Mathematical Excalibur

Volume 13, Number 5 — *January-February, 2009*

Olympiad Corner

The following were the problems of the Final Round (Part 2) of the Austrian Mathematical Olympiad 2008.

First Day: June 6th, 2008

Problem 1. Prove the inequality

$$\sqrt{a^{1-a}b^{1-b}c^{1-c}} \le \frac{1}{3}$$

holds for all positive real numbers a, b and c with $a+b+c = 1$.

Problem 2. (a) Does there exist a polynomial $P(x)$ with coefficients in integers, such that $P(d) = 2008/d$ holds for all positive divisors of 2008?

(b) For which positive integers n does a polynomial $P(x)$ with coefficients in integers exists, such that $P(d) = n/d$ holds for all positive divisors of n?

Problem 3. We are given a line g with four successive points P, Q, R, S, reading from left to right. Describe a straight-edge and compass construction yielding a square $ABCD$ such that P lies on the line AD, Q on the line BC, R on the line AB and S on the line CD.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 7, 2009***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Generating Functions

Kin Yin Li

In some combinatorial problems, we may be asked to determine a certain sequence of numbers $a_0, a_1, a_2, a_3, \ldots$. We can associate such a sequence with the following series

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

This is called the *generating function* of the sequence. Often the geometric series $1/(1-t) = 1 + t + t^2 + t^3 + \cdots$ for $|t| < 1$ and its square

$$1/(1-t)^2 = (1 + t + t^2 + t^3 + \cdots)^2$$
$$= 1 + 2t + 3t^2 + 4t^3 + 5t^4 + \cdots$$

will be involved in our discussions.

Below we will provide examples to illustrate how generating functions can solve some combinatorial problems.

Example 1. Let $a_0=1$, $a_1=1$ and

$$a_n = 4a_{n-1} - 4a_n \text{ for } n \ge 2.$$

Find a formula for a_n in terms of n.

Solution. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots$. Then we have

$$\begin{aligned} f(x) - 1 - x &= a_2x^2 + a_3x^3 + a_4x^4 + \cdots \\ &= (4a_1 - 4a_0)x^2 + (4a_2 - 4a_1)x^3 + \cdots \\ &= (4a_1x^2 + 4a_2x^3 + \cdots) - (4a_0x^2 + 4a_1x^3 + \cdots) \\ &= 4x(f(x) - 1) - 4x^2 f(x). \end{aligned}$$

Solving for $f(x)$ and taking $|x| < ½$,

$$\begin{aligned} f(x) &= (1-3x)/(1-2x)^2 \\ &= 1/(1-2x) - x/(1-2x)^2 \\ &= \sum_{n=0}^{\infty}(2x)^n - x\sum_{n=1}^{\infty} n(2x)^{n-1} \\ &= \sum_{n=0}^{\infty}(2^n - n2^{n-1})x^n. \end{aligned}$$

Therefore, $a_n = 2^n - n\,2^{n-1}$.

Example 2. Find the number a_n of ways n dollars can be changed into 1 or 2 dollar coins (regardless of order). For example, when $n = 3$, there are 2 ways, namely three 1 dollar coins or one 1 dollar coin and one 2 dollar coin.

Solution. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots$. To study this infinite series, let $|x| < 1$. For each way of changing n dollars into r 1 dollar and s 2 dollar coins, we can record it as $x^r x^{2s} = x^n$. Now r and s may be any nonnegative integers. Adding all the recorded terms for all nonnegative integers n, then factoring, we get

$$\sum_{r=0}^{\infty} x^r \sum_{s=0}^{\infty} x^{2s} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} x^{r+2s} = \sum_{n=0}^{\infty} a_n x^n = f(x).$$

On the other hand,

$$\begin{aligned} \sum_{r=0}^{\infty} x^r \sum_{s=0}^{\infty} x^{2s} &= \frac{1}{1-x}\cdot\frac{1}{1-x^2} = \frac{1}{(1-x)^2(1-x)} \\ &= \frac{1}{2}\left(\frac{1}{(1-x)^2} + \frac{1}{1-x^2}\right) \\ &= \tfrac{1}{2}\left((1+2x+3x^2+\cdots)+(1+x^2+x^4+\cdots)\right) \\ &= 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + \cdots. \\ &= \sum_{n=0}^{\infty}([n/2]+1)x^n. \end{aligned}$$

Therefore, $a_n = [n/2] + 1$.

Example 3. Let n be a positive integer. Find the number a_n of polynomials $P(x)$ with coefficients in $\{0,1,2,3\}$ such that $P(2) = n$.

Solution. Let $f(t)$ be the generating function of the sequence $a_0, a_1, a_2, a_3, \ldots$. Let $P(x) = c_0 + c_1x + \cdots + c_kx^k$ with $c_i \in \{0,1,2,3\}$. Now $P(2) = n$ if and only if $c_0 + 2c_1 + \cdots + 2^kc_k = n$. Taking $t \in (-1,1)$, we can record this as

$$t^n = t^{c_0} t^{2c_1} \cdots t^{2^k c_k}.$$

Note 2^ic_i is one of the four numbers 0, 2^i, 2^{i+1}, $3\cdot 2^i$. Adding all the recorded terms for all nonnegative integers n and all possible $c_0, c_1, \ldots, c_k \in \{0,1,2,3\}$, then factoring on the right, we have

$$f(t) = \sum_{n=0}^{\infty} a_n t^n = \prod_{i=0}^{\infty}(1 + t^{2^i} + t^{2^{i+1}} + t^{3\cdot 2^i}).$$

Using $1+s+s^2+s^3=(1-s^4)/(1-s)$, we see

$$\begin{aligned} f(t) &= \frac{1-t^4}{1-t}\cdot\frac{1-t^8}{1-t^2}\cdot\frac{1-t^{16}}{1-t^4}\cdot\frac{1-t^{32}}{1-t^8}\cdots \\ &= \frac{1}{1-t}\cdot\frac{1}{1-t^2}. \end{aligned}$$

As in example 2, we get $a_n = [n/2] + 1$.

For certain problems, instead of using the generating function of a_0, a_1, a_2, a_3, …, we may consider the series

$$x^{a_0}+x^{a_1}+x^{a_2}+x^{a_3}+\cdots.$$

Example 4. (*1998 IMO Shortlisted Problem*) Let a_0, a_1, a_2, … be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i+2a_j+4a_k$, where i, j and k are not necessarily distinct. Determine a_{1998}.

Solution. For $|x|<1$, let $f(x)=\sum_{i=0}^{\infty}x^{a_i}$.

The given condition implies

$$f(x)f(x^2)f(x^4)=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}.$$

Replacing x by x^2, we get

$$f(x^2)f(x^4)f(x^8)=\frac{1}{1-x^2}.$$

From these two equations, we get $f(x)=(1+x)f(x^8)$. Repeating this recursively, we get

$$f(x)=(1+x)(1+x^8)(1+x^{8^2})(1+x^{8^3})\cdots.$$

In expanding the right side, we see the exponents a_0, a_1, a_2, … are precisely the nonnegative integers whose base 8 representations have only digit 0 or 1. Since $1998=2+2^2+2^3+2^6+2^7+2^8+2^9+2^{10}$, so $a_{1998}=8+8^2+8^3+8^6+8^7+8^8+8^9+8^{10}$.

For our next examples, we need some identities involving p-th roots of unity, where p is a positive integer. These are complex numbers λ, which are all the solutions of the equation $z^p=1$. For a real θ, we will use the common notation $e^{i\theta}=\cos\theta+i\sin\theta$. Since the equation is of degree p, there are exactly p p-th roots of unity. We can easily check that they are $e^{i\theta}$ with $\theta=0$, $2\pi/p$, $4\pi/p$, …, $2(p-1)\pi/p$.

Below let λ be any p-th root of unity, other than 1. When we have a series

$$B(z)=b_0+b_1z+b_2z^2+b_3z^3+\cdots,$$

sometimes we need to find the value of $b_p+b_{2p}+b_{3p}+\cdots$. We can use the fact

$$1+\lambda^j+\lambda^{2j}+\cdots+\lambda^{(p-1)j}=\frac{1-\lambda^{pj}}{1-\lambda^j}=0$$

(for any j not divisible by p) to get

$$\frac{1}{p}\sum_{j=0}^{p-1}B(\lambda^j)=b_p+b_{2p}+b_{3p}+\cdots. \quad (*)$$

For p odd, we have the factorization

$$1+t^p=(1+t)(1+\lambda t)\cdots(1+\lambda^{p-1}t) \quad (**)$$

since both sides have $-1/\lambda^i$ (i=0,1,…,$p-1$) as roots and are monic of degree p.

Example 5. Can the set $\mathbb{N}$ of all positive integers be partitioned into more than one, but still a finite number of arithmetic progressions with no two having the same common differences?

Solution. (*Due to Donald J. Newman*) Assume the set $\mathbb{N}$ can be partitioned into sets $S_1, S_2,\ldots,S_k$, where $S_i=\{a_i+nd_i: n\in\mathbb{N}\}$ with $d_1>d_2>\cdots>d_k$. Then for $|z|<1$,

$$\sum_{n=1}^{\infty}z^n=\sum_{n=1}^{\infty}z^{a_1+nd_1}+\sum_{n=1}^{\infty}z^{a_2+nd_2}+\cdots+\sum_{n=1}^{\infty}z^{a_k+nd_k}.$$

Summing the geometric series, this gives

$$\frac{z}{1-z}=\frac{z^{a_1}}{1-z^{d_1}}+\frac{z^{a_2}}{1-z^{d_2}}+\cdots+\frac{z^{a_k}}{1-z^{d_k}}.$$

Letting z tend to $e^{2\pi i/d_1}$, we see the left side has a finite limit, but the right side goes to infinity. That gives a contradiction.

Example 6. (*1995 IMO*) Let p be an odd prime number. Find the number of subsets A of the set $\{1,2,\ldots,2p\}$ such that

(i) A has exactly p elements, and

(ii) the sum of all the elements in A is divisible by p.

Solution. Consider the polynomial

$$F_a(x)=(1+ax)(1+a^2x)(1+a^3x)\cdots(1+a^{2p}x)$$

When the right side is expanded, let $c_{n,k}$ count the number of terms of the form $(a^{i_1}x)(a^{i_2}x)\cdots(a^{i_k}x)$, where i_1, i_2, …, i_k are integers such that $1\le i_1<i_2<\cdots<i_k\le 2p$ and $i_1+i_2+\cdots+i_k=n$. Then

$$F_a(x)=1+\sum_{k=1}^{2p}\left(\sum_{n=1}^{\infty}c_{n,k}a^n\right)x^k.$$

Now, in terms of $c_{n,k}$, the answer to the problem is $C=c_{p,p}+c_{2p,p}+c_{3p,p}+\cdots$.

To get C, note the coefficient of x^p in $F_a(x)$ is $\sum_{n=1}^{\infty}c_{n,p}a^n$. By (*) above, we see

$$C=\frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty}c_{n,p}\omega^{nj}.$$

Now the right side is the coefficient of x^p in $\frac{1}{p}\sum_{j=0}^{p-1}F_{\omega^j}(x)$, which equals

$$\frac{1}{p}\sum_{j=0}^{p-1}(1+\omega^jx)(1+\omega^{2j}x)\cdots(1+\omega^{2pj}x).$$

For $j=0$, the term is $(1+x)^{2p}$. For $1\le j\le p-1$, using (**) with $\lambda=\omega^j$ and $t=\lambda x$, we see the j-th term is $(1+x^p)^2$. Using these, we have

$$\frac{1}{p}\sum_{j=0}^{p-1}F_{\omega^j}(x)=\frac{1}{p}[(1+x)^{2p}+(p-1)(1+x^p)^2].$$

Therefore, the coefficient of x^p is

$$C=\frac{1}{p}\left[\binom{2p}{p}+2(p-1)\right].$$

So far all generating functions were in one variable. For the curious mind, next we will look at an example involving a two variable generating function

$$f(x,y)=\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}a_{i,j}x^iy^j$$

of the simplest kind.

Example 7. An $a\times b$ rectangle can be tiled by a number of $p\times1$ and $1\times q$ types of rectangles, where a, b, p, q are fixed positive integers. Prove that a is divisible by p or b is divisible by q. (Here a $k\times1$ and a $1\times k$ rectangles are considered to be different types.)

Solution. Inside the (i,j) cell of the $a\times b$ rectangle, let us put the term x^iy^j for $i=1,2,\ldots,a$ and $j=1,2,\ldots,b$. The sum of the terms inside a $p\times1$ rectangle is

$$x^iy^j+\cdots+x^{i+p-1}y^j=(1+x+\cdots+x^{p-1})\,x^iy^j,$$

if the top cell is at (i,j), while the sum of the terms inside a $1\times q$ rectangle is

$$x^iy^j+\cdots+x^iy^{j+q-1}=x^iy^j\,(1+y+\cdots+y^{q-1}),$$

if the leftmost cell is at (i,j). Now take

$$x=e^{2\pi i/p} \text{ and } y=e^{2\pi i/q}.$$

Then both sums become 0. If the desired tiling is possible, then the total sum of all terms in the $a\times b$ rectangle would be

$$0=\sum_{i=1}^{a}\sum_{j=1}^{b}x^iy^j=xy\frac{(1-x^a)(1-y^b)}{(1-x)(1-y)}.$$

This implies that a is divisible by p or b is divisible by q.

For the readers who like to know more about generating functions, we recommend two excellent references:

T. Andreescu and Z. Feng, *A Path to Combinatorics for Undergraduates*, Birkhäuser, Boston, 2004.

M. Novaković, *Generating Functions*, The IMO Compendium Group, 2007 (www.imomath.com)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***March 7, 2009.***

Problem 316. For every positive integer $n > 6$, prove that in every n-sided convex polygon $A_1A_2\ldots A_n$, there exist $i \neq j$ such that

$$|\cos\angle A_i - \cos\angle A_j| < \frac{1}{2(n-6)}.$$

Problem 317. Find all polynomial $P(x)$ with integer coefficients such that for every positive integer n, 2^n-1 is divisible by $P(n)$.

Problem 318. In ΔABC, side BC has length equal to the average of the two other sides. Draw a circle passing through A and the midpoints of AB, AC. Draw the tangent lines from the centroid of the triangle to the circle. Prove that one of the points of tangency is the incenter of ΔABC.
(*Source: 2000 Chinese Team Training Test*)

Problem 319. For a positive integer n, let S be the set of all integers m such that $|m| < 2n$. Prove that whenever $2n+1$ elements are chosen from S, there exist three of them whose sum is 0.
(*Source: 1990 Chinese Team Training Test*)

Problem 320. For every positive integer $k > 1$, prove that there exists a positive integer m such that among the rightmost k digits of 2^m in base 10, at least half of them are 9's.
(*Source: 2005 Chinese Team Training Test*)

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Solutions

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Problem 311. Let $S = \{1,2,\ldots,2008\}$. Prove that there exists a function $f : S \to$ {red, white, blue, green} such that there does not exist a 10-term arithmetic progression $a_1,a_2,\ldots,a_{10}$ in S satisfying $f(a_1) = f(a_2) = \cdots = f(a_{10})$.

Solution 1. **Kipp JOHNSON** (Valley Catholic School, teacher, Beaverton, Oregon, USA) and **PUN Ying Anna** (HKU Math, Year 3).

The number of 10-term arithmetic progressions in S is the same as the number of ordered pairs (a,d) such that a, d are in S and $a+9d \le 2008$. Since $d \le 2007/9=223$ and for each such d, a goes from 1 to $2008-9d$, so there are at most

$$4^{(2008-10)} \times 4 \times \sum_{d=1}^{223}(2008-9d)$$

$$= 4^{1999} \times 223000$$

functions $f : S \to$ {red, white, blue, green} such that there exists a 10-term arithmetic progression $a_1,a_2,\ldots,a_{10}$ in S satisfying $f(a_1) = f(a_2) = \cdots = f(a_{10})$, while there are more (namely 4^{2008}) functions from S to {red, white, blue, green}. So the desired function exists.

Solution 2. **G.R.A. 20 Problem Solving Group** (Roma, Italy).

Replace red, white, blue, green by 0, 1, 2, 3 respectively. It can be seen by a direct checking that f: $\{1,2,\ldots,2048\} \to \{0,1,2,3\}$ given by

$$f(n) = \left[\frac{n-1}{8}\right]_{\text{mod } 2} + 2\left[\frac{n-1}{128}\right]_{\text{mod } 2}$$

avoids any 9-term arithmetic progression having the same value (where $k_{\text{mod } 2}$ is 0 if k is even and 1 if k is odd). The range of f is $((0^8 1^8)^8 (2^8 3^8)^8)^8$, where for any string x, x^8 denotes the string obtained by putting eight copies of the string x one after another in a row and $f(n)$ is the n-th digit in the specified string.

Commended solvers: **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

Problem 312. Let $x, y, z > 1$. Prove that

$$\frac{x^4}{(y-1)^2} + \frac{y^4}{(z-1)^2} + \frac{z^4}{(x-1)^2} \ge 48.$$

Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **Kipp JOHNSON** (Valley Catholic School, teacher, Beaverton, Oregon, USA), **Kelvin LEE** (Trinity College, University of Cambridge, Year 2), **LEUNG Kai Chung** (HKUST Math, Year 2), **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery)**, MA Ka Hei** (Wah Yan College, Kowloon), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and **PUN Ying Anna** (HKU Math, Year 3).

Let $x = a + 1$, $y = b + 1$ and $z = c + 1$. Applying the *AM-GM* inequality twice, we have

$$\frac{x^4}{(y-1)^2} + \frac{y^4}{(z-1)^2} + \frac{z^4}{(x-1)^2}$$

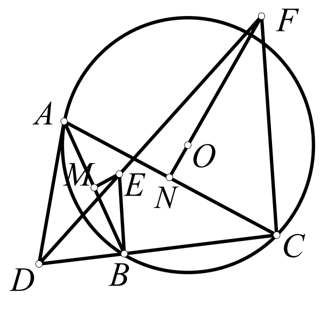
$$= \frac{(a+1)^4}{b^2} + \frac{(b+1)^4}{c^2} + \frac{(c+1)^4}{a^2}$$

$$\ge 3\left(\frac{(a+1)^4(b+1)^4(c+1)^4}{a^2b^2c^2}\right)^{1/3}$$

$$\ge 3\left(\frac{(2\sqrt{a})^4(2\sqrt{b})^4(2\sqrt{c})^4}{a^2b^2c^2}\right)^{1/3} = 48.$$

Commended solvers: **CHUNG Ping Ngai** (La Salle College, Form 5), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **NG Ngai Fung** (STFA Leung Kau Kui College, Form 6), **Paolo PERFETTI** (Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Dimitar TRENEVSKI** (Yahya Kemal College, Skopje, Macedonia) and **TSOI Kwok Wing** (PLK Centenary Li Shiu Chung Memorial College, Form 6).

Problem 313. In ΔABC, $AB < AC$ and O is its circumcenter. Let the tangent at A to the circumcircle cut line BC at D. Let the perpendicular lines to line BC at B and C cut the perpendicular bisectors of sides AB and AC at E and F respectively. Prove that D, E, F are collinear.



Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **CHUNG Ping Ngai** (La Salle College, Form 5), **Kelvin LEE** (Trinity College, University of Cambridge, Year 2), **NG Ngai Fung** (STFA Leung Kau Kui College, Form 6) and **PUN Ying Anna** (HKU Math, Year 3).

Let M be the midpoint of AB and N be the midpoint of AC. Using $\angle ABE = \angle ABC - 90°$, $\angle FCA = 90° - \angle ABC$ and the sine law, we have

$$\frac{BE}{CF}=\frac{BM/\cos\angle ABE}{CN/\cos\angle FCA}$$

$$=\frac{\frac{1}{2}AB/\sin\angle ABC}{\frac{1}{2}AC/\sin\angle BCA}=\frac{AB^2}{AC^2}.$$

From $\Delta DCA\sim\Delta DAB$, we see

$$\frac{DA}{DC}=\frac{DB}{DA}=\frac{\sin\angle DAB}{\sin\angle DBA}=\frac{\sin\angle ACB}{\sin\angle ABC}=\frac{AB}{AC}.$$

So

$$\frac{BE}{CF}=\frac{AB^2}{AC^2}=\frac{DA}{DC}\cdot\frac{DB}{DA}=\frac{DB}{DC}.$$

Then $\angle BDE=\angle CDF$. Therefore D,E,F are collinear.

Commended solvers: **Stefan LOZANOVSKI and Bojan JOVESKI** (Private Yahya Kemal College, Skopje, Macedonia).

Problem 314. Determine all positive integers x, y, z satisfying $x^3-y^3=z^2$, where y is a prime, z is not divisible by 3 and z is not divisible by y.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5) and **PUN Ying Anna** (HKU Math, Year 3).

Suppose there is such a solution. Then

$$z^2=x^3-y^3=(x-y)(x^2+xy+y^2)$$
$$=(x-y)((x-y)^2+3xy). \qquad (*)$$

Since y is a prime, z is not divisible by 3 and z is not divisible by y, (*) implies $(x,y)=1$ and $(x-y,3)=1$. Then

$$(x^2+xy+y^2,x-y)=(3xy,x-y)=1. \qquad (**)$$

Now (*) and (**) imply

$$x-y=m^2,\ x^2+xy+y^2=n^2 \text{ and } z=mn$$

for some positive integers m and n. Consequently,

$$4n^2=4x^2+4xy+4y^2=(2x+y)^2+3y^2.$$

Then $3y^2=(2n+2x+y)(2n-2x-y)$. Since y is prime, there are 3 possibilities:

(1) $2n+2x+y=3y^2$, $2n-2x-y=1$
(2) $2n+2x+y=3y$, $2n-2x-y=y$
(3) $2n+2x+y=y^2$, $2n-2x-y=3$.

In (1), subtracting the equations leads to $3y^2-1=2(2x+y)=2(2m^2+3y)$. Then

$$m^2+1=3y^2-6y-3m^2\equiv 0 \pmod 3).$$

However, $m^2+1\equiv 1$ or $2 \pmod 3$. We get a contradiction.

In (2), subtracting the equations leads to $x=0$, contradiction.

In (3), subtracting the equations leads to $y^2-3=2(2x+y)=2(2m^2+3y)$, which can be rearranged as $(y-3)^2-4m^2=12$. This leads to $y=7$ and $m=1$. Then $x=8$ and $z=13$. Since $8^3-7^3=13^2$, this gives the only solution.

Commended solvers: **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

Problem 315. Each face of 8 unit cubes is painted white or black. Let n be the total number of black faces. Determine the values of n such that in every way of coloring n faces of the 8 unit cubes black, there always exists a way of stacking the 8 unit cubes into a $2\times2\times2$ cube C so the numbers of black squares and white squares on the surface of C are the same.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5) and **PUN Ying Anna** (HKU Math, Year 3).

The answer is $n=23$ or 24 or 25. First notice that if n is a possible value, then so is $48-n$. This is because we can interchange all the black and white coloring and the condition can still be met by symmetry. Hence, without loss of generality, we may assume $n\le 24$.

For the 8 unit cubes, there are totally 24 pairs of opposite faces. In each pair, no matter how the cubes are stacked, there is one face on the surface of C and one face hidden.

If $n\le 22$, there is a coloring that has $[n/2]$ pairs with both opposite faces black. Then at least $[n/2]$ black faces will be hidden so that there can be at most $n-[n/2]\le 11$ black faces on the surface of C. This contradicts the existence of a stacking with 12 black and 12 white squares on the surface of C. So only $n=23$ or 24 is possible.

Next, start with an arbitrary stacking. Let b be the number of black squares on the surface of C. For each of the 8 unit cubes, take an axis formed by the centers of a pair of opposite faces and rotate the cube about that axis by 90˚. Then take an axis formed by the centers of another pair of opposite faces of the same cube and rotate the cube about the axis by 90˚ twice. These three 90˚ rotations switch the three exposed faces with the three hidden faces. So after doing the twenty-four 90˚ rotations for the 8 unit cubes, there will be $n-b$ black squares on the surface of C.

For $n=23$ or 24 and $b\le n$, the average of b and $n-b$ is 11.5 or 12, hence 12 is between b and $n-b$ inclusive.

Finally, observe that after each of the twenty-four 90˚ rotations, one exposed square will be hidden and one hidden square will be exposed. So the number of black squares on the surface of C can only increase by one, stay the same or decrease by one.

Therefore, at a certain moment, there will be exactly 12 black squares (and 12 white squares) on the surface of C.

Commended solvers: **G.R.A. 20 Problem Solving Group** (Roma, Italy) and **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

Olympiad Corner

(continued from page 1)

Second Day: June 7th, 2008

Problem 4. Determine all functions f mapping the set of positive integers to the set of non-negative integers satisfying the following conditions:

(1) $f(mn)=f(m)+f(n)$,
(2) $f(2008)=0$, and
(3) $f(n)=0$ for all $n\equiv 39 \pmod{2008}$.

Problem 5. Which positive integers are missing in the sequence $\{a_n\}$, with

$$a_n=n+[\sqrt{n}]+[\sqrt[3]{n}]$$

for all $n\ge 1$? ($[x]$ denotes the largest integer less than or equal to x, i.e. g with $g\le x<g+1$.)

Problem 6. We are given a square $ABCD$. Let P be a point not equal to a corner of the square or to its center M. For any such P, we let E denote the common point of the lines PD and AC, if such a point exists. Furthermore, we let F denote the common point of the lines PC and BD, if such a point exists.

All such points P, for which E and F exist are called acceptable points. Determine the set of all acceptable points, for which the line EF is parallel to AD.