Mathematical Excalibur

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Olympiad Corner

The following were the problems of the first day of the 2008 Chinese Girls' Math Olympiad.

Problem 1. (a) Determine if the set $\{1,2,\cdots,96\}$ can be partitioned into 32 sets of equal size and equal sum.

(b) Determine if the set $\{1,2,\cdots,99\}$ can be partitioned into 33 sets of equal size and equal sum.

Problem 2. Let $\varphi(x) = ax^3+bx^2+cx+d$ be a polynomial with real coefficients. Given that $\varphi(x)$ has three positive real roots and that $\varphi(x) < 0$, prove that $2b^3 + 9a^2d - 7abc \le 0$.

Problem 3. Determine the least real number a greater than 1 such that for any point P in the interior of square $ABCD$, the area ratio between some two of the triangles PAB, PBC, PCD, PDA lies in the interval $[1/a, a]$.

Problem 4. Equilateral triangles ABQ, BCR, CDS, DAP are erected outside the (convex) quadrilateral $ABCD$. Let X, Y, Z, W be the midpoints of the segments PQ, QR, RS, SP respectively. Determine the maximum value of

$$\frac{XZ+YW}{AC+BD}.$$

Acknowledgment: Thanks to Elina Chiu, Dept. of Math., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 3, 2009***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Remarks on IMO 2009

Leung Tat-Wing

2009 IMO Hong Kong Team Leader

The 50th International Mathematical Olympiad (IMO) was held in Bremen, Germany from 10th to 22nd July 2009. I arrived Bremen amid stormy and chilly (16°*C*) weather. Our other team members arrived three days later. The team eventually obtained 1 gold, 2 silver and 2 bronze medals, ranked (unofficially) 29 out of 104 countries/regions. This was the first time more than 100 countries participated. Our team, though not among the strongest teams, did reasonably well. But here I mainly want to give some remarks about this year's IMO, before I forget.

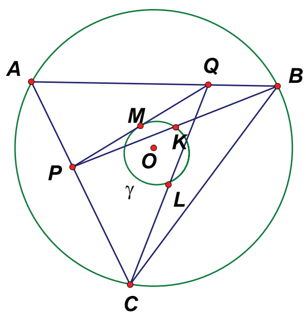
First, the problems of the contest:

Problem 1. Let n be a positive integer and let $a_1, a_2, \ldots, a_k$ $(k \ge 2)$ be distinct integers in the set $\{1,2,\ldots,n\}$ such that n divides $a_i(a_{i+1}-1)$ for $i=1,\ldots,k-1$. Prove that n does not divide $a_k(a_1-1)$.

This nice and easy number theory problem was the only number theory problem in the contest. Indeed it is not easy to find a sequence satisfying the required conditions, especially when k is close to n, or n is prime. Since adding the condition n divides $a_k(a_1-1)$ should be impossible, it was natural to prove the statement by contradiction. Clearly $2 \le k \le n$, and we have $a_1 \equiv a_1a_2 \pmod n$, $a_2 \equiv a_2a_3 \pmod n$, …, $a_{k-1} \equiv a_{k-1}a_k$ (mod n). The extra condition $a_k \equiv a_ka_1$ (mod n) would in fact "complete the circle". Now $a_1 \equiv a_1a_2$ (mod n). Using the second condition, we get $a_1 \equiv a_1a_2 \equiv a_1a_2a_3 \pmod n$ and so on, until we get $a_1 \equiv a_1a_2\cdots a_k \pmod n$. However, in a circle every point is a starting point. So starting from a_2, using the second condition we have $a_2 \equiv a_2a_3 \pmod n$. By the third condition, we then have $a_2 \equiv a_2a_3a_4$ (mod n). As now the circle is complete, we eventually have $a_2 \equiv a_2a_3\cdots a_ka_1$ (mod n). Arguing in this manner we eventually have $a_1 \equiv a_2 \equiv \cdots \equiv a_k$ (mod n), which is of course a contradiction!

Problem 2. Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that PQ is tangent to the circle Γ. Prove that OP=OQ.

The nice geometry problem was supposed to be a medium problem, but it turned out it was easier than what the jury had thought. The trick was to understand the relations involved. A very nice solution provided by one of our members went as follows.



As $KM \| BQ$ (midpoint theorem), we have $\angle AQP = \angle QMK$. Since PQ is tangent to Γ, we have $\angle QMK = \angle MLK$ (angle of alternate segment). Therefore, $\angle AQP = \angle MLK$. By the same argument, we have $\angle APQ = \angle MKL$. Hence, $\Delta APQ \sim \Delta MKL$. Therefore,

$$\frac{AP}{AQ} = \frac{MK}{ML} = \frac{2MK}{2ML} = \frac{BQ}{CP}.$$

This implies $AP \cdot PC = AQ \cdot QB$. But by considering the power of P with respect to the circle ABC, we have

$$AP \cdot PC = (R+OP)(R-OP) = R^2 - OP^2,$$

where R is the radius of the circumcircle of ΔABC.

Likewise,

$$AQ \cdot QB = (R+OQ)(R-OQ) = R^2 - OQ^2.$$

These force $OP^2 = OQ^2$, or $OP = OQ$, done!

Problem 3. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \text{ and } s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

This was one of the two hard problems (3 and 6). Fortunately, it turned out that it was still within reach.

One trouble is of course the notation. Of course, s_{s_1} stands for the s_1^{th} term of the s_i sequence and so on. Starting from an arithmetic progression (AP) with common difference d, then it is easy to check that both

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \text{ and } s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are APs with common difference d^2. The question is essentially proving the "converse". So the first step is to prove that the common differences of the two APs s_{s_i} and s_{s_i+1} are in fact the same, say s. It is not too hard to prove and is intuitively clear, for two lines of different slopes will eventually meet and cross each other, violating the condition of strictly increasing sequence. The next step is the show the difference between two consecutive terms of s_i is indeed $\sqrt{s}$, (thus s is a square). One can achieve this end by the method of descent, or max/min principle, etc.

Problem 4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

This problem was also relatively easy. It is interesting to observe that an isosceles triangle can be the starting point of an IMO problem. With geometric software such as *Sketchpad*, one can easily see that $\angle CAB$ should be 60° or 90°. To prove the statement of the problem, one may either use synthetic method or coordinate method. One advantage of using the coordinate method is after showing the possible values of $\angle CAB$, one can go back to show these values do work by suitable substitutions. Some contestants lost marks either because they missed some values of $\angle CAB$ or forgot to check the two possible cases do work.

Problem 5. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths a, $f(b)$ and $f(b+f(a)-1)$. (A triangle is *non-degenerate* if its vertices are not collinear.)

The Jury worried if the word <u>triangle</u> may be allowed to be degenerate in some places. But I supposed all our secondary school students would consider only non-degenerate triangles. This was a nice problem in functional inequality (triangle inequality). One proves the problem by establishing several basic properties of f. Indeed the first step is to prove $f(1)=1$, which is not entirely easy. Then one proceeds to show that f is injective and/or $f(f(x)) = x$, etc, and finally shows that the only possible function is the identity function $f(x) = x$ for all x.

Problem 6. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s=a_1+a_2+\cdots+a_n$. A grasshopper is to jump along the real axis, starting from the point O and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

It turned out that this problem was one of the most difficult problems in IMO history. Only three of the 564 contestants received full scores. (Perhaps it was second to problem 3 posed in IMO 2007, for which only 2 contestants received full scores.)

When I first read the solution provided by the Problem Committee, I felt I was reading a paper of analysis. Without reading the solution, of course I would say we could try to prove the problem by induction, as the cases of small n were easy. The trouble was how to establish the induction step. Later the Russians provided a solution by induction, by separating the problem into sub-cases min $M < a_n$ or min $M \geq a_n$, and then applying the principle-hole principle, etc judiciously to solve the problem. Terry Tao said (jokingly) that the six problems were easy. But in his blog, he admitted that he had spent sometime reading the problem and he even wrote an article about it (I have not seen the article.)

The two hard problems (3 and 6) were more combinatorial and/or algebraic in nature. I had a feeling that this year the Jury has been trying to avoid hard number theory problems, which were essentially corollaries of deep theorems (for example, IMO 2003 problem 6 by the Chebotarev density theorem or IMO 2008 problem 3 by a theorem of H. Iwaniec) or hard geometry problem using sophisticated geometric techniques (like IMO 2008 problem 6).

The Germans ran the program vigorously (obstinately). They had an organization (Bildung und Begabung) that looked after the entire event. They had also prepared a very detailed shortlist problem set and afterwards prepared very detailed marking schemes for each problem. The coordinators were very professional and they studied the problems well. Thus, there were not too many arguments about how many points should be awarded for each problem.

Three of the problems (namely 1, 2 and 4) were relatively easy, problems 3 and 5 were not too hard, so although problem 6 was hard, contestants still scored relatively high points. This explained why the cut-off scores were not low, 14 for bronze, 24 for silver and 32 for gold.

It might seem that we still didn't do the hard problems too well. But after I discussed with my team members, I found that they indeed had the potential and aptitude to do the hard problems. What may still be lacking are perhaps more sophisticated skills and/or stronger will to tackle such problems.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***October 3, 2009.***

Problem 326. Prove that $3^{4^5}+4^{5^6}$ is the product of two integers, each at least 10^{2009}.

Problem 327. Eight pieces are placed on a chessboard so that each row and each column contains exactly one piece. Prove that there is an even number of pieces on the black squares of the board.
(*Source: 1989 USSR Math Olympiad*)

Problem 328. *(Due to Tuan Le, Fairmont High School, Anaheim, Ca., USA)* Let $a,b,c > 0$. Prove that

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2}+\frac{\sqrt{b^3+c^3}}{b^2+c^2}+\frac{\sqrt{c^3+a^3}}{c^2+a^2}$$
$$\geq \frac{6(ab+bc+ca)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}.$$

Problem 329. Let $C(n,k)$ denote the binomial coefficient with value $n!/(k!(n-k)!)$. Determine all positive integers n such that for all $k = 1, 2, \cdots, n-1$, we have $C(2n,2k)$ is divisible by $C(n,k)$.

Problem 330. In ΔABC, $AB = AC = 1$ and $\angle BAC = 90°$. Let D be the midpoint of side BC. Let E be a point inside segment CD and F be a point inside segment BD. Let M be the point of intersection of the circumcircles of ΔADE and ΔABF, other than A. Let N be the point of intersection of the circumcircle of ΔACE and line AF, other than A. Let P be the point of intersection of the circumcircle of ΔAMN and line AD, other than A. Determine the length of segment AP with proof.
(*Source: 2003 Chinese IMO team test*)

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Solutions

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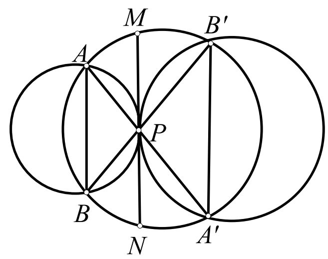
Problem 321. Let *AA', BB'* and *CC'* be three non-coplanar chords of a sphere and let them all pass through a common point P inside the sphere. There is a (unique) sphere S_1 passing through *A, B, C, P* and a (unique) sphere S_2 passing through *A', B', C', P*.

If S_1 and S_2 are externally tangent at P, then prove that *AA'=BB'=CC'*.

Solution. **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and **Jim Robert STUDMAN** (Hanford, Washington, USA).

Consider the intersection of the 3 spheres with the plane through *A, A', B, B'* and *P*.



Let *MN* be the common external tangent through *P* to the circle through *A, B, P* and the circle through *A', B'P* as shown above. We have $\angle ABP = \angle APM = \angle A'PN = \angle A'B'P = \angle A'B'B = \angle BAA' = \angle BAP$. Hence, *AP=BP*. Similarly, *A'P = B'P*. So *AA' = AP+A'P = BP+B'P = BB'*. Similarly, *BB' = CC'*.

Other commended solvers: **CHUNG Ping Ngai** (La Salle College, Form 6) and **LAM Cho Ho** (CUHK Math Year 1).

Problem 322. (*Due to Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam*) Let *a, b, c* be positive real numbers satisfying the condition $a+b+c = 3$. Prove that

$$\frac{a^2(b+1)}{a+b+ab}+\frac{b^2(c+1)}{b+c+bc}+\frac{c^2(a+1)}{c+a+ca}\geq 2.$$

Solution. **CHUNG Ping Ngai** (La Salle College, Form 6), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and the proposer independently**.**

Observe that

$$\frac{a^2(b+1)}{a+b+ab} = a-\frac{ab}{a+b+ab}. \qquad (*)$$

Applying the AM-GM inequality twice, we have

$$\frac{ab}{a+b+ab}\leq\frac{ab}{3\sqrt[3]{a^2b^2}}=\frac{\sqrt[3]{ab}}{3}\leq\frac{a+b+1}{9}.$$

By (*), we have

$$\frac{a^2(b+1)}{a+b+ab}\geq a-\frac{a+b+1}{9}=\frac{8a-b-1}{9}.$$

Adding two other similar inequalities and using $a+b+c = 3$ on the right, we get the desired inequality.

Other commended solvers: **LAM Cho Ho** (CUHK Math Year 1), **Manh Dung NGUYEN** (Special High School for Gifted Students, HUS, Vietnam), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Stefan STOJCHEVSKI** (Yahya Kemal College, Skopje, Macedonia), **Jim Robert STUDMAN** (Hanford, Washington, USA) and **Dimitar TRENEVSKI** (Yahya Kemal College, Skopje, Macedonia).

Problem 323. Prove that there are infinitely many positive integers n such that 2^n+2 is divisible by n.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 6), **LAM Cho Ho** (CUHK Math Year 1) and **WONG Ka Fai** (Wah Yan College Kowloon, Form 4).

We will prove the stronger statement that there are infinitely many positive even integers n such that 2^n+2 is divisible by n and also that 2^n+1 is divisible by $n-1$. Call such n a good number. Note $n = 2$ is good. Next, it suffices to prove that if n is good, then the larger integer $m = 2^n+2$ is also good.

Suppose n is good. Since n is even and $m = 2^n+2$ is twice an odd integer, so $m = nj$ for some odd integer j. Also, the odd integer $m-1 = 2^n+1 = (n-1)k$ for some odd integer k. Using the factorization $a^i+1 = (a+1)(a^{i-1}-a^{i-2}+\cdots+1)$ for positive odd integer i, we see that

$$\begin{aligned}2^m+2 &= 2(2^{(n-1)k}+1)\\ &= 2(2^{n-1}+1)\,(2^{(n-1)(k-1)}-\cdots+1)\end{aligned}$$

is divisible by $2(2^{n-1}+1) = m$ and

$$2^m+1 = 2^{nj}+1 = (2^n+1)(2^{n(j-1)}-\cdots+1)$$

is divisible by $2^n+1= m-1$. Therefore, m is also good.

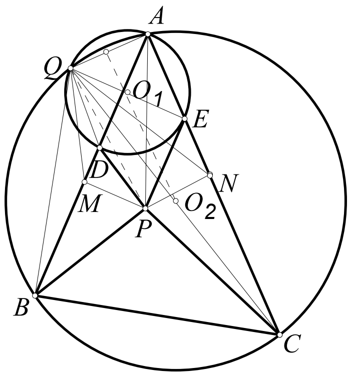
Problem 324. *ADPE* is a convex quadrilateral such that $\angle ADP = \angle AEP$. Extend side *AD* beyond *D* to a point *B* and extend side *AE* beyond *E* to a point *C* so that $\angle DPB = \angle EPC$. Let O_1 be the circumcenter of *ΔADE* and let O_2 be the circumcenter of *ΔABC*.

If the circumcircles of *ΔADE* and *ΔABC* are not tangent to each other, then prove that line O_1O_2 bisects line segment AP.

Solution. **Jim Robert STUDMAN** (Hanford, Washington, USA).

Let the circumcircle of ΔADE and the circumcircle of ΔABC intersect at A and Q.

Observe that line O_1O_2 bisects chord AQ and $O_1O_2 \perp AQ$. Hence, line O_1O_2 bisects line segment AP will follow if we can show that $O_1O_2 \parallel PQ$, or equivalently that $PQ \perp AQ$.



Let points M and N be the feet of perpendiculars from P to lines AB and AC respectively. Since $\angle ANP = 90° = \angle AMP$, points A, N, P, M lie on a circle Γ with AP as diameter. We claim that $\angle MQN = \angle MAN$. This would imply Q is also on circle Γ, and we would have $PQ \perp AQ$ as desired.

Since we are given $\angle ADP = \angle AEP$, we get $\angle BDP = \angle CEP$. This combines with the given fact $\angle DPB = \angle EPC$ imply ΔDPB and ΔEPC are similar, which yields $DB/EC = DP/EP = DM/EN$.

Since A,E,D,Q are concyclic, we have

$$\begin{aligned}\angle BDQ &= 180° - \angle ADQ \\ &= 180° - \angle AEQ = \angle CEQ.\end{aligned}$$

This and $\angle DBQ = \angle ABQ = \angle ACQ = \angle ECQ$ imply ΔDQB and ΔEQC are similar. So we have $QD/QE = DB/EC$. Combining with the equation at the end of the last paragraph, we get

$$QD/QE = DM/EN.$$

Using ΔDQB and ΔEQC are similar, we get $\angle MDQ = \angle BDQ = \angle CEQ = \angle NEQ$. These imply ΔMDQ and ΔNEQ are similar. Then $\angle MQD = \angle NQE$.

Finally, for the claim, we now have

$$\begin{aligned}\angle MQN &= \angle MQD + \angle DQN \\ &= \angle NQE + \angle DQN \\ &= \angle DQE \\ &= \angle DAE \\ &= \angle MAN.\end{aligned}$$

Comments: Some solvers used a bit of homothety to simplify the proof.

Other commended solvers: **CHUNG Ping Ngai** (La Salle College, Form 6), **LAM Cho Ho** (CUHK Math Year 1), **NG Ngai Fung** (STFA Leung Kau Kui College, Form 7).

Problem 325. On a plane, n distinct lines are drawn. A point on the plane is called a *k-point* if and only if there are exactly k of the n lines passing through the point. Let $k_2, k_3, \ldots, k_n$ be the numbers of 2-points, 3-points, …, n-points on the plane, respectively.

Determine the number of regions the n lines divided the plane into in terms of n, $k_2, k_3, \ldots, k_n$.
(*Source: 1998 Jiangsu Province Math Competition*)

Solution. **LAM Cho Ho** (CUHK Math Year 1).

Take a circle of radius r so that all intersection points of the n lines are inside the circle and none of the n lines is tangent to the circle. Now each line intersects the circle at two points. These $2n$ points on the circle are the vertices of a convex $2n$-gon (call it M) as we go around the circle, say clockwise. Let the n lines partition the interior of M into P_3 triangles, P_4 quadrilaterals, ⋯, P_j j-gons, ⋯. These polygonal regions are all convex since the angles of these regions, which were formed by intersecting at least two lines, are all less than 180°. By convexity, no two sides of any polygonal region are parts of the same line. So we have $P_j = 0$ for $j > 3n$.

Consider the sum of all the angles of these regions partitioning M. On one hand, it is $180°(P_3+2P_4+3P_5+\cdots)$ by counting region by region. On the other hand, it also equals $360°(k_2+k_3+\cdots+k_n)+(2n-2)180°$ by counting all the angles around each vertices of the regions. Cancelling 180°, we get

$$P_3+2P_4+3P_5+\cdots = 2(k_2+k_3+\cdots+k_n)+(2n-2).$$

Next, consider the total number of all the edges of these regions partitioned M (with each of the edges inside M counted twice). On one hand, it is $3P_3+4P_4+5P_5+\cdots$ by counting region by region. On the other hand, it is also $(4k_2+6k_3+\cdots 2nk_n)+4n$ by counting the number of edges around the k-points and around the vertices of M. The $4n$ term is due to the $2n$ edges of M and each vertex of M (being not a k-point) issues exactly one edge into the interior of M. So we have

$$3P_3+4P_4+5P_5+\cdots = 4k_2+6k_3+\cdots 2nk_n+4n.$$

Subtracting the last two displayed equations, we can obtain

$$P_3+P_4+P_5+\cdots = k_2+2k_3+(n-1)k_n+n+1.$$

Finally, the number of regions these n lines divided the plane into is the limit case r tends to infinity. Hence, it is exactly $k_2+2k_3+\cdots+(n-1)k_n+n+1$.

Other commended solvers: **CHUNG Ping Ngai** (La Salle College, Form 6) and **YUNG Fai**.

Remarks on IMO 2009

(continued from page 2)

As I found out from the stronger teams (Chinese, Japanese, Korean, or Thai, etc.), they were obviously more heavily or vigorously trained. For instance, a Thai boy/girl had to go through more like 10 tests to be selected as a team member.

Another thing I learned from the meeting was several countries were interested to host the event (South-East Asia countries and Asia-Minor countries). In fact, one country is going to host three international competitions of various subjects in a row for three years. Apparently they think hosting these events is good for gifted education.

The first IMO was held in Romania in 1959. Throughout these 51 years, only one year IMO was not held (1980). To commemorate the fiftieth anniversary of IMO in 2009, six notable mathematicians related to IMO (B. Bollabas, T. Gowers, L. Lovasz, S. Smirnov, T. Tao and J. C. Yoccoz) were invited to talk to the contestants. Of course, Yoccoz, Gowers and Tao were Fields medalists. The afternoon of celebration then became a series of (rather) heavy lectures (not bad). They described the effects of IMOs on them and other things. The effect of IMO on the contestants is to be seen later, of course!