Mathematical Excalibur

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Olympiad Corner

The following were the problems of the Hong Kong Team Selection Test 2, which was held on November 8, 2008 for the 2009 IMO.

Problem 1. Let f:$Z\to Z$ (Z is the set of all integers) be such that $f(1) = 1$, $f(2) = 20$, $f(-4) = -4$ and

$$f(x+y) = f(x) + f(y) + axy(x+y) + bxy + c(x+y) + 4$$

for all $x,y \in Z$, where a, b and c are certain constants.

(a) Find a formula for $f(x)$, where x is any integer.

(b) If $f(x) \geq mx^2 + (5m+1)x + 4m$ for all non-negative integers x, find the greatest possible value of m.

Problem 2. Define a *k-clique* to be a set of k people such that every pair of them know each other (knowing is mutual). At a certain party, there are two or more 3-cliques, but no 5-clique. Every pair of 3-cliques has at least one person in common. Prove that there exist at least one, and not more than two persons at the party, whose departure (or simultaneous departure) leaves no 3-clique remaining.

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On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 10, 2009***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Double Counting

Law Ka Ho, Leung Tat Wing and Li Kin Yin

There are often different ways to count a quantity. By counting it in two ways (i.e. double counting), we thus obtain the same quantity in different forms. This often yields interesting equalities and inequalities. We begin with some simple examples.

Below we will use the notation $C_r^n = n!/(r!(n-r)!)$.

Example 1. (*IMO HK Prelim 2003*) Fifteen students join a summer course. Every day, three students are on duty after school to clean the classroom. After the course, it was found that every pair of students has been on duty together exactly once. How many days does the course last for?

Solution. Let the answer be k. We count the total number of pairs of students were on duty together in the k days. Since every pair of students was on duty together exactly once, this is equal to $C_2^{15} \times 1 = 105$. On the other hand, since 3 students were on duty per day, this is also equal to $C_2^3 \times k = 3k$. Hence $3k = 105$ and so $k = 35$.

Example 2. (*IMO 1987*) Let $p_n(k)$ be the number of permutations of the set $\{1, 2, \ldots, n\}$, $n \geq 1$, which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k \cdot p_n(k) = n!.$$

(*Remark*: A *permutation* f of a set S is a one-to-one mapping of S onto itself. An element i in S is called a *fixed point* of the permutation f if $f(i) = i$.)

Solution. Note that the left hand side of the equality is the total number of fixed points in all permutations of $\{1,2,\ldots, n\}$. To show that this number is equal to $n!$, note that there are $(n-1)!$ permutations of $\{1, 2, \ldots, n\}$ fixing 1, $(n-1)!$ permutations fixing 2, and so on, and $(n-1)!$ permutations fixing n. It follows that the total number of fixed points in all permutations is equal to $n \cdot (n-1)! = n!$.

The simplest combinatorial identity is perhaps $C_r^n = C_{n-r}^n$. While this can be verified algebraically, we can give a proof in a more combinatorial flavour: to choose r objects out of n, it is equivalent to choosing $n-r$ objects out of n to be discarded. There are C_r^n ways to do the former and C_{n-r}^n ways to do the latter. So the two quantities must be equal.

Example 3. Interpret the following equalities from a combinatorial point of view:

(a) $C_k^n = C_{k-1}^{n-1} + C_k^{n-1}$

(b) $C_1^n + 2C_2^n + \cdots + nC_n^n = n \cdot 2^{n-1}$

Solution. (a) On one hand, the number of ways to choose k objects out of n objects is C_k^n. On the other hand, we may count by including the first object or not. If we include the first object, we need to choose $k-1$ objects from the remaining $n-1$ objects and there are C_{k-1}^{n-1} ways to do so.

If we do not include the first object, we need to choose k objects from the remaining $n-1$ objects and there are C_k^{n-1} ways to do so. Hence

$$C_k^n = C_{k-1}^{n-1} + C_k^{n-1}.$$

(b) Suppose that from a set of n people, we want to form a committee with a chairman of the committee. On one hand, there are n ways to choose a chairman, and for each of the remaining $n-1$ persons we may or may not include him in the committee. Hence there are $n \cdot 2^{n-1}$ ways to finish the task.

On the other hand, we may choose k people to form a committee ($1 \leq k \leq n$), which can be done in C_k^n ways, and for each of these ways there are k ways to select the chairman. Hence the number of ways to finish the task is also equal to

$$C_1^n + 2C_2^n + \cdots + nC_n^n.$$

Example 4. (*IMO 1989*) Let n and k be positive integers and let S be a set of n points in the plane such that:

(i) no three points of S are collinear, and

(ii) for every point P of S, there are at least k points of S equidistant from P.

Prove that $k < \frac{1}{2} + \sqrt{2n}$.

Solution. Solving for n, the desired inequality is equivalent to $n > k(k-1)/2 + 1/8$. Since n and k are positive integers, this is equivalent to $n - 1 \geq C_2^k$. Now we join any two vertices of S by an *edge* and count the number of edges in two ways.

On one hand, we have C_2^n edges. On the other hand, from any point of S there are at least k points equidistant from it. Hence if we draw a circle with the point as centre and with the distance as radius then there are at least C_2^k chords as edges. The total number of such chords, counted with multiplicities, is at least nC_2^k. Any two circles can have at most one common chord and hence there could be a maximum C_2^n chords (for every possible pairs of circles) counted twice. Therefore,

$$nC_2^k - C_2^n \leq C_2^n,$$

which simplifies to $n - 1 \geq C_2^k$. (Note that collinearity was not needed.)

Example 5. (*IMO 1998*) In a competition, there are m contestants and n judges, where $n \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

Solution. We begin by considering pairs of judges who agree on certain contestants. We study this from two perspectives.

For contestant i, $1 \leq i \leq m$, suppose there are x_i judges who pass him, and y_i judges who fail him. On one hand, the number of pairs of judges who agree on him is

$$C_2^{x_i} + C_2^{y_i} = \frac{x_i^2 - x_i + y_i^2 - y_i}{2}$$

$$\geq \frac{(x_i + y_i)^2/2}{2} - \frac{x_i + y_i}{2}$$

$$= \frac{1}{4}n^2 - \frac{n}{2} = \frac{1}{4}\left[(n-1)^2 - 1\right].$$

Since n is odd and $C_2^{x_i} + C_2^{y_i}$ is an integer, it is at least $(n-1)^2/4$.

On the other hand, there are n judges and each pair of judges agree on at most k contestants. Hence the number of pairs of judges who agree on a certain contestant is at most kC_2^n. Thus,

$$kC_2^n \geq \sum_{i=1}^{m}(C_2^{x_i} + C_2^{y_i}) \geq \frac{m(n-1)^2}{4},$$

which can be simplified to obtain the desired result.

Some combinatorial problems in mathematical competitions can be solved by double counting certain ordered triples. The following are two such examples.

Example 6. (*CHKMO 2007*) In a school there are 2007 male and 2007 female students. Each student joins not more than 100 clubs in the school. It is known that any two students of opposite genders have joined at least one common club. Show that there is a club with at least 11 male and 11 female members.

Solution. Assume on the contrary that every club either has at most 10 male members or at most 10 female members. We shall get a contradiction via double counting certain ordered triples.

Let S be the number of ordered triples of the form (m, f, c), where m denotes a male student, f denotes a female student and c denotes a club. On one hand, since any two students of opposite genders have joined at least one common club, we have

$$S \geq 2007^2 = 4028049.$$

On the other hand, we can consider two types of clubs: let X be the set of clubs with at most 10 male members, and Y be the set of clubs with at least 11 male members (and hence at most 10 female members). Note that there are at most $10\times2007\times100=2007000$ triples (m, f, c) with $c \in X$, because there are 2007 choices for f, then at most 100 choices for c (each student joins at most 100 clubs), and then at most 10 choices for m (each club $c \in X$ has at most 10 male members). In exactly the same way, we can show that there are at most 2007000 triples (m, f, c) with $c \in Y$. This gives

$$S \leq 2007000+2007000=4014000,$$

a contradiction.

Example 7. (*2004 IMO Shortlisted Problem*) There are 10001 students at a university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose the following conditions hold:

(i) Each pair of students is in exactly one club.

(ii) For each student and each society, the student is in exactly one club of the society.

(iii) Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is in exactly m societies.

Find all possible values of k.

Solution. An ordered triple (a, C, S) will be called <u>*acceptable*</u> if a is a student, C is a club and S is a society such that $a \in C$ and $C \in S$. We will count the number of acceptable ordered triples in two ways.

On one hand, for every student a and society S, by (ii), there is a unique club C such that (a, C, S) is acceptable. Hence, there are $10001k$ acceptable ordered triples.

On the other hand, for every club C, let the number of members in C be denoted by $|C|$. By (iii), C is in exactly $(|C|-1)/2$ societies. So there are $|C|(|C|-1)/2$ acceptable ordered triples with C as the second coordinates. Let Γ be the set of all clubs. Hence, there are

$$\sum_{C\in\Gamma}\frac{|C|(|C|-2)}{2}$$

acceptable ordered triples. By (i), this is equal to the number of pairs of students, which is 10001×5000. Therefore,

$$10001k = \sum_{C\in\Gamma}\frac{|C|(|C|-2)}{2}$$

$$= 10001\times5000,$$

which implies $k = 5000$.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***January 10, 2009.***

Problem 311. Let $S = \{1,2,\ldots,2008\}$. Prove that there exists a function $f : S \to$ {red, white, blue, green} such that there does not exist a 10-term arithmetic progression $a_1,a_2,\ldots,a_{10}$ in S satisfying $f(a_1) = f(a_2) = \cdots = f(a_{10})$.

Problem 312. Let $x, y, z > 1$. Prove that

$$\frac{x^4}{(y-1)^2}+\frac{y^4}{(z-1)^2}+\frac{z^4}{(x-1)^2}\ge 48.$$

Problem 313. In ΔABC, $AB < AC$ and O is its circumcenter. Let the tangent at A to the circumcircle cut line BC at D. Let the perpendicular lines to line BC at B and C cut the perpendicular bisectors of sides AB and AC at E and F respectively. Prove that D, E, F are collinear.

Problem 314. Determine all positive integers x, y, z satisfying $x^3 - y^3 = z^2$, where y is a prime, z is not divisible by 3 and z is not divisible by y.

Problem 315. Each face of 8 unit cubes is painted white or black. Let n be the total number of black faces. Determine the values of n such that in every way of coloring n faces of the 8 unit cubes black, there always exists a way of stacking the 8 unit cubes into a $2\times2\times2$ cube C so the numbers of black squares and white squares on the surface of C are the same.

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Solutions

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Problem 306. Prove that for every integer $n \ge 48$, every cube can be decomposed into n smaller cubes, where every pair of these small cubes does not have any common interior point and has possibly different sidelengths.

Solution. **G.R.A. 20 Problem Solving Group** (Roma, Italy) and **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

For such an integer n, we will say cubes are <u>*n-decomposable*</u>. Let <u>*r-cube*</u> mean a cube with sidelength r. If a r-cube C is n-decomposable, then we can first decompose C into 8 $r/2$-cubes and then decompose one of these $r/2$-cubes into n cubes to get a total of $n+7$ cubes so that C is $(n+7)$-decomposable.

Let C be a 1-cube. All we need to show is C is n-decomposable for $48 \le n \le 54$.

For $n=48$, decompose C to 27 1/3-cubes and then decompose 3 of these, each into 8 1/6-cubes.

For $n=49$, cut C by two planes parallel to the bottom at height 1/2 and 1/6 from the bottom, which can produce 4 1/2-cubes at the top layer, 9 1/3-cubes in the middle layer and 36 1/6-cubes at the bottom layer.

For $n=50$, decompose C to 8 1/2-cubes and then decompose 6 of these, each into 8 1/4-cubes.

For $n=51$, decompose C into 8 1/2-cubes, then take 3 of these 1/2-cubes on the top half to form a L-shaped prism and cut out 5 1/3-cubes and 41 1/6-cubes.

For $n=52$, decompose C into 1 3/4-cube and 37 1/4-cubes, then decompose 2 1/4-cubes, each into 8 1/8-cubes.

For $n=53$, decompose C to 27 1/3-cubes and then decompose 1 of these into 27 1/9-cubes.

For $n=54$, decompose C into 8 1/2-cubes, then take 2 of the adjacent 1/2-cubes, which form a $1\times1/2\times1/2$ box, from which we can cut 2 3/8-cubes, 4 1/4-cubes and 42 1/8-cubes.

Comments: Interested readers may find more information on this problem by visiting mathworld.wolfram.com and by searching for <u>*Cube Dissection*</u> .

Problem 307. Let

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

be a polynomial with real coefficients such that $a_0 \ne 0$ and for all real x,

$$f(x)f(2x^2) = f(2x^3+x).$$

Prove that $f(x)$ has no real root.

Solution. **José Luis DÍAZ-BARRERO** (Universitat Politècnica de Catalunya, Barcelona, Spain), **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **Ozgur KIRCAK and Bojan JOVESKI** (Jahja Kemal College, Skopje, Macedonia)**, LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery)**, NG Ngai Fung** (STFA Leung Kau Kui College, Form 6), **O Kin Chit Alex** (G.T. Ellen Yeung College) and **Fai YUNG**.

For such polynomial $f(x)$, let k be largest such that $a_k\ne0$. Then

$$f(x)f(2x^2) = a_0^2 2^n x^{3n} + \cdots + a_k^2 2^{n-k} x^{3(n-k)},$$

$$f(2x^3+x) = a_0 2^n x^{3n} + \cdots + a_k x^{n-k},$$

where the terms are ordered by decreasing degrees. This can happen only if $n - k = 0$. So $f(0) = a_n \ne 0$. Assume $f(x)$ has a real root $x_0 \ne 0$. The equation $f(x)f(2x^2) = f(2x^3+x)$ implies that if x_n is a real root, then $x_{n+1} = 2x_n^3 + x_n$ is also a real root. Since this sequence is strictly monotone, this implies $f(x)$ has infinitely many real roots, which is a contradiction.

Commended solvers: **Simon YAU Chi Keung** (City U).

Problem 308. Determine (with proof) the greatest positive integer $n >1$ such that the system of equations

$$(x+1)^2+y_1^2=(x+2)^2+y_2^2=\cdots=(x+n)^2+y_n^2.$$

has an integral solution $(x,y_1, y_2, \cdots, y_n)$.

Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **Ozgur KIRCAK and Bojan JOVESKI** (Jahja Kemal College, Skopje, Macedonia) and **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

We will show the greatest such n is 3. For $n = 3$, $(x, y_1, y_2, y_3) = (-2, 0, 1, 0)$ is a solution. For $n \ge 4$, assume the system has an integral solution. Since $x+1$, $x+2$, …, $x+n$ are of alternate parity, so $y_1, y_2, \ldots, y_n$ are also of alternate parity. Since $n \ge 4$, y_k is even for $k = 2$ or 3. Consider

$$(x+k-1)^2+y_{k-1}^2=(x-k)^2+y_k^2=(x+k+1)^2+y_{k+1}^2.$$

The double of the middle expression equals the sum of the left and right expressions. Eliminating common terms in that equation, we get

$$2y_k^2 = y_{k-1}^2 + y_{k+1}^2 + 2. \qquad (*)$$

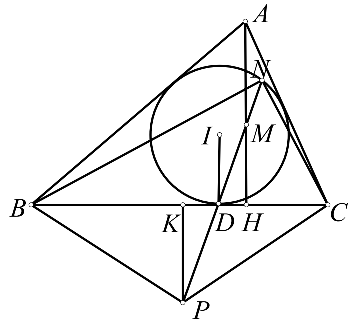
Now y_{k-1} and y_{k+1} are odd. Then the left side of (*) is 0 (mod 8), but the right side is 4 (mod 8), a contradiction.

Commended solvers: **O Kin Chit Alex** (G.T. (Ellen Yeung) College), **Raúl A. SIMON** (Santiago, Chile) and **Simon**

YAU Chi Keung (City U).

Problem 309. In acute triangle ABC, $AB > AC$. Let H be the foot of the perpendicular from A to BC and M be the midpoint of AH. Let D be the point where the incircle of ΔABC is tangent to side BC. Let line DM intersect the incircle again at N. Prove that $\angle BND = \angle CND$.

Solution.



Let I be the center of the incircle. Let the perpendicular bisector of segment BC cut BC at K and cut line DM at P. To get the conclusion, it is enough to show $DN{\cdot}DP=DB{\cdot}DC$ (which implies B,P,C,N are concyclic and since $PB = PC$, that will imply $\angle BND = \angle CND$).

Let sides $BC=a$, $CA=b$ and $AB=c$. Let $s = (a+b+c)/2$, then $DB = s-b$ and $DC = s-c$. Let r be the radius of the incircle and $[ABC]$ be the area of triangle ABC. Let $\alpha = \angle CDN$ and $AH = h_a$. Then $[ABC]$ equals

$$ah_a/2 = rs = \sqrt{s(s-a)(s-b)(s-c)}.$$

Now

$$DK = DB - KB = \frac{a+c-b}{2} - \frac{a}{2} = \frac{c-b}{2},$$

$$DH = DC - HC = \frac{a+b-c}{2} - b\cos\angle ACB$$

$$= \frac{a+b-c}{2} - \frac{a^2+b^2-c^2}{2a}$$

$$= \frac{(c-b)(b+c-a)}{2a} = \frac{(c-b)(s-a)}{a}.$$

Moreover, $DN = 2r \sin \alpha$, $DP = DK/(\cos \alpha) = (c-b)/(2\cos \alpha)$. So

$$DN \cdot DP = r(c-b)\tan\alpha = r(c-b)\frac{MH}{DH}$$

$$= r(c-b)\frac{h_a/2}{(c-b)(s-a)/a}$$

$$= r\frac{ah_a/2}{s-a} = \frac{rsrs}{s(s-a)} = \frac{[ABC]^2}{s(s-a)}$$

$$= (s-b)(s-c) = DB \cdot DC.$$

Problem 310. (*Due to Pham Van Thuan*) Prove that if p, q are positive real numbers such that $p + q = 2$, then

$$3p^q q^p + p^p q^q \le 4.$$

***Solution* 1. Proposer's Solution.**

As $p, q > 0$ and $p + q = 2$, we may assume $2 > p \ge 1 \ge q > 0$. Applying Bernoulli's inequality, which asserts that if $x > -1$ and $r \in [0,1]$, then $1+rx \ge (1+x)^r$, we have

$$p^p = pp^{p-1} \ge p(1+(p-1)^2) = p(p^2-2p+2),$$
$$q^q \le 1+q(q-1) = 1+(2-p)(1-p) = p^2-3p+3,$$
$$p^q \le 1+q(p-1) = 1+(2-p)(p-1) = -p^2+3p-1,$$
$$q^p = qq^{p-1} \le q(1+(p-1)(q-1)) = p(2-p)^2.$$

Then

$$\begin{aligned} &3p^q q^p + p^p q^q - 4 \\ &\le 3(-p^2+3p-1)p(2-p)^2 \\ &\qquad + p(p^2-2p+2)(p^2-3p+3) - 4 \\ &= -2p^5+16p^4-40p^3+36p^2-6p-4 \\ &= -2(p-1)^2(p-2)((p-2)^2-5) \le 0. \end{aligned}$$

(To factor with $p-1$ and $p-2$ was suggested by the observation that $(p,q) = (1,1)$ and $(p,q) \to (2,0)$ lead to equality cases.)

Comments: The case $r = m/n \in \mathbb{Q} \cap [0,1]$ of Bernoulli's inequality follows by applying the *AM-GM* inequality to $a_1,\ldots,a_n$, where $a_1 = \cdots = a_m = 1+x$ and $a_{m+1} = \cdots = a_n = 1$. The case $r \in [0,1] \setminus \mathbb{Q}$ follows by taking rational m/n converging to r.

***Solution* 2. LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

Suppose $2 > p \ge 1 \ge q > 0$. Applying Bernoulli's inequality with $1+x = p/q$ and $r = p/2$, we have

$$\left(\frac{p}{q}\right)^{p/2} \le 1 + \frac{p}{2}\left(\frac{p}{q} - 1\right) = \frac{p^2+q^2}{2q}.$$

Multiplying both sides by q and squaring both sides, we have

$$p^p q^q \le (p^2+q^2)^2/4.$$

Similarly, applying Bernoulli's inequality with $1+x = q/p$ and $r = p/2$, we can get $p^p q^q \le p^2q^2$. So

$$\begin{aligned} 3p^q q^p + p^p q^q &\le (p^4 + 14p^2q^2 + q^4)/4 \\ &= (p^4+6p^2q^2+q^4+4pq(2pq))/4 \\ &\le (p^4+6p^2q^2+q^4+4pq(p^2+q^2))/4 \\ &= (p+q)^4/4 = 4. \end{aligned}$$

Commended solvers: **Paolo Perfetti** (Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Olympiad Corner

(continued from page 1)

Problem 3. Prove that there are infinitely many primes p such that $N_p = p^2$, where N_p is the total number of solutions to the equation

$$3x^3+4y^3+5z^3-y^4z \equiv 0 \pmod{p}.$$

Problem 4. Two circles C_1, C_2 with different radii are given in the plane, they touch each other externally at T. Consider any points $A \in C_1$ and $B \in C_2$, both different from T, such that $\angle ATB = 90°$.

(a) Show that all such lines AB are concurrent.

(b) Find the locus of midpoints of all such segments AB.

Double Counting

(continued from page 2)

Example 8. (*2003 IMO Shortlisted Problem*) Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. Let $A = (a_{ij})_{1\le i,j\le n}$ be the matrix with entries

$$a_{ij} = \begin{cases} 1, & \text{if } x_i + y_j \ge 0; \\ 0, & \text{if } x_i + y_j < 0. \end{cases}$$

Suppose that B is an $n{\times}n$ matrix with entries 0 or 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A=B$.

Solution. Let $A = (a_{ij})_{1\le i,j\le n}$. Define

$$S = \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i + y_j)(a_{ij} - b_{ij}).$$

On one hand, we have

$$S = \sum_{i=1}^{n} x_i\left(\sum_{j=1}^{n} a_{ij} - \sum_{j=1}^{n} b_{ij}\right) + \sum_{j=1}^{n} y_j\left(\sum_{i=1}^{n} a_{ij} - \sum_{i=1}^{n} b_{ij}\right) = 0.$$

On the other hand, if $x_i+y_j \ge 0$, then $a_{ij} = 1$, which implies $a_{ij}-b_{ij} \ge 0$; if $x_i+y_j < 0$, then $a_{ij} = 0$, which implies $a_{ij}-b_{ij} \le 0$. Hence, $(x_i+y_j)(a_{ij}-b_{ij}) \ge 0$ for all i,j. Since $S = 0$, all $(x_i+y_j)(a_{ij}-b_{ij}) = 0$.

In particular, if $a_{ij}=0$, then $x_i+y_j < 0$ and so $b_{ij} = 0$. Since a_{ij}, b_{ij} are 0 or 1, so $a_{ij} \ge b_{ij}$ for all i,j. Finally, since the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A, so $a_{ij} = b_{ij}$ for all i,j.