Mathematical Excalibur

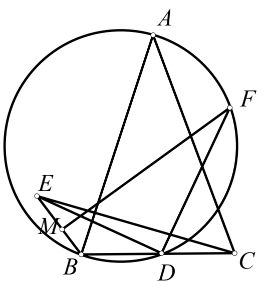
Volume 15, Number 3 *October 2010 - December, 2010*

Olympiad Corner

Below are the problems of the 2010 Chinese Girls' Math Olympiad, which was held on August 10-11, 2010.

Problem 1. Let n be an integer greater than two, and let $A_1, A_2, \ldots, A_{2n}$ be pairwise disjoint nonempty subsets of $\{1,2,\ldots,n\}$. Determine the maximum value of $\sum_{i=1}^{2n}\frac{|A_i\cap A_{i+1}|}{|A_i|\cdot|A_{i+1}|}$. (Here we set $A_{2n+1}=A_1$. For a set X, let $|X|$ denote the number of elements in X.)

Problem 2. In $\triangle ABC$, $AB=AC$. Point D is the midpoint of side BC. Point E lies outside $\triangle ABC$ such that $CE\perp AB$ and $BE=BD$. Let M be the midpoint of segment BE. Point F lies on the minor arc AD of the circumcircle of $\triangle ABD$ such that $MF\perp BE$. Prove that $ED\perp FD$.



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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 14, 2011***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO Shortlisted Problems

Kin Y. Li

Every year, before the IMO begins, a problem selection committee collects problem proposals from many nations. Then it prepares a short list of problems for the leaders to consider when the leaders meet at the IMO site. The following were some of the interesting shortlisted problems in past years that were not chosen. Perhaps some of the ideas may reappear in later proposals in coming years.

Example 1. (*1985 IMO Proposal by Israel*) For which integer $n \ge 3$ does there exist a regular n-gon in the plane such that all its vertices have integer coordinates in a rectangular coordinate system?

Solution. Let A_i have coordinates (x_i,y_i), where x_i, y_i are integers for $i=1,2,\cdots,n$. In the case $n = 3$, if $A_1A_2A_3$ is equilateral, then on one hand, its area is

$$\frac{\sqrt{3}}{4}A_1A_2^2=\frac{\sqrt{3}}{4}\left((x_1-x_2)^2+(y_1-y_2)^2\right),$$

which is irrational. On the other hand, its area is also

$$\frac{\left|\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}\right|}{2}=\pm\frac{1}{2}\begin{vmatrix}x_2-x_1 & y_2-y_1\\ x_3-x_1 & y_3-y_1\end{vmatrix},$$

which is rational. Hence, the case $n = 3$ leads to contradiction. The case $n = 4$ is true by taking (0,0),(0,1),(1,1) and (1,0). The case $n = 6$ is false since $A_1A_3A_5$ would be equilateral.

For the other cases, suppose $A_1A_2\cdots A_n$ is such a regular n-gon with minimal side length. For $i=1,2,\cdots,n$, define point B_i so that $A_iA_{i+1}A_{i+2}B_i$ is a parallelogram (where $A_{n+1}=A_1$ and $A_{n+2}=A_2$). Since $A_{i+1}A_{i+2}$ is parallel to A_iA_{i+3} (where $A_{n+3}=A_3$) and $A_{i+1}A_{i+2} < A_iA_{i+3}$, we see B_i is between A_i and A_{i+3} on the segment A_iA_{i+3}. In particular, B_i is inside $A_1A_2\cdots A_n$.

Next the coordinates of B_i are $(x_{i+2}-x_{i+1}+x_i,\ y_{i+2}-y_{i+1}+y_i)$, both of which are integers. Using A_iA_{i+3} is parallel to $A_{i+1}A_{i+2}$, by subtracting coordinates, we can see $B_i \ne B_{i+1}$ and B_iB_{i+1} is parallel to $A_{i+1}A_{i+2}$. By symmetry, $B_1B_2\cdots B_n$ is a regular n-gon inside $A_1A_2\cdots A_n$. Hence, the side length of $B_1B_2\cdots B_n$ is less than the side length of $A_1A_2\cdots A_n$. This contradicts the side length of $A_1A_2\cdots A_n$ is supposed to be minimal. Therefore, n=4 is the only possible case.

Example 2. (*1987 IMO Proposal by Yugoslavia*) Prove that for every natural number k ($k \ge 2$) there exists an irrational number r such that for every natural number m,

$$[r^m] \equiv -1 \pmod{k}.$$

(Here $[x]$ denotes the greatest integer less than or equal to x.)

(*Comment:* The congruence equation is equivalent to $[r^m]+1$ is divisible by k. Since $[r^m] \le r^m < [r^m]+1$, we want to add a small amount $\delta\in(0,1]$ to r^m to make it an integer divisble by k. If we can get $\delta = s^m$ for some $s\in(0,1)$, then some algebra may lead to a solution.)

Solution. If I have a quadratic equation

$$f(x) = x^2-akx+bk = 0$$

with a, b integers and irrational roots r and s such that $s\in(0,1)$, then $r+s=ak\equiv 0 \pmod{k}$ and $rs=bk\equiv 0 \pmod{k}$. Using

$$r^{m+1}+s^{m+1}=(r+s)(r^m+s^m)-rs(r^{m-1}+s^{m-1}),$$

by induction on m, we see r^m+s^m is also an integer as cases m=0,1 are clear. So

$$[r^m]+1= r^m+s^m \equiv (r+s)^m \equiv 0 \pmod{k}.$$

Finally, to get such a quadratic, we compute the discriminant $\Delta= a^2k^2-4bk$. By taking $a = 2$ and $b = 1$, we have

$$(2k-2)^2 < \Delta= 4k^2-4k < (2k-1)^2.$$

This leads to roots r, s irrational and

$$\frac{1}{2} < s = \frac{2k-\sqrt{\Delta}}{2} < 1.$$

In the next example, we will need to compute the exponent e of a prime number p such that p^e is the largest power of p dividing $n!$. The formula is

$$e=\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\cdots.$$

Basically, since $n!=1\times 2\times\cdots\times n$, we first factor out p from numbers between 1 to n that are divisible by p (this gives $[n/p]$ factors of p), then we factor out another p from numbers between 1 to n that are divisible by p^2 (this give $[n/p^2]$ more factors of p) and so on.

Example 3. (*1983 and 1991 IMO Proposal by USSR*) Let a_n be the last nonzero digit (from left to right) in the decimal representation of $n!$. Prove that the sequence $a_1, a_2, a_3, \ldots$ is not periodic after a finite number of terms (equivalently $0.a_1a_2a_3\ldots$ is irrational).

Solution. Assume beginning with the term a_M, the sequence becomes periodic with period t. Then for $m\ge M$, we have $a_{m+t}=a_m$. To get a contradiction, we will do it in steps.

Step 1. For every positive integer k, $(10^k)! = (10^k-1)!\times 10^k$ implies

$$a_{10^k}=a_{10^k-1}.$$

Step 2. We can get integers $k>m\ge M$ such that 10^k-10^m is a multiple of t as follow. We factor t into the form 2^r5^sw, where w is an integer relatively prime to 10. By Euler's theorem, $10^{\varphi(w)}-1$ is a number divisible by w. Choose $m=\max\{M,r,s\}$ and $k=m+\varphi(w)$. Then $10^k-10^m=2^m5^m(10^{\varphi(w)}-1)$ is a multiple of t, say $10^k-10^m=ct$ for some integer c.

Step 3. Let $n=10^k-1+ct$. By periodicity, we have

$$a_n=a_{10^k-1}=a_{10^k}=a_{n+1}.$$

Let $a_n=d$, that is the last nonzero digit of $n!$ is d. Since $(n+1)!=(n+1)\times n!$ and the last nonzero digit of $n+1=2\times 10^k-10^m$ is 9, we see $a_{n+1}=a_n$ implies the units digit of $9d$ is d. Checking d=1 to 9, we see only $d=5$ is possible. So $n!$ ends in 50…0.

Step 4. By step 3, we see the prime factorization of $n!$ is of the form 2^r5^sw with w relatively prime to 10 and $s\ge r+1>r$. However,

$$r=\left[\frac{n}{2}\right]+\left[\frac{n}{2^2}\right]+\left[\frac{n}{2^3}\right]+\cdots >\left[\frac{n}{5}\right]+\left[\frac{n}{5^2}\right]+\left[\frac{n}{5^3}\right]+\cdots=s.$$

This is a contradiction and we are done.

Example 4. (*2001 IMO Proposal by Great Britain*) Let ABC be a triangle with centroid G. Determine, with proof, the position of the point P in the plane of ABC such that

$$AP\cdot AG+BP\cdot BG+CP\cdot CG$$

is minimum, and express this minimum value in terms of the side lengths of ABC.

Solution. (*Due to the late Professor Murray Klamkin*) Use a vector system with the origin taken to be the centroid of ABC. Denoting the vector from the origin to the point X by $\boldsymbol{X}$, we have

$$\begin{aligned}&AP\cdot AG+BP\cdot BG+CP\cdot CG\\&=|\boldsymbol{A}-\boldsymbol{P}||\boldsymbol{A}|+|\boldsymbol{B}-\boldsymbol{P}||\boldsymbol{B}|+|\boldsymbol{C}-\boldsymbol{P}||\boldsymbol{C}|\\&\ge|(\boldsymbol{A}-\boldsymbol{P})\cdot\boldsymbol{A}|+|(\boldsymbol{B}-\boldsymbol{P})\cdot\boldsymbol{B}|+|(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{C}|\\&=|\boldsymbol{A}|^2+|\boldsymbol{B}|^2+|\boldsymbol{C}|^2\quad(\text{since }\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}=\boldsymbol{0})\\&=(BC^2+CA^2+AB^2)/3.\end{aligned}$$

Equality holds if and only if

$$|\boldsymbol{A}-\boldsymbol{P}||\boldsymbol{A}|=|(\boldsymbol{A}-\boldsymbol{P})\cdot\boldsymbol{A}|,$$
$$|\boldsymbol{B}-\boldsymbol{P}||\boldsymbol{B}|=|(\boldsymbol{B}-\boldsymbol{P})\cdot\boldsymbol{B}|$$
and $$|\boldsymbol{C}-\boldsymbol{P}||\boldsymbol{C}|=|(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{C}|,$$

which is equivalent to P is on the lines GA, GB and GC, i.e. $P=G$.

The next example is a proof of a theorem of Fermat. It is (the contrapositive of) an infinite descent argument that Fermat might have used.

Example 5. (*1978 IMO Proposal by France*) Prove that for any positive integers x, y, z with $xy-z^2=1$ one can find nonnegative integers a, b, c, d such that $x=a^2+b^2$, $y=c^2+d^2$ and $z=ac+bd$. Set $z=(2n)!$ to deduce that for any prime number $p=4n+1$, p can be represented as the sum of squares of two integers.

Solution. We will prove the first statement by induction on z. If z=1, then $(x,y)=(1,2)$ or $(2,1)$ and we take $(a,b,c,d)=(0,1,1,1)$ or $(1,1,0,1)$ respectively.

Next for integer $w>1$, suppose cases $z=1$ to $w-1$ are true. Let positive integers u,v,w satisfy $uv-w^2=1$ with $w>1$. Note $u=v$ leads to w=0, which is absurd. Also $u=w$ leads to w=1, again absurd. Due to symmetry in u, v, we may assume $u<v$. Let $x=u$, $y=u+v-2w$ and $z=w-u$. Since

$$uv=w^2+1>w^2=uv-1>u^2-1,$$

so $y\ge 2(uv)^{1/2}-2w>0$ and $z=w-u>0$. Next we can check $xy-z^2=uv-w^2=1$. By inductive hypothesis, we have

$$x=a^2+b^2,\ y=c^2+d^2,\ z=ac+bd.$$

So $u=x=a^2+b^2$, $w=x+z=a^2+b^2+ac+bd=a(a+c)+b(b+d)$ and $v=y-u+2w=(a+c)^2+(b+d)^2$. This completes the proof of the first statement.

For the second statement, we have

$$\begin{aligned}z^2&=(2n)!(2n)(2n-1)\cdots 1\\&=(2n)!(p-(2n+1))\cdots(p-4n)\\&\equiv(-1)^{2n}(4n)!=(p-1)!\equiv-1\pmod{p},\end{aligned}$$

where the last congruence is by Wilson's theorem. This implies z^2+1 is a multiple of p, i.e. $z^2+1=py$ for some positive integer y. By the first statement, we see $p=a^2+b^2$ for some positive integers a and b.

Example 6. (*1997 IMO Proposal by Russia*) An infinite arithmetic progression whose terms are positive integers contains the square of an integer and the cube of an integer. Show that it contains the sixth power of an integer.

Solution. Let a be the first term and d be the common difference. We will prove by induction on d. If d=1, then the terms are consecutive integers, hence the result is true. Next, suppose d>1. Let $r=\gcd(a,d)$ and $h=d/r$, then $\gcd(a/r,h)=1$. We have two cases.

Case 1: gcd(r,h) = 1. Then $\gcd(a,h)=1$. Since there exist x^2 and y^3 in the progression, so x^2 and $y^3\equiv a \pmod{d}$. Since h divides d, x^2 and $y^3\equiv a\pmod{h}$. From $\gcd(a,h)=1$, we get $\gcd(y,h)=1$. Then there exists an integer t such that $ty\equiv x\pmod{h}$. So

$$t^6a^2\equiv t^6y^6\equiv x^6\equiv a^3\pmod{h}.$$

Since $\gcd(a,h)=1$, we may cancel a^2 to get $t^6\equiv a\pmod{h}$.

Since $\gcd(r,h)=1$, there exists an integer k such that $kh\equiv -t\pmod{r}$. Then we have $(t+kh)^6\equiv 0\equiv a\pmod{r}$ and also $(t+kh)^6\equiv a\pmod{h}$. Since $\gcd(r,h)=1$ and $rh=d$, we get $(t+kh)^6\equiv a\pmod{d}$. Hence, $(t+kh)^6$ is in the progression.

Case 2: gcd(r,h) > 1. Let p be a prime dividing $\gcd(r,h)$. Then p divides r, which divides a and d. Let p^m be the greatest power of p dividing a and p^n be the greatest power of p dividing d. Since $d=rh$, p divides h and $\gcd(a,d)=r$, we see $n>m\ge 1$.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***January 14, 2011.***

Problem 356. *A* and *B* alternately color points on an initially colorless plane as follow. *A* plays first. When *A* takes his turn, he will choose a point not yet colored and paint it red. When *B* takes his turn, he will choose 2010 points not yet colored and paint them blue. When the plane contains three red points that are the vertices of an equilateral triangle, then *A* wins. Following the rules of the game, can *B* stop *A* from winning?

Problem 357. Prove that for every positive integer n, there do not exist four integers a, b, c, d such that $ad=bc$ and $n^2 < a < b < c < d < (n+1)^2$.

Problem 358. $ABCD$ is a cyclic quadrilateral with AC intersects BD at P. Let E, F, G, H be the feet of perpendiculars from P to sides AB, BC, CD, DA respectively. Prove that lines EH, BD, FG are concurrent or are parallel.

Problem 359. (*Due to Michel BATAILLE*) Determine (with proof) all real numbers x,y,z such that $x+y+z \ge 3$ and

$$x^3+y^3+z^3+x^4+y^4+z^4 \le 2(x^2+y^2+z^2).$$

Problem 360. (*Due to Terence ZHU, Affiliated High School of Southern China Normal University*) Let n be a positive integer. We call a set S of at least n distinct positive integers a <u>*n-divisible*</u> set if among every n elements of S, there always exist two of them, one is divisible by the other.

Determine the least integer m (in terms of n) such that every n-divisible set S with m elements contains n integers, one of them is divisible by all the remaining $n-1$ integers.

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Solutions

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Problem 351. Let S be a unit sphere with center O. Can there be three arcs on S such that each is a 300° arc on some circle with O as center and no two of the arcs intersect?

Solution. **Andy LOO** (St. Paul's Co-ed College).

The answer is no. Assume there exist three such arcs l_1, l_2 and l_3. For k=1,2,3, let C_k be the unit circle with center O that l_k is on. Since l_k is a 300° arc on C_k, <u>every point P on C_k is on l_k or its reflection point with respect to O is on l_k</u>. Let P_{ij} and P_{ji} be the intersection points of C_i and C_j. (Since P_{ij} and P_{ji} are reflection points with respect to O, if P_{ij} does not lie on both l_i and l_j, then P_{ji} will be on l_i and l_j, contradiction.) So we may let P_{ij} be the point on l_i and not on l_j and P_{ji} be the point on l_j and not on l_i.

Now P_{21} and P_{31} are on C_1 and outside of l_1, so $\angle P_{21}OP_{31} < 60°$. Hence the length of arcs $P_{21}P_{31}$ and $P_{12}P_{13}$ are equal and are less than $\pi/3$ (and similarly for $\angle P_{32}OP_{12}$, $\angle P_{13}OP_{23}$ and their arcs). Denote the distance (i.e. the length of shortest path) between P and Q on S by $d(P,Q)$. We have

$$\begin{aligned}\pi &= d(P_{12},P_{21})\\ &\le d(P_{12},P_{32})+d(P_{32},P_{31})+d(P_{31},P_{21})\\ &< \pi/3+\pi/3+\pi/3=\pi,\end{aligned}$$

which is absurd.

Other commended solvers: **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College).

Problem 352. (*Proposed by Pedro Henrique O. PANTOJA, University of Lisbon, Portugal*) Let a, b, c be real numbers that are at least 1. Prove that

$$\frac{a^2bc}{\sqrt{bc}+1}+\frac{b^2ca}{\sqrt{ca}+1}+\frac{c^2ab}{\sqrt{ab}+1} \ge \frac{3}{2}.$$

Solution. **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA).

From $a^2\sqrt{bc} \ge \sqrt{bc} \ge 1$, we get

$$\sum_{cyclic}\frac{a^2bc}{\sqrt{bc}+1} \ge \sum_{cyclic}\frac{a^2bc}{2\sqrt{bc}} = \sum_{cyclic}\frac{a^2\sqrt{bc}}{2} \ge \frac{3}{2}.$$

Moreover, we will prove the stronger fact: if $a, b, c > 0$ and $abc \ge 1$, then the inequality still holds. From $k = abc \ge 1$, we get

$$\frac{a^2bc}{\sqrt{bc}+1} = \frac{ka^{3/2}}{\sqrt{k}+a^{1/2}} \ge \frac{a^{3/2}}{1+a^{1/2}}, \quad (*)$$

where the inequality can be checked by cross-multiplication. For $x > 0$, define

$$f(x) = \frac{x^{3/2}}{1+x^{1/2}} - \frac{5}{8}\ln x.$$

Its derivative is

$$f'(x) = \frac{(\sqrt{x}-1)(8x^{3/2}+20x+15x^{1/2}+5)}{8x(\sqrt{x}+1)^2}.$$

This shows $f(1)=1/2$ is the minimum value of f, since $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$. Then by (*),

$$\sum_{cyclic}\frac{a^2bc}{\sqrt{bc}+1} \ge \sum_{cyclic}\frac{a^{3/2}}{1+a^{1/2}} \ge \frac{3}{2}+\frac{5}{8}\ln abc \ge \frac{3}{2}.$$

Other commended solvers: **Samuel Liló ABDALLA** (ITA-UNESP, São Paulo, Brazil), **CHAN Chiu Yuen Oscar** (Wah Yan College Hong Kong), **Ozgur KIRCAK** (Jahja Kemal College, Skopje, Macedonia), **LAM Lai Him** (HKUST Math UG Year 2), **Andy LOO** (St. Paul's Co-ed College), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Salem MALIKIĆ** (Student, University of Sarajevo, Bosnia and Herzegovina), **NG Chau Lok** (HKUST Math UG Year 1), **Thien NGUYEN** (Luong The Vinh High School, Dong Nai, Vietnam), **O Kin Chit Alex** (GT(Ellen Yeung) College), **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Karatapanis SAVVAS** (3rd Senior High School of Rhoades, Greece), **TRAN Trong Hoang Tuan John** (Bac Lieu Specialized Secondary School, Vietnam), **WONG Chi Man** (CUHK Info Engg Grad), **WONG Sze Nga** (Diocesan Girls' School), **WONG Tat Yuen Simon** and **POON Lok Wing** (Carmel Divine Grace Foundation Secondary School) and **Simon YAU.**

Problem 353. Determine all pairs (x, y) of integers such that $x^5-y^2=4$.

Solution. **Ozgur KIRCAK** (Jahja Kemal College, Skopje, Macedonia), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy), **Anderson TORRES** (São Paulo, Brazil) and **Ghaleo TSOI Kwok-Wing** (University of Warwick, Year 1).

Let x, y take on values −5 to 5. We see $x^5 \equiv 0$, 1 or 10 (mod 11), but $y^2+4 \equiv 2$, 4, 5, 7, 8 or 9 (mod 11). Therefore, there can be no solution.

Other commended solvers: **Andy LOO** (St. Paul's Co-ed College).

Problem 354. For 20 boxers, find the least number n such that there exists a schedule of n matches between pairs of them so that for every three boxers, two of them will face each other in one of the matches.

Solution. **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College) and **Andy LOO** (St. Paul's Co-ed College).

Among the boxers, let A be a boxer that will be in the least number of matches, say m matches. For the $19-m$ boxers that do not have a match with A, each pair of them with A form a triple. Since A doesn't play them, every one of these $(19-m)(18-m)/2$ pairs must play each other in a match by the required condition.

For the m boxers that have a match with A, each of them (by the minimal condition on A) has at least m matches. Since each of these matches may be counted at most twice, we get at least $(m+1)m/2$ more matches. So

$$n \geq \frac{(19-m)(18-m)}{2} + \frac{(m+1)m}{2}$$
$$= (m-9)^2 + 90 \geq 90.$$

Finally, $n = 90$ is possible by dividing the 20 boxers into two groups of 10 boxers and in each group, every pair is scheduled a match. This gives a total of 90 matches.

Other commended solvers: **WONG Sze Nga** (Diocesan Girls' School).

Problem 355. In a plane, there are two *similar* convex quadrilaterals $ABCD$ and $AB_1C_1D_1$ such that C, D are inside $AB_1C_1D_1$ and B is outside $AB_1C_1D_1$ Prove that if lines BB_1, CC_1 and DD_1 concur, then $ABCD$ is cyclic. Is the converse also true?

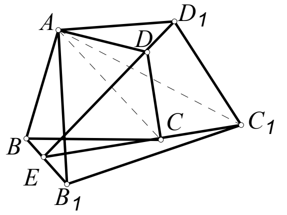
Solution. **CHAN Chiu Yuen Oscar** (Wah Yan College Hong Kong) and **LEE Shing Chi** (SKH Lam Woo Memorial Secondary School).

Since $ABCD$ and $AB_1C_1D_1$ are similar, we have

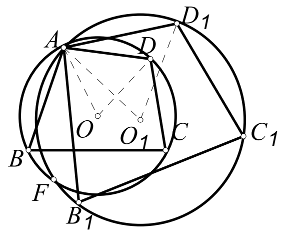
$$\frac{AB}{AB_1} = \frac{AC}{AC_1} = \frac{AD}{AD_1}. \qquad (1)$$

Also, $\triangle ABC$ and $\triangle AB_1C_1$ are similar. Then $\angle BAC = \angle B_1AC_1$. Subtracting $\angle B_1AC$ from both sides, we get $\angle BAB_1 = \angle CAC_1$. Similarly, $\angle CAC_1 = \angle DAD_1$. Along with (1), these give us $\triangle BAB_1$, $\triangle CAC_1$ and $\triangle DAD_1$ are similar. So

$$\angle AB_1B = \angle AC_1C = \angle AD_1D. \qquad (2)$$



Now if lines BB_1, CC_1 and DD_1 concur at E, then (2) can be restated as $\angle AB_1E = \angle AC_1E = \angle AD_1E$. These imply A, B_1, C_1, D_1, E are concyclic. So $AB_1C_1D_1$ is cyclic. Then by similarity, $ABCD$ is cyclic.



For the converse, suppose $ABCD$ is cyclic, then $AB_1C_1D_1$ is cyclic by similarity. Let the two circumcircles intersect at A and F. Let O be the circumcenter of $ABCD$ and O_1 be the circumcenter of $AB_1C_1D_1$. It follows $\triangle AOD$ and $\triangle AO_1D_1$ are similar. Hence $\angle AOD = \angle AO_1D_1$. From this we get

$$\angle AFD = \frac{1}{2}\angle AOD = \frac{1}{2}\angle AOD_1 = \angle AFD_1.$$

This implies line DD_1 passes through F. Similarly, lines BB_1 and CC_1 pass through F. Therefore, lines BB_1, CC_1 and DD_1 concur.

Other commended solvers: **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College).

Olympiad Corner

(continued from page 1)

Problem 3. Prove that for every given positive integer n, there exists a prime p and an integer m such that

(a) $p \equiv 5 \pmod 6$;
(b) $p \nmid n$;
(c) $n \equiv m^3 \pmod p$.

Problem 4. Let $x_1, x_2, \ldots, x_n$ (with $n \geq 2$) be real numbers such that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

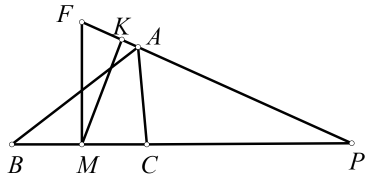
Prove that

$$\sum_{k=1}^{n}\left(1 - \frac{k}{\sum_{i=1}^{n} i x_i^2}\right)^2 \frac{x_k^2}{k} \leq \left(\frac{n-1}{n+1}\right)^2 \sum_{k=1}^{n} \frac{x_k^2}{k}.$$

Determine when equality holds.

Problem 5. Let $f(x)$ and $g(x)$ be strictly increasing linear functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x)$ is an integer if and only if $g(x)$ is an integer. Prove that for any real number x, $f(x) - g(x)$ is an integer.

Problem 6. In acute $\triangle ABC$, $AB > AC$. Let M be the midpoint of side BC. The exterior angle bisector of $\angle BAC$ meets ray BC at P. Points K and F lie on line PA such that $MF \perp BC$ and $MK \perp PA$. Prove that $BC^2 = 4PF \cdot AK$.



Problem 7. Let n be an integer greater than or equal to 3. For a permutation $p = (x_1, x_2, \ldots, x_n)$ of $(1,2,\ldots,n)$, we say x_j *lies between* x_i *and* x_k if $i < j < k$. (For example, in the permutation (1,3,2,4), 3 lies between 1 and 4, and 4 does not lie between 1 and 2.) Set $S=\{p_1, p_2,\ldots,p_m\}$ consists of (distinct) permutations p_i of $(1,2,\ldots,n)$. Suppose that among every three distinct numbers in $\{1,2,\ldots,n\}$, one of these numbers does not lie between the other two numbers in every permutation $p_i \in S$. Determine the maximum value of m.

Problem 8. Determine the least odd number $a > 5$ satisfying the following conditions: There are positive integers m_1, m_2, n_1, n_2 such that $a = m_1^2 + n_1^2$, $a^2 = m_2^2 + n_2^2$ and $m_1 - n_1 = m_2 - n_2$.

IMO Shortlisted Problems

(continued from page 2)

Then p^m divides a and d, hence all terms a, $a+d$, $a+2d$, ⋯ of the progression. In particular, p^m divides x^2 and y^3. Hence, m is a multiple of 6.

Consider the arithmetic progression obtained by dividing all terms of a, $a+d$, $a+2d$,⋯ by p^6. All terms are positive integers, the common difference is $d/p^6 < d$ and also contains $(x/p^3)^2$ and $(y/p^2)^3$. By induction hypothesis, this progression contains a sixth power j^6. Then $(pj)^6$ is a sixth power in a, $a+d$, $a+2d$,⋯ and we are done.