − m − n such point P and every good rectangle has such a vertex.

If m + n ≥ 18, then there are at most 99 − m − n ≤ 81 good rectangles. Otherwise, m + n ≤ 17. Now every good rectangle has a vertex on the x-axis and a vertex on the y-axis other than O. So there are at most mn ≤ (m+n)²/4 ≤ 81 rectangles by the AM-GM inequality. The claim follows.

**Solution 2. G.R.A. 20 Problem Solving Group (Roma, Italy).**

Let \( f(x) = x(x-1)/2 \). We will prove that if there are N lattice points, there are at most \((N/12)^2\) such rectangles. For \( N = 100 \), we have \((10)^2 = 45^2 = 2025\) (this bound is attained when the 100 points form a 10×10 square).

Suppose the N points are distributed on \( m \) lines parallel to an axis. Say the number of points in the \( m \) lines are \( r_1, r_2, \ldots, r_m \), arranged in increasing order. Now the two lines with \( r_j \) and \( r_j \) points can form no more than \( f(\min(r_j, r_j)) \) rectangles. Hence, the number of rectangles is at most

\[
\sum_{i=j}^{m} f(\min(r_i, r_j)) = \sum_{i=1}^{m} (m-i) f(r_i) \leq \sum_{i=1}^{m} (m-i) \left( \frac{N}{m} \right) = f(m) \left( \frac{N}{m} \right)^2 \leq f(\sqrt{N})^2.
\]

The second inequality follows by expansion and usage of the AM-GM inequality. The first one can be proved by expanding and simplifying it to

\[
2m \sum_{i=1}^{m} (m-i) f(r_i) \leq (m-1) \sum_{i=1}^{m} f(r_i) - 1. \quad (*)
\]

We will prove this by induction on \( m \).

For \( m = 2, 4r_2(r_2 - 1) \leq (r_2 + 1)(r_2 - 1) \) follows from \( 1 \leq r_1 \leq r_2 \). For the inductive step, we suppose \((*)\) is true. To do the \((m+1)\)-st case of \((*)\), observe that \( r_i \leq r_{m+1} \) implies

\[
m \sum_{i=1}^{m} r_i(r_i - 1) \leq m(r_{m+1} - 1) \sum_{i=1}^{m} r_i,
\]

\[
m \sum_{i=1}^{m} r_i(r_i - 1) \leq m r_{m+1} \sum_{i=1}^{m} (r_i - 1),
\]

\[
2 \sum_{i=1}^{m} (m+1-i) r_i(r_i - 1) \leq m r_{m+1} (r_{m+1} - 1) + \sum_{i=1}^{m} r_i(r_i - 1).
\]

Let \( L(m) \) and \( R(m) \) denote the left and right sides of \((*)\) respectively. Adding the last three inequalities, it turns out we get \( L(m+1) - L(m) \leq R(m+1) - R(m) \). Now \((*)\) holds, so \( L(1) \leq R(m) \). Adding these, we get \( L(m) \leq R(m) \).

**Problem 305.** A circle \( \Gamma_2 \) is internally tangent to the circumcircle \( \Gamma_1 \) of \( \Delta PAB \) at \( P \) and side \( AB \) at \( C \). Let \( E, F \) be the intersection of \( \Gamma_2 \) with sides \( PA, PB \) respectively. Let \( EF \) intersect \( PC \) at \( D \). Lines \( PD, AD \) intersect \( \Gamma_1 \) again at \( G, H \) respectively. Prove that \( F, G, H \) are collinear.

**Solution.** CHEUNG Wang Chi (Magdalen College, University of Cambridge, England), Glenier L. BELLO-BURGUET (I.E.S. Hermanos D’Elhuyar, Spain), NGUYEN Tho Tung (High School for Gifted Education, Ha Noi University of Education) and PUN Ying Anna (HKU Math Year 3).

Let \( PT \) be the external tangent to both circles at \( P \). We have

\[
\angle PAB = \angle BPT = \angle PEF,
\]

which implies \( EF \parallel AB \). Let \( O \) be the center of \( \Gamma_2 \). Since \( OC \perp AB \) (because \( AB \) is tangent to \( \Gamma_2 \) at \( C \)), we deduce that \( OC \perp EF \) and therefore \( OC \) is the perpendicular bisector of \( EF \). Hence \( C \) is the midpoint of arc \( EC \). Then \( PC \) bisects \( \angle PEF \). On the other hand,

\[
\angle HDF = \angle HAB = \angle HPB = \angle HPF,
\]

which implies \( H, P, D, F \) are concyclic.

Therefore,

\[
\angle HDF = \angle DPF = \angle EPD = \angle APG = \angle AHG = \angle DHG,
\]

which implies \( F, G, H \) are collinear.

**Remarks.** A few solvers got \( EF \parallel AB \) by observing there is a homothety with center \( P \) sending \( \Gamma_2 \) to \( \Gamma_1 \) so that \( E \) goes to \( A \) and \( F \) goes to \( B \).

Commended solvers: Victor FONG (CUHK Math Year 2) and Salem MALIKIC (Sarajevo College, Sarajevo, Bosnia and Herzegovina).

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**Olympiad Corner (continued from page 1)**

**Problem 4.** Find all functions \( f: (0, \infty) \to (0, \infty) \) (so, \( f \) is a function from the positive real numbers to the positive real numbers) such that

\[
\frac{(f(w))^2 + (f(x))^2}{(f(y))^2 + (f(z))^2} = \frac{w^2 + x^2}{y^2 + z^2}
\]

for all positive real numbers \( w, x, y, z \) satisfying \( wx = yz \).

**Problem 5.** Let \( n \) and \( k \) be positive integers with \( k \geq n \) and \( k \) an even number. Let \( 2n \) lamps labeled 1, 2, \ldots, 2n be given, each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let \( N \) be the number of such sequences consisting of \( k \) steps and resulting in the state where lamp 1 through \( n \) are all on, and lamps \( n+1 \) through \( 2n \) are all off.

Let \( M \) be the number of such sequences consisting of \( k \) steps, resulting in the state where lamps 1 through \( n \) are all on, and lamps \( n+1 \) through \( 2n \) are all off, but where none of the lamps \( n+1 \) and \( 2n \) is ever switched on.

Determine the ratio \( N/M \).

**Problem 6.** Let \( ABCD \) be a convex quadrilateral with \( |BA| \neq |BC| \). Denote the incircles of triangles \( ABC \) and \( ADC \) by \( \omega_1 \) and \( \omega_2 \) respectively. Suppose that there exists a circle \( \omega \) tangent to the ray \( BA \) beyond \( A \) and to the ray \( BC \) beyond \( C \), which is also tangent to the lines \( AD \) and \( CD \). Prove that the common external tangents of \( \omega_1 \) and \( \omega_2 \) intersect on \( \omega \).