Mathematical Excalibur

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Olympiad Corner

Here are the Asia Pacific Math Olympiad problems on March 2010.

Problem 1. Let ABC be a triangle with $\angle BAC \neq 90°$. Let O be the circumcenter of triangle ABC and let Γ be the circumcircle of triangle BOC. Suppose that Γ intersects the line segment AB at P different from B, and the line segment AC at Q different from C. Let ON be a diameter of the circle Γ. Prove that the quadrilateral $APNQ$ is a parallelogram.

Problem 2. For a positive integer k, call an integer a *pure k-th power* if it can be represented as m^k for some integer m. Show that for every positive integer n there exist n distinct positive integers such that their sum is a pure 2009-th power, and their product is a pure 2010-th power.

Problem 3. Let n be a positive integer. n people take part in a certain party. For any pair of the participants, either the two are acquainted with each other or they are not. What is the maximum possible number of the pairs for which the two are not acquainted but have a common acquaintance among the participants?

(continued on page 4)

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On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 21, 2010***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

Ramsey Numbers and Generalizations

Law Ka Ho

The following problem is classical: among any 6 people, there exist 3 who know each other or 3 who don't know each other (we assume if A knows B, then B knows A). When 6 is replaced by 5, this is no longer true, as can be seen by constructing a counterexample. We write $R(3,3)=6$ and this is called a *Ramsey number*. In general, $R(m,n)$ denotes the smallest positive integer k such that, among any k people, there exist m who know each other or n who don't know each other.

How do we know that $R(m,n)$ exists for all m, n? A key result is the following.

Theorem 1. For any m, $n > 1$, we have $R(m,n) \le R(m-1,n)+R(m,n-1)$.

Proof. Take $R(m-1,n)+R(m,n-1)$ people. We need to show that there exist m people who know each other or n people who don't know each other. If a person X knows $R(m-1,n)$ others, then among the people X knows, there exist either $m-1$ who know each other (so that together with m, there are m people who know each other) or n people who don't know each other, so we are done. Similarly, if X doesn't know $R(m,n-1)$ others, we are also done. But one of these two cases must occur because the total number of 'others' is $R(m-1,n)+R(m,n-1)-1$.

Using Theorem 1, one can easily show (by induction on $m+n$) that $R(m,n) \le C_{m-1}^{m+n-2}$. This establishes an upper bound on $R(m,n)$. To establish a lower bound, we need a counter-example. While construction of counter-examples is in general very difficult, the probabilistic method (see Vol. 14, No. 3) may be able to help us in getting a non-constructive proof. Yet to get the exact value of a Ramsey number, the lower and upper bounds must match, which is extremely difficult. For m, $n > 3$, fewer than 10 values of $R(m,n)$ are known:

$R(3,4)=9$, $R(3,5)=14$, $R(3,6)=18$
$R(3,7)=23$, $R(3,8)=28$, $R(3,9)=36$
$R(4,4)=18$, $R(4,5)=25$

Even $R(5,5)$ is unknown at present. The best lower and upper bounds obtained so far are respectively 43 and 49. Paul Erdös once made the following remark.

> *Suppose an evil alien would tell mankind "Either you tell me [the value of R(5,5)] or I will exterminate the human race."… It would be best in this case to try to compute it, both by mathematics and with a computer.*
>
> *If he would ask [for the value of R(6,6)], the best thing would be to destroy him before he destroys us, because we couldn't [determine R(6,6)].*

Problems related to the Ramsey numbers occur often in mathematical competitions.

Example 2. (CWMO 2005) There are n new students. Among any three of them there exist two who know each other, and among any four of them there exist two who do not know each other. Find the greatest possible value of n.

Solution. The answer is 8. First, n can be 8 if the 8 students are numbered 1 to 8 and student i knows student j if and only if $|i-j| \not\equiv 1,4 \pmod 8$. Next, suppose $n=9$ is possible. Then no student may know 6 others, for among the 6 either 3 don't know each other or 3 know each other (so together with the original student there exist 4 who know each other). Similarly, it cannot happen that a student doesn't know 4 others. Hence each student knows exactly 5 others. But this is impossible, because if we sum the number of others whom each student know, we get $9\times5=45$, which is odd, yet each pair of students who know each other is counted twice.

Remark. The answer to the above problem is $R(3,4)-1$, as can be seen by comparing with the definition of $R(3,4)$.

The Ramsey number can be generalised in many different directions. One is to increase the number of statuses from 2 (know or don't know) to more than 2, as the following example shows.

Example 3. (IMO 1964) Seventeen people correspond by mail with one another — each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

Solution. Suppose the three topics are *A*, *B* and *C*. Pick any person; he writes to 16 others. By the pigeonhole principle, he writes to 6 others on the same topic, say *A*. If any two of the 6 people write to each other on *A*, then we are done. If not, then these 6 people write to each other on *B* or *C*. Since $R(3,3)=6$, either 3 of them write to each other on *B*, or 3 of them write to each other on *C*. In any case there exist 3 people who write to each other about the same topic.

Remark. The above problem proves $R(3,3,3) \le 17$, where $R(m,n,p)$ is defined analogously as $R(m,n)$ except that there are now three possible statuses instead of two. It can be shown that $R(3,3,3)=17$ by constructing a counterexample when there are only 16 people.

Another direction of generalization is to generalise '*m* people who know each other' or '*n* people who don't know each other' to other structures. (Technically, the *graph Ramsey number* $R(G,H)$ is the smallest positive integer *k* such that when every two of *k* points are joined together by a red or blue edge, there must exist a red copy of *G* or a blue copy of *H*. Hence $R(m,n)=R(K_m,K_n)$, where K_m denotes the complete graph on *m* vertices, i.e. *m* points among which every two are joined by an edge).

Example 4. *N* people attend a meeting, and some of them shake hands with each other. Suppose that each person shakes hands with at most 100 other people, and among any 50 people there exist at least two who have shaken hands with each other. Find the greatest possible value of *N*.

Solution. The answer is 4949. We first show that *N* = 4949 is possible: suppose there are 49 groups of 101 people each, and two people shake hands if and only if they are in the same group. It is easy to check that the requirements of the question are satisfied. Now suppose *N* = 4950 and each person shakes hands with at most 100 others. We will show that there exist 50 people who have not shaken hands with each other, thus contradicting the given condition. To do this, pick a first person P_1 and cross out all those who have shaken hands with him. Then pick P_2 from the rest and again cross out those who have shaken hands with him, and so on. In this way, at most 100 people are crossed out each time. After P_{49} is chosen, at least $4950-49-49\times 100=1$ person remains, so we will be able to choose P_{50}. Because of the 'crossing out' algorithm, we see that no two of P_1, P_2, …, P_{50} have shaken hands with each other.

Remark. By identifying each person with a point and joining two points by a red line if two people have shaken hands and a blue line otherwise, we see that the above problem proves $R(K_{1,100},K_{50})=4950$. Here $K_{1,100}$ is the graph on 101 points by joining 1 point to the other 100 points.

The *Van der Waerden number* $W(r,k)$ is the smallest positive integer *N* such that if each of 1, 2, …, *N* is assigned one of *r* colours, then there exist a monochromatic *k*-term arithmetic progression. The following example shows that we have $W(2,3) \le 325$.

Example 5. If each of the integers 1, 2, …, 325 is assigned red or blue colour, there exist three integers *p*, *q*, *r* which are assigned the same colour and which form an arithmetic progression.

Solution. Divide the 325 integers into 65 groups $G_1 = \{1, 2, 3, 4, 5\}$, $G_2 = \{6, 7, 8, 9, 10\}$, …, $G_{65} = \{321, 322, 323, 324, 325\}$. There are $2^5 = 32$ possible colour patterns for each group. Hence there exist three groups G_a and G_b, $1 \le a < b \le 33$, whose colour patterns are the same. We note that $2b-a \le 65$ and that *a*, *b*, $2b-a$ form an arithmetic progression. Now two of the first three numbers of G_a are of the same colour, say, the first and third are red (it can be seen that the proof goes exactly the same way if it is the first and second, or second and third). If the fifth is also red, then we are done. Otherwise, the first and third numbers of both G_a and G_b (recall that they have identical colour patterns) are red while the fifth is blue. If the fifth number of G_{2b-a} is red, then it together with the first number of G_a and the third number of G_b form a red arithmetic progression; if it is blue, then it together with the fifth numbers of G_a and G_b form a blue arithmetic progression.

It can be shown via a two-dimensional inductive argument that $W(r,k)$ exists for all *r*, *k*. We see that the existence of Ramsey numbers and van der Waerden numbers are very similar: both say that the desired structure exists in a sufficiently large population.

An analogy to this (though not mathematically rigorous) is that when there are sufficiently many stars in the sky, one can form from them whatever picture one wishes. (This is one of the lines in the movie *A Beautiful Mind*!)

Yet another generalization of the van der Waerden Theorem (which says that $W(r,k)$ exists for all *r*, *k*) is the *Hales-Jewett Theorem*. The exact statement of the theorem is rather technical, but we can look at an informal version here. We are familiar with the two-person tic-tac-toe game played on a 3×3 square in two dimensions. We also have the two-person tic-tac-toe game played on a $4\times 4\times 4$ cube in three dimensions (try it out at http://www.mathdb.org/fun/games/tie_toe/e_tie_toe.htm!). Both games can end in a draw. However, it is easy to see that a two-person tic-tac-toe game played on a 2×2 square in two dimensions cannot end in a draw. The Hales-Jewett Theorem says that for any *n* and *k*, the *k*-person tic-tac-toe game played on an $n\times n\times\cdots\times n$ (*D* factors of *n*, where *D* is the dimension) hypercube cannot end in a draw when *D* is large enough! (For instance, we have just seen that when *n* = 2 and *k* = 2, then *D* = 2 is large enough, while when *n* = 3 and *k* = 2, then *D* = 2 is not large enough.) In case *k* = 2 (i.e. a two-person game) and when *D* is large enough so that a draw is impossible, it can be shown (via a so-called strategy stealing argument) that the first player has a winning strategy!

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***May 21, 2010.***

Problem 341. Show that there exists an infinite set S of points in the 3-dimensional space such that every plane contains at least one, but not infinitely many points of S.

Problem 342. Let $f(x)=a_nx^n+\cdots+a_1x+p$ be a polynomial with coefficients in the integers and degree $n>1$, where p is a prime number and

$$|a_n|+|a_{n-1}|+\cdots+|a_1|<p.$$

Then prove that $f(x)$ is not the product of two polynomials with coefficients in the integers and degrees less than n.

Problem 343. Determine all ordered pairs (a,b) of positive integers such that $a\neq b$, $b^2+a=p^m$ (where p is a prime number, m is a positive integer) and a^2+b is divisible by b^2+a.

Problem 344. $ABCD$ is a cyclic quadrilateral. Let M, N be midpoints of diagonals AC, BD respectively. Lines BA, CD intersect at E and lines AD, BC intersect at F. Prove that

$$\left|\frac{BD}{AC}-\frac{AC}{BD}\right|=\frac{2MN}{EF}.$$

Problem 345. Let $a_1, a_2, a_3, \cdots$ be a sequence of integers such that there are infinitely many positive terms and also infinitely many negative terms. For every positive integer n, the remainders of $a_1, a_2, \cdots, a_n$ upon divisions by n are all distinct. Prove that every integer appears exactly one time in the sequence.

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Solutions

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Problem 336. (*Due to Ozgur Kircak, Yahya Kemal College, Skopje, Macedonia*) Find all distinct pairs (x,y) of integers satisfying the equation

$$x^3+2009y=y^3+2009x.$$

Solution. **CHOW Tseung Man** (True Light Girls' College), **CHUNG Ping Ngai** (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **D. Kipp JOHNSON** (Valley Catholic School, Beaverton, Oregon, USA), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Emanuele NATALE** (Università di Roma "Tor Vergata", Roma, Italy), **Pedro Henrique O. PANTOJA** (UFRN, Natal, Brazil), **PUN Ying Anna** (HKU), **TSOI Kwok Wing** (PLK Centenary Li Shiu Chung Memorial College), **Simon YAU Chi-Keung** and **Fai YUNG**.

All pairs (x,x) satisfy the equation. If (x,y) satisfies the equation and $x\neq y$, then

$$x^2+xy+y^2=\frac{x^3-y^3}{x-y}=2009\equiv 2 \pmod 3.$$

However, $x^2+xy+y^2\equiv x^2-2xy+y^2=(x-y)^2\equiv 0$ or 1 (mod 3). So there are no solutions with $x\neq y$.

Problem 337. In triangle ABC, $\angle ABC=\angle ACB=40°$. P and Q are two points inside the triangle such that $\angle PAB=\angle QAC=20°$ and $\angle PCB=\angle QCA=10°$. Determine whether B, P, Q are collinear or not.

***Solution* 1.** **CHUNG Ping Ngai** (La Salle College, Form 6) and **HUNG Ka Kin Kenneth** (Diocesan Boys' School).

Let $\angle PBA=a$, $\angle PBC=b$, $\angle QBA=a'$ and $\angle QBC=b'$. By the trigonometric form of Ceva's theorem, we have

$$1=\frac{\sin\angle PBA}{\sin\angle PBC}\frac{\sin\angle PCB}{\sin\angle PCA}\frac{\sin\angle PAC}{\sin\angle PAB}=\frac{\sin a\sin 10^\circ\sin 80^\circ}{\sin b\sin 30^\circ\sin 20^\circ},$$

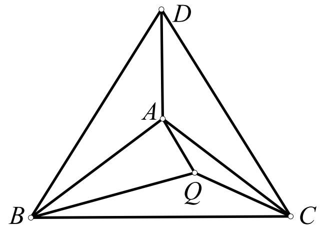
$$1=\frac{\sin\angle QBA}{\sin\angle QBC}\frac{\sin\angle QCB}{\sin\angle QCA}\frac{\sin\angle QAC}{\sin\angle QAB}=\frac{\sin a'\sin 30^\circ\sin 20^\circ}{\sin b'\sin 10^\circ\sin 80^\circ}.$$

As $\sin10°\sin80°=\sin10°\cos10°=½\sin20°=\sin30°\sin20°$, we obtain $\sin a=\sin b$ and $\sin a'=\sin b'$. Since $0<a,b,a',b'<90°$ and $a+b=40°=a'+b'$, we get $a=b=a'=b'=20°$, i.e. $\angle PBA=\angle PBC=\angle QBA=\angle QBC$. Therefore, B, P, Q are collinear.

***Solution* 2**. **LEE Kai Seng**.

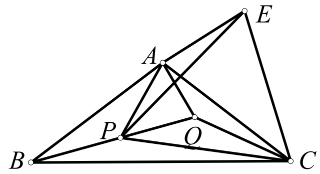
We will show B,P,Q collinear by proving lines BQ and BP bisect $\angle ABC$.

Draw an equilateral triangle BDC with D on the same side of BC as A. Since $\angle ABC=\angle ACB=40°$, $AB=AC$. Then both D and A are equal distance from B and C. So DA bisects $\angle BDC$. We have



$\angle QCD=60°-\angle BCQ=30°=\angle ADC.$

Also, $\angle DCA=\angle QCD-\angle QCA=20°=\angle QAC$, which implies $QA||CD$. Then $AQCD$ is an isosceles trapezoid, so $AD=QC$. This with $BD=BC$ and $\angle BDA=30°=\angle QCB$ imply $\Delta BDA\cong\Delta QCB$. Then $BA=BQ$. Since $\angle BAQ=\angle BAC-\angle QAC=100°-20°=80°$, we get $\angle ABQ=20°=½\angle ABC$. So BQ bisects $\angle ABC$.



Extend BA to a point E so that $BE=BC$. Then $\angle BCE=½(180°-\angle ABC)=70°$. Next, we will show ΔEPC is equilateral.

We have $\angle PCE=\angle BCE-\angle PCB=60°$, $\angle ACE=\angle BCE-\angle BCA=30°=½\angle PCE$. So CA bisects $\angle PCE$. Next, $\angle CAE=180°-\angle BAC=80°=\angle BAC-\angle BAP=\angle CAP$. Then $\Delta CAE\cong\Delta CAP$. So $CE=CP$ and ΔEPC is equilateral. Then B, P are equal distance from E and C. Hence BP bisects $\angle ABC$.

Other commended solvers: **CHAN Chun Wai** (St. Paul's College), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **PUN Ying Anna** (HKU).

Problem 338. Sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_0=1$, $b_0=0$ and for $n=0,1,2,\ldots$,

$$a_{n+1}=7a_n+6b_n-3,$$
$$b_{n+1}=8a_n+7b_n-4.$$

Prove that a_n is a perfect square for all $n=0,1,2,\ldots$

***Solution* 1.** **CHUNG Ping Ngai** (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **D. Kipp JOHNSON** (Valley Catholic School, Beaverton, Oregon, USA), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam), **O Kin Chit Alex** (G.T. (Ellen Yeung) College), **Ercole SUPPA** (Teramo, Italy) and **YEUNG Chun Wing** (St. Paul's College).

Solving for b_n in the first equation and putting it into the second equation, we have

$a_{n+2}=14a_{n+1}-a_n-6$ for $n=0,1,2,\ldots$ (*)

with $a_0=1$ and $a_1=4$. Let $d_n=a_n-½$. Then (*) becomes $d_{n+2}=14d_{n+1}-d_n$. Since the roots of $x^2-14x+1=0$ are $7\pm4\sqrt{3}$, we get d_n is of the form $\alpha(7-4\sqrt{3})^n+\beta(7+4\sqrt{3})^n$. Using $d_0=½$ and $d_1=3½$, we get $\alpha=¼$ and $\beta=¼$. So

$$a_n=d_n+\frac{1}{2}=\frac{2+(7-4\sqrt{3})^n+(7+4\sqrt{3})^n}{4}.$$

Now, consider the sequence $\{c_n\}$ of positive integers, defined by $c_0=1$, $c_1=2$ and

$c_{n+2}=4c_{n+1}-c_n$ for $n=0,1,2,\ldots.$ (**)

Since the roots of $x^2-4x+1=0$ are $2\pm\sqrt{3}$, as above we get

$$c_n=\frac{(2-\sqrt{3})^n+(2+\sqrt{3})^n}{2}.$$

Squaring c_n, we see $a_n=c_n^2$.

***Solution* 2**. **William CHAN** and **Invisible MAK** (Carmel Alison Lam Foundation Secondary School).

The equations imply

$a_{n+2}=14a_{n+1}-a_n-6$ for $n=0,1,2,\ldots$ (*)

We will prove $a_na_{n+2}=(a_{n+1}+3)^2$ by math induction. The case $n=0$ is $1\times49=(4+3)^2$. Suppose $a_{n-1}a_{n+1}=(a_n+3)^2$. Then

$$\begin{aligned}&a_na_{n+2}-(a_{n+1}+3)^2\\&=a_n(14a_{n+1}-a_n-6)-(a_{n+1}+3)^2\\&=14a_na_{n+1}-a_n^2-6a_n-a_{n+1}^2-6a_{n+1}-9\\&=(14a_n-a_{n+1}-6)a_{n+1}-(a_n+3)^2\\&=a_{n-1}a_{n+1}-(a_n+3)^2\\&=0.\end{aligned}$$

This completes the induction.

Next, we will show all a_n's are perfect squares. Now $a_0=1^2$ and $a_1=2^2$. Suppose $a_{n-1}=r^2$ and $a_n=s^2$, we get $a_{n+1}=(a_n+3)^2/r^2$ and $a_{n+2}=(a_{n+1}+3)^2/s^2$. Since the square root of a positive integer is an integer or an irrational number, a_{n+1} and a_{n+2} are perfect squares. By mathematical induction, the result follows.

Other commended solvers: **PUN Ying Anna** (HKU), **TSOI Kwok Wing** (PLK Centenary Li Shiu Chung Memorial College).

Problem 339. In triangle ABC, $\angle ACB=90°$. For every n points inside the triangle, prove that there exists a labeling of these points as $P_1, P_2, \ldots, P_n$ such that

$$P_1P_2^2+P_2P_3^2+\cdots+P_{n-1}P_n^2\le AB^2.$$

Solution. **Federico BUONERBA** (Università di Roma "Tor Vergata", Roma, Italy), **HUNG Ka Kin Kenneth** (Diocesan Boys' School) and **PUN Ying Anna** (HKU).

We will prove the following more general result:

Let ABC be a triangle with $\angle ACB=90°$. For every n points inside or on the sides of the triangle, there exists a labeling of these points as $P_1, P_2, \ldots, P_n$ such that

$$AP_1^2+P_1P_2^2+\cdots+P_{n-1}P_n^2+P_nB^2\le AB^2.$$

We prove this by induction on n. For the case $n=1$, since $\angle AP_1B\ge90°$, the cosine law gives $AP_1^2+P_1B^2\le AB^2$.

Next we assume all cases less than n are true. For the case n, we can divide the original right triangle into two right triangles by taking the altitude from C to H on the hypotenuse AB. We can assume that the two smaller right triangles AHC and BHC contain $m>0$ and $n-m>0$ points respectively (otherwise, one of these two smaller triangles contains all the points and we keep dividing in the same way the smaller right triangle which contains all the points). Since $m<n$ and $n-m<n$, by the induction hypothesis, there exist a labeling of points in triangle AHC as $P_1, P_2, \ldots, P_m$ such that

$$AP_1^2+P_1P_2^2+\cdots+P_{m-1}P_m^2+P_mC^2\le AC^2$$

and a labeling of points in triangle BHC as $P_{m+1}, P_{m+2}, \ldots, P_m$ such that

$$CP_{m+1}^2+P_{m+1}P_{m+2}^2+\cdots+P_nB^2\le CB^2.$$

Since $\angle P_mCP_{m+1}\le90°$, the cosine law gives $P_mP_{m+1}^2\le P_mC^2+CP_{m+1}^2$. Then

$$\begin{aligned}&AP_1^2+P_1P_2^2+\cdots+P_{n-1}P_n^2+P_nB^2\\&\le AC^2+CB^2=AB^2.\end{aligned}$$

Problem 340. Let k be a given positive integer. Find the least positive integer N such that there exists a set of $2k+1$ distinct positive integers, the sum of all its elements is greater than N and the sum of any k elements is at most $N/2$.

Solution. **CHAN Chun Wai** (St. Paul's College), **CHOW Tseung Man** (True Light Girls' College), **CHUNG Ping Ngai** (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **PUN Ying Anna** (HKU).

Let $a_1, a_2, \ldots, a_{2k+1}$ be such a set of $2k+1$ of positive integers arranged in increasing order. We have

$$\sum_{i=1}^{2k+1}a_i\ge N+1\ge2\sum_{i=k+2}^{2k+1}a_i+1.$$

Then

$$\begin{aligned}a_{k+1}&\ge\sum_{i=k+2}^{2k+1}a_i-\sum_{i=1}^{k}a_i+1\\&=\sum_{i=1}^{k}(a_{i+k+1}-a_i)+1\\&\ge\sum_{i=1}^{k}(k+1)+1\\&=k^2+k+1.\end{aligned}$$

Also,

$$\begin{aligned}\frac{N}{2}&\ge\sum_{i=k+2}^{2k+1}a_i=\sum_{i=k+2}^{2k+1}(a_i-a_{k+1})+\sum_{i=k+2}^{2k+1}a_{k+1}\\&\ge\sum_{i=k+2}^{2k+1}(i-k-1)+k(k^2+k+1)\\&=\frac{2k^3+3k^2+3k}{2}.\end{aligned}$$

Now all inequalities above become equality if we take $a_i=k^2+i$ for $i=1, 2, \ldots, 2k+1$. So the least positive value of N is $2k^3+3k^2+3k$.

Olympiad Corner

(continued from page 1)

Problem 4. Let ABC be an acute triangle satisfying the condition $AB>BC$ and $AC>BC$. Denote by O and H the circumcenter and orthocenter, respectively, of the triangle ABC. Suppose that the circumcircle of the triangle AHC intersects the line AB at M different from A, and that the circumcircle of the triangle AHB intersects the line AC at N different from A. Prove that the circumcenter of the triangle MNH lies on the line OH.

Problem 5. Find all functions f from the set $\boldsymbol{R}$ of real numbers into $\boldsymbol{R}$ which satisfy for all $x, y, z\in\boldsymbol{R}$ the identity

$$\begin{aligned}&f(f(x)+f(y)+f(z))\\&=f(f(x)-f(y))+f(2xy+f(z))+2f(xz-yz).\end{aligned}$$